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Some Bitopological Properties via Grills

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Abstract

In this paper we generate a bitopological space from the old one by using the notion of a grill and study the relation between these spaces. Given a bitopological space (X, τ_1, τ_2) (bts, for short) and a grill $\mathcal{G}$ on X, we introduce a new local function $\Phi_{12}(A) = \{x \in X : O_x \cap A \in \mathcal{G} \ \forall \ O_x \in \tau_{12}(x)\}$, where $\tau_{12} = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}$ is a supra topology [7] generated by τ_1 and τ_2, (X, τ_{12}) is a supra topological space associate to the bts (X, τ_1, τ_2). We show that $\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A)$, where $\Phi_i(A) : P(X) \longrightarrow P(X)$, $(i = 1, 2)$, the given local functions associated to the spaces (X, τ_i). We show that the operator $cl^*_{12}(A) = A \cup \Phi_{12}(A)$ is a supra closure operator [6, 8] and then induces a supra topology τ^*_{12} which is finer than τ_{12}, τ^*_{12} is not a topology in general. The properties and the relations between the spaces (X, τ_{12}),(X, τ^*_{12}) and (X, τ^*_1, τ^*_2) have investigated.

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1 Introduction

The idea of a grill on a topological space was first introduced by Choquet [4] in 1947. It is observed that from literature that the concept of grills is a powerful supporting tool, like nets and filters, in dealing with many a topological concept quite effectively. For instance proximity spaces, closure spaces and the theory of compactifications and similar other extensions problems are seen to have been tackled excellently by sheer use of grills (see [3, 2, 12, 14] for details). In this paper, given a bts (X, τ_1, τ_2) and its associated supra topological space (X, τ_{12}) [6]. Also, let $\mathcal{G}$ be a grill on a space X, we introduce a new local function, $\Phi_{12} : P(X) \longrightarrow P(X)$ and show that $\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A)$. By making use of this function, we generate a family τ^*_{12} which is a supra topology and may be not a topology in general. The family τ^*_{12} is finer than τ_1, τ_2 and τ_{12}. The properties of the operator Φ_{12} have obtained. Also, we investigate the relations between τ_1, τ_2 and τ_{12}. We show that the operator $cl^*_{12}(A) = A \cup \Phi_{12}(A)$ is a supra closure operator and this operator induces the family τ^*_{12}. This paper contains 5 sections. Section 2 is a Preliminary section. Section 3 concerns with the notion of the local function Φ_{12} and some of its properties. Section 4 devoted to more properties of the bts's and its relation with the grill. In section 5, the suitability between the family τ_{12} and the grill $\mathcal{G}$ have obtained. Finally, some topological concepts related to such notion have given.

In what follows, by a space X, we shall mean a bts (X, τ_1, τ_2) ($\tau_i - cl$ or $\overline{A}^i$) and ($\tau_i - int$ or A^{io}), $i = 1, 2$, respectively, denote the τ_i-closure and τ_i-interior of A in X. Also, the power set of X will be denoted by $P(X)$ and A' or $X \setminus A$ will stand for the complement of A . A collection $\mathcal{G}$ of a nonempty subsets of a space X is called a grill [14] on X, if it satisfies the following conditions:

1. $\emptyset \notin \mathcal{G}$,
2. $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$, and
3. $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any $x \in X$, we shall let $\tau_i(x)$ (resp. $\tau_{12}(x)$) to denote the collection of all τ_i (τ_{12}-) open nbd of x, $i = 1, 2$.

2 Preliminaries

This section contains the notions which will be needed in the sequel, for more information see [4, 12, 14].

Definition 2.1. *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. We define a mapping $\Phi : P(X) \longrightarrow P(X)$, denoted by $\Phi_{\mathcal{G}}(A, \tau)$ or simply $\Phi(A)$, and defined by $\Phi_{\mathcal{G}}(A)= \{x \in X : O_x \cap A \in \mathcal{G} \ \forall \ O_x \in \tau(x)\}$, $\forall \ A \in P(X)$.*

Proposition 2.1. *Let (X, τ) be a topological space. Then*

1. *If $\mathcal{G}$ is any grill on X, then Φ is an increasing function in the sense that $A \subseteq B \ \Rightarrow \ \Phi(A) \subseteq \Phi(B)$ and if $\mathcal{G}_1$, $\mathcal{G}_2$ are two grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\Phi(\mathcal{G}_1) \subseteq \Phi(\mathcal{G}_2) \ \ \forall \ A \subseteq X$.*

2. *For any grill $\mathcal{G}$ on X and any $A \subseteq X$ if $A \notin \mathcal{G}$, then $\Phi(A) = \emptyset$.*

Proposition 2.2. *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$,*

1. *$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.*
2. *$\Phi(\phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A)$.*

If $\mathcal{G}$ is a grill on X. Define a mapping $cl^* : P(X) \longrightarrow P(X)$ by $cl^*(A) = A \cup \Phi(A)\ \forall\ A \subseteq X$. Then we have:

Theorem 2.1. *The above map cl^* satisfies Kuratowski′s closure axioms.*

Definition 2.2. *Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ (say) on X given by:*

$$\tau_{\mathcal{G}} = \{U \subseteq X : cl^*(U') = U'\},$$

where for any $A \subseteq X$, $cl^*(A) = A \cup \Phi(A) = \tau_{\mathcal{G}} - cl(A)$, U' is the complement of U.

Theorem 2.2.
1. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on a space X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}$.*
2. *If $\mathcal{G}$ is a grill on a space X and $B \notin \mathcal{G}$, then B is closed in $(X, \tau_{\mathcal{G}})$.*
3. *For any subset A of a apace X and any grill $\mathcal{G}$ on X, $\Phi(A)$ is $\tau_{\mathcal{G}}$-closed.*

Definition 2.3. *[9]. A bts is a triple (X, τ_1, τ_2) where τ_1 and τ_2 are arbitrary topologies on X.*

Definition 2.4. *[5]. Let (X, τ_1, τ_2) be a bts. Then $A \subseteq X$ is said to be pairwise open (P.open, for short) if $A = U_1 \cup U_2$, $U_i \in \tau_i$, $(i = 1, 2)$. A set A is P.closed if its complement A' is P.open.*
Note that the notion of P.open sets as well as P.closed sets has studied in [7, 6] under the name of τ_{12}-open and τ_{12}-closed.

Definition 2.5. *[1]. A family $\eta \subseteq P(X)$ is said to be a supra topology on X if η contains $X, \emptyset$ and η closed under arbitrary union. The elements of η are supraopen sets and their complements are said to be supraclosed sets.*

Proposition 2.3. *[6, 5] Let (X, τ_1, τ_2) be a bts. The family of all P.open subsets of X, denoted by*

$$\tau_{12} = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}$$

is a supra topology on X and (X, τ_{12}) is the supra topological space associated to the bts (X, τ_1, τ_2).

Definition 2.6. *[8]. An operator $C : P(X) \longrightarrow P(X)$ is a supra closure closure operator if it satisfies the following conditions for all $A, B \subseteq X$.*

Sc1 $C(\emptyset) = \emptyset$.

Sc2 $A \subseteq C(A)$.

Sc3 $C(A \cup B) \supseteq C(A) \cup C(B)$.

Sc4 $C(C(A)) = C(A)$.

Proposition 2.4. *[7, 6]. Let (X, τ_1, τ_2) be a bts. The operator $cl_{12} : P(X) \longrightarrow P(X)$ defined by $cl_{12}(A) = \overline{A}^1 \cap \overline{A}^2$ is a supra closure operator s.t $\tau_{12} = \{A \subseteq X : cl_{12}(A') = A'\}$.*

Proposition 2.5. *[6]. Let (X, τ_1, τ_2) be a bts. The operator $int_{12} : P(X) \longrightarrow P(X)$ defined by $int_{12}(A) = A^{o1} \cup A^{o2}$ is a supra interior operator s.t $\tau_{12} = \{A \subseteq X : int_{12}(A) = A\}$.*

Now, we prove the following two proposition.

Proposition 2.6. *Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. Then*

1. $\tau_1, \tau_2 \subseteq \tau_{12}$.
2. $cl_{12}(A) = X \backslash int_{12}(X \backslash A)$.
3. $int_{12}(A) = X \backslash cl_{12}(X \backslash A)$.
4. *A is P.open* $\Leftrightarrow A = int_{12}(A)$.
5. *A is P.closed* $\Leftrightarrow A = cl_{12}(A)$.

Proof.
Part (1) follows by definition of τ_{12}. we prove the parts 2 and 4. The proof of the others are similar.

(2) $cl_{12}(A) = \overline{A}^1 \cap \overline{A}^2 = X \backslash (X \backslash A)^{o1} \cap X \backslash (X \backslash A)^{o2} = X \backslash ((X \backslash A)^{o1} \cup (X \backslash A)^{o2}) = X \backslash (int_{12}(X \backslash A))$.

(4) Let A be P.open. Then $A = U_1 \cup U_2$, $U_i \in \tau_i$ $(i = 1, 2)$. It follows that $int_{12}(A) = int_{12}(U_1 \cup U_2) = (U_1 \cup U_2)^{o1} \cup (U_1 \cup U_2)^{o2} \supseteq U_1^{o1} \cup U_2^{o1} \cup U_1^{o2} \cup U_2^{o2} = U_1 \cup U_2^{o1} \cup U_1^{o2} \cup U_2 \supseteq U_1 \cup U_2 = A$. Hence $A \subseteq int_{12}(A)$, but clearly $int_{12}(A) \subseteq A$. So $A = int_{12}(A)$. Conversely, let $A = int_{12}(A)$. Then $A = A^{o1} \cup A^{o2}$, $A^{oi} \in \tau_i$ $(i = 1, 2)$ and consequently A is P.open.

Proposition 2.7. *Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. Then $x \in cl_{12}(A) \Leftrightarrow \forall\, O_x \in \tau_{12}, O_x \cap A \neq \phi$.*

Proof.
Let $x \in cl_{12}(A)$ and $O_x \cap A = \emptyset$ for some $O_x \in \tau_{12}(x)$. Then $O_x = O_x^1 \cup O_x^2$, $O_x^i \in \tau_i$ $(i = 1, 2)$. It follows that, $(O_x^1 \cup O_x^2) \cap A = \emptyset \Rightarrow O_x^1 \cap A = \emptyset$ and $O_x^2 \cap A = \emptyset$. Now $x \in O_x \Rightarrow x \in O_x^1$ or $x \in O_x^2$. If $x \in O_x^1$, $O_x^1 \cap A = \emptyset$, we have $x \notin \overline{A}^1$ and consequently $x \notin cl_{12}(A)$, which is a contradiction. Also, $x \in O_x^2$, $O_x^2 \cap A = \emptyset$, gives $x \notin \overline{A}^2 \Rightarrow x \notin cl_{12}(A)$, which is also a contradiction. Conversely, let $O_x \cap A \neq \emptyset$ $\forall$ $O_x \in \tau_{12}(x)$ and let $x \notin cl_{12}(A)$. Then $x \notin \overline{A}^1$ or $x \notin \overline{A}^2$. So, $x \notin \overline{A}^1 \Rightarrow \exists$ $O_x \in \tau_1 \subseteq \tau_{12}$ s.t $O_x \cap A = \emptyset$. Also, $x \notin \overline{A}^2 \Rightarrow \exists$ $O_x \in \tau_2 \subseteq \tau_{12}$ s.t $O_x \cap A = \emptyset$. In both cases we have a contradiction.

3 Bitopological spaces and the operator Φ_{12}

In this section, we consider (X, τ_1, τ_2) as a bts, and (X, τ_{12}) its associated supra topological space and $\mathcal{G}$ be a grill on X.

Definition 3.1. *Let (X, τ_1, τ_2) be a bts, $\mathcal{G}$ be a grill on a space X and $A \subseteq X$. Then the operator $\Phi_{12} : P(X) \to P(X)$ given by $\Phi_{12}(A) = \{x \in X : O_x \cap A \in \mathcal{G}\ \forall\ O_x \in \tau_{12}(x)\}$ is a local function associated with $\mathcal{G}$.*

Proposition 3.1. *Let (X, τ_1, τ_2) be a bts. Then:*

(i) *If $\mathcal{G}$ is any grill on X, then $\Phi_{12}$is an increasing function,i.e. $A \subseteq B(\subseteq X) \Rightarrow \Phi_{12}(A) \subseteq \Phi_{12}(B)$.*

(ii) *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\Phi_{12}^{\mathcal{G}_1}(A) \subseteq \Phi_{12}^{\mathcal{G}_2}(A)$ $\forall A \subseteq X$.*

(iii) *For any grill $\mathcal{G}$ on X and $A \subseteq X$, if $A \notin \mathcal{G}$, then $\Phi_{12}(A) = \emptyset$.*

Proof. *It follows from the definition of the local function Φ_{12}.*

Proposition 3.2. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$*

(i) $\Phi_{12}(A \cup B) \supseteq \Phi_{12}(A) \cup \Phi_{12}(B)$,

(ii) $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A) = cl_{12}(\Phi_{12}(A)) \subseteq cl_{12}(A)$.

Proof.
(i) Since $A, B \subseteq A \cup B$, by Proposition 3.1 (i), $\Phi_{12}(A) \subseteq \Phi_{12}(A \cup B)$ and $\Phi_{12}(B) \subseteq \Phi_{12}(A \cup B)$. It follows that $\Phi_{12}(A) \cup \Phi_{12}(B) \subseteq \Phi_{12}(A \cup B)$.
(ii) To prove that $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A)$ let $x \in \Phi_{12}(\Phi_{12}(A))$. Then $O_x \cap \Phi_{12}(A) \in \mathcal{G}, \forall O_x \in \tau_{12}(X)$. So, $O_x \cap \Phi_{12}(A) \neq \emptyset$ and consequently there exists

$y \in O_x \cap \Phi_{12}(A)$. Then $y \in O_x$ and $y \in \Phi_{12}(A)$. Thus, $O_y \cap A \in \mathcal{G}$ for all $O_y \in \tau_{12}(y)$. Since $y \in O_x$, $O_x \cap A \in \mathcal{G}$, so $x \in \Phi_{12}(A)$ and therefore $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A)$.

Clearly, $\Phi_{12}(A) \subseteq cl_{12}(\Phi_{12}(A))$, so, we prove that $cl_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A)$. Thus, let $x \in cl_{12}(\Phi_{12}(A))$. Then $\forall O_x \in \tau_{12}(x)$; $O_x \cap \Phi_{12}(A) \neq \emptyset$. So, there exists $y \in O_x \cap \Phi_{12}(A)$. It follows that $y \in O_x$ and $y \in \Phi_{12}(A)$. So, for all $O_y \in \tau_{12}(y)$, $O_y \cap A \in \mathcal{G}$. Hence $O_x \cap A \in \mathcal{G}$ and this yields $x \in \Phi_{12}(A)$. Finally, we have $\Phi_{12}(A) \supseteq cl_{12}(\Phi_{12}(A))$ and consequently $\Phi_{12}(A) = cl_{12}(\Phi_{12}(A))$. Now to complete the proof of part (ii), we show that $\Phi_{12}(A) \subseteq cl_{12}(A)$. So, let $x \notin cl_{12}(A)$. Then there exists $O_x \in \tau_{12}(x)$ such that $O_x \cap A = \emptyset$, then $x \notin \Phi_{12}(A)$ and consequently $\Phi_{12}(A) \subseteq cl_{12}(A)$.

Remark 3.1. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Let (X, τ_1^*, τ_2^*) be a bts induced by $\mathcal{G}$, where*

$$\tau_1^* = \{A \subseteq X : cl_1^*(X \backslash A) = X \backslash A\},$$

$$\tau_2^* = \{A \subseteq X : cl_2^*(X \backslash A) = X \backslash A\},$$

$$cl_i^*(A) = A \cup \Phi_i(A) \ (i = 1, 2) \ \textit{and}$$

$$\Phi_i(A) = \{x \in X : O_x \cap A \in \mathcal{G} \ \forall \ O_x \in \tau_i(x)\}.$$

Also, note that $\tau_i \subseteq \tau_i^*$.

Lemma 3.1. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Let $\Phi_{12} : P(X) \to P(X)$ be a local function. Then*

$$\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A) \ \forall \ , A \subseteq X.$$

Proof.
Let $x \notin \Phi_1(A) \cap \Phi_2(A)$. Then $x \notin \Phi_1(A)$ or $x \notin \Phi_2(A)$. If $x \notin \Phi_1(A)$, then there exists $O_x \in \tau_1 \subseteq \tau_{12}$ such that $O_x \cap A \notin \mathcal{G}$. Hence $x \notin \Phi_{12}(A)$. Similarly, if $x \notin \Phi_2(A)$, then there exists $O_x \in \tau_2 \subseteq \tau_{12}$ such that $O_x \cap A \notin \mathcal{G}$. Hence $x \notin \Phi_{12}(A)$. So, in both cases, $\Phi_{12}(A) \subseteq \Phi_1(A) \cap \Phi_2(A)$. On the other hand, if $x \notin \Phi_{12}(A)$, then there exists $O_x \in \tau_{12}(x)$ such that $O_x \cap A \notin \mathcal{G}$. Now, $O_x \in \tau_{12}(x) \Rightarrow O_x = O_x^1 \cup O_x^2$ ($O_x^i \in \tau_i, i = 1, 2$) $\Rightarrow (O_x^1 \cup O_x^2) \cap A \notin \mathcal{G} \Rightarrow O_x^i \cap A \notin \mathcal{G}$ (since $\mathcal{G}$ is a grill). Now, $x \in O_x \Rightarrow x \in O_x^1$ or $x \in O_x^2 \Rightarrow O_x^1 \cap A \notin \mathcal{G}$ or $O_x^2 \cap A \notin \mathcal{G} \Rightarrow x \notin \Phi_1(A)$ or $x \notin \Phi_2(A) \Rightarrow x \notin \Phi_1(A) \cap \Phi_2(A)$. Hence the result.

The following theorem gives the properties of the local function Φ_{12} in terms of the local functions Φ_1 and Φ_2.

Theorem 3.1. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Then, the local function $\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A)$ satisfies the following properties.*

(i) $\Phi_{12}(\phi) = \phi$,

(ii) $A \subseteq B \Rightarrow \Phi_{12}(A) \subseteq \Phi_{12}(B)$,

(iii) $\Phi_{12}(A) \cup \Phi_{12}(B) \subseteq \Phi_{12}(A \cup B)$

(iv) $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A) = cl_{12}(\Phi_{12}(A)) \subseteq cl_{12}(A)$.

Proof.
(i)$\Phi_{12}(\phi) = \Phi_1(\phi) \cap \Phi_2(\phi) = \phi$.
(ii) Let $A \subseteq B$. *Then* $\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A) \subseteq \Phi_1(B) \cap \Phi_2(B) = \Phi_{12}(B)$ *(by using the properties of* Φ_1, Φ_2*).*
(iii) Follows from (ii).
(iv)

$$\begin{aligned}\Phi_{12}(\Phi_{12}(A)) &= \Phi_1(\Phi_{12}(A)) \cap \Phi_2(\Phi_{12}(A)) \\ &= \Phi_1(\Phi_1(A) \cap \Phi_2(A)) \cap \Phi_2(\Phi_1(A) \cap \Phi_2(A)) \\ &\subseteq \Phi_1(\Phi_1(A)) \cap \Phi_1(\Phi_2(A)) \cap \Phi_2(\Phi_1(A)) \cap \Phi_2(\Phi_2(A) \\ &\subseteq \Phi_1(A) \cap \Phi_1(\Phi_2(A)) \cap \Phi_2(\Phi(A)) \cap \Phi_2(A). \\ &\subseteq \Phi_1(A) \cap \Phi_2(A) = \Phi_{12}(A).\end{aligned}$$

Hence $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A)$.
Clearly, $\Phi_{12}(A) \subseteq cl_{12}(\Phi_{12}(A))$.
On the other hand,

$$\begin{aligned}cl_{12}(\Phi_{12}(A)) &= \overline{\Phi_{12}(A)}^1 \cap \overline{\Phi_{12}(A)}^2 \\ &= \overline{\Phi_1(A) \cap \Phi_2(A)}^1 \cap \overline{\Phi_1(A) \cap \Phi_2(A)}^2 \\ &\subseteq \overline{\Phi_1(A)}^1 \cap \overline{\Phi_2(A)}^1 \cap \overline{\Phi_1(A)}^2 \cap \overline{\Phi_2(A)}^2 \\ &= \Phi_1(A) \cap \overline{\Phi_2(A)}^1 \cap \overline{\Phi_1(A)}^2 \cap \Phi_2(A) (\text{ since } \overline{\Phi_i(A)}^i = \Phi_i(A), \text{ i=1,2}) \\ &= \Phi_1(A) \cap \Phi_2(A) = \Phi_{12}(A).\end{aligned}$$

Hence $\Phi_{12}(A) = cl_{12}(\Phi_{12}(A))$.

Finally, we show that $\Phi_{12}(A) \subseteq cl_{12}(A)$. *Since,* $\Phi_{12}(A) = \Phi_1(A) \cap \Phi_2(A) \subseteq \overline{A}^1 \cap \overline{A}^2 = cl_{12}(A)$ *(since* $\Phi_i(A) \subseteq \overline{A}^i$*, i=1,2).*

If $\mathcal{G}$ *is a grill on a space* (X, τ_1, τ_2). *Define a mapping* $cl^*_{12} : P(X) \to P(X)$ *by* $cl^*_{12}(A) = A \cup \Phi_{12}(A)\ \forall A \subseteq X$. *Then we have the following theorem.*

Theorem 3.2. *The above map cl^*_{12} is a supra closure operator which induces the supra topology $\tau^*_{12} = \{A \subseteq X : cl^*_{12}(X \backslash A) = X \backslash A\}$.*

Proof.
Let $cl^*_{12}(A) = A \cup \Phi_{12}(A)$. *Then*
(SC1) $cl^*_{12}(\phi) = \phi \cup \Phi_{12}(\phi) = \phi$.
(SC2) Clearly, $A \subseteq cl^*_{12}(A)$
Note that if $A \subseteq B$, *then* $cl^*_{12}(A) = A \cup \Phi_{12}(A) \subseteq B \cup \Phi_{12}(B) = cl^*_{12}(B)$, *i.e.*

$A \subseteq B \Rightarrow cl^*_{12}(A) \subseteq cl^*_{12}(B)$.
(SC3) $cl^*_{12}(A) \cup cl^*_{12}(B) \subseteq cl^*_{12}(A \cup B)$ (follows from the above note).
(SC4) The proof follows by using the properties of Φ_1, Φ_2 and by using (SC2). Hence cl^*_{12} is a supra closure operator.

it is easy to show that the family

$$\tau^*_{12} = \{A \subseteq X : cl^*_{12}(X \backslash A) = X \backslash A\}$$

is a supra topology on X it is not a topology in general (see Example 3.1 below).

Definition 3.2. *Corresponding to a grill $\mathcal{G}$ on a bts (X, τ_1, τ_2) there exists a unique supra topology τ^*_{12}(say) on X given by*

$$\tau^*_{12} = \{U \subseteq X : cl^*_{12}(X \backslash U) = X \backslash U\}$$

which is finer than τ_{12} and $cl^*_{12}(A) = A \cup \Phi_{12}(A) = \tau^*_{12} - cl(A) \ \forall \ A \subseteq X$.

Theorem 3.3. *Let (X, τ_1, τ_2) be a bts, $\mathcal{G}$ be a grill on X and $A \subseteq X$. Then*

$$cl^*_{12}(A) = A \cup \Phi_{12}(A) = cl^*_1(A) \cap cl^*_2(A).$$

Proof.
Since $cl^*_{12}(A) = A \cup \Phi_{12}(A)$, then
$$cl^*_{12}(A) = A \cup (\Phi_1(A) \cap \Phi_2(A))$$
$$= (A \cup \Phi_1(A)) \cap (A \cup \Phi_2(A))$$
$$= cl^*_1(A) \cap cl^*_2(A).$$

Note that the above Theorem means that we can established the same supra topology from a bts (X, τ_1, τ_2) by using two equivalent methods. The first follows from the local function Φ_{12} and the other by using the closure operators cl^*_1, cl^*_2 induced by the local functions Φ_1, Φ_2.

Theorem 3.4. *Let (X, τ_1, τ_2) be a bts, $\mathcal{G}$ be a grill on X. Let $(X, {\tau_1}^*, {\tau_2}^*)$ be a bts induced by $\mathcal{G}$ and the local functions Φ_1 and Φ_2. Then*

$$\tau^*_{12} = \{U_1 \cup U_2 : U_i \in \tau^*_i, i = 1, 2\}.$$

Proof.
Let $A \in \tau^*{}_{12}$. Then $cl^*_{12}(X \backslash A) = X \backslash A$
$\Rightarrow X \backslash A = cl^*_1(X \backslash A) \cap cl^*_2(X \backslash A)$
$\Rightarrow X \backslash A = X \backslash int^*_1(A) \cap X \backslash int^*_2(A)$
$\Rightarrow A = int^*_1(A) \cup int^*_2(A) = U_1 \cup U_2,\ U_i \in \tau^*_i$.

Conversely, let $A == U_1 \cup U_2,\ U_i \in \tau^*_i$. Then $cl^*_{12}(X \backslash A) = cl^*_{12}(X \backslash U_1 \cap X \backslash U_2) = cl^*_1(X \backslash U_1 \cap X \backslash U_2) \cap cl^*_2(X \backslash U_1 \cap X \backslash U_2) \subseteq cl^*_1(X \backslash U_1) \cap cl^*_1(X \backslash U_2) \cap cl^*_2(X \backslash U_1) \cap cl^*_2(X \backslash U_2) = X \backslash U_1 \cap cl^*_1(X \backslash U_2) \cap cl^*_2(X \backslash U_1) \cap X \backslash U_2 = X \backslash U_1 \cap X \backslash U_2 = X \backslash A$. But, $X \backslash A \subseteq cl^*_{12}(X \backslash A)$. Hence $cl^*_{12}(X \backslash A) = X \backslash A$ and consequently $A \in \tau^*_{12}$.

Remark 3.2. *Let (X, τ_1, τ_2) be a bts, $\mathcal{G}$ be a grill on X. Then*

(1) $\Phi_{12}(A \cup B) \neq \Phi_{12}(A) \cup \Phi_{12}(B)$ *in general.*

(2) $cl^*_{12}(A \cup B) \neq cl^*_{12}(A) \cup cl^*_{12}(B)$ *in general.*

(3) τ^*_{12} *which induced by* cl^*_{12} *may be not a topology in general but it is a supra topology finer than* τ_1, τ_2 *and* τ_{12}.

(4) $\tau_1 \cup \tau_2 \subseteq \tau_{12} \subseteq \tau^*_{12}$.

Example 3.1. *Let $X = \{a, b, c\}$, τ_1 and τ_2 be two topologies on X such that $\tau_1 = \{\emptyset, X, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a, c\}\}$. Let $\mathcal{G} = P(X)\backslash\emptyset$. Now, let $A = \{b\}$, $B = \{c\}$, then $\tau_{12} = \{\emptyset, X, \{a, b\}, \{a, c\}\}$ is a supra topology, since $\{a, b\} \cap \{a, c\} \notin \tau_{12}$. Since $\Phi_{12}(\{b\}) = \{b\}$, $\Phi_{12}(\{c\}) = \{c\}$ and $\Phi_{12}(\{b, c\}) = X$, then $\Phi_{12}(A \cup B) \neq \Phi_{12}(A) \cup \Phi_{12}(B)$ and consequently, $cl^*_{12}(A \cup B) \neq cl^*_{12}(A) \cup cl^*_{12}(B)$. Also, $\tau^*_{12} = \tau_{12}, \tau_1, \tau_2 \subseteq \tau_{12}$ and τ^*_{12} is a supra topology.*

Proposition 3.3. **(a)** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two grills on a space X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$. Then $\tau^*_{12}(\mathcal{G}_2) \subseteq \tau^*_{12}(\mathcal{G}_1)$*

(b) *If $\mathcal{G}$ is a grill on a space (X, τ_1, τ_2) and $B \notin \mathcal{G}$, then B is P.closed in (X, τ^*_1, τ^*_2).*

(c) *For any subset A of a space X and any grill $\mathcal{G}$ on X, $\Phi_{12}(A)$ is P.closed in (X, τ^*_1, τ^*_2) or it is τ^*_{12}-closed.*

Proof.

(a) Let $U \in \tau^*_{12}(\mathcal{G}_2)$. Then $X\backslash U = \tau^*_{12}(\mathcal{G}_2) - cl(X\backslash U) = X\backslash U \cup \Phi^{\mathcal{G}_2}_{12}(X\backslash U)$
$\Rightarrow \Phi^{\mathcal{G}_2}_{12}(X\backslash U) \subseteq X\backslash U$
$\Rightarrow \Phi^{\mathcal{G}_1}_{12}(X\backslash U) \subseteq \Phi^{\mathcal{G}_2}_{12}(X\backslash U) \subseteq X\backslash U$ *(by Proposition 3.1 (ii))*
$\Rightarrow X\backslash U = X\backslash U \cup \Phi^{\mathcal{G}_1}_{12}(X\backslash U) = \tau^*_{12}(\mathcal{G}_1) - cl(X\backslash U)$
$\Rightarrow U \in \tau^*_{12}(\mathcal{G}_1)$, *i.e.* $\tau^*_{12}(\mathcal{G}_2) \subseteq \tau^*_{12}(\mathcal{G}_1)$.
(b) By Proposition 3.1(iii), $B \notin \mathcal{G} \Rightarrow \Phi_{12}(B) = \phi \Rightarrow cl^*_{12}(B) = B \Rightarrow B$ is a τ^*_{12}-closed or P.closed in (X, τ^*_1, τ^*_2). Another proof, let $B \notin \mathcal{G}$. Then B is τ^*_1-closed and τ^*_2-closed. So, $\Phi_1(B) \subseteq B$ and $\Phi_2(B) \subseteq B \Rightarrow \Phi_{12}(B) = \Phi_1(B) \cap \Phi_2(B) \subseteq B \Rightarrow \Phi_{12}(B) \subseteq B$. Hence $B = B \cup \Phi_{12}(B) = cl^*_{12}(B) \Rightarrow B$ is P.closed in (X, τ^*_1, τ^*_2).
(c) Since $cl^*_{12}(\Phi_{12}(A)) = \Phi_{12}(A) \cup \Phi_{12}(\Phi_{12}(A)) \Rightarrow cl^*_{12}(\Phi_{12}(A)) = \Phi_{12}(A)$(since $\Phi_{12}(\Phi_{12}(A)) \subseteq \Phi_{12}(A)$) $\Rightarrow \tau^*_{12} - cl(\Phi_{12}(A)) = \Phi_{12}(A) \Rightarrow \Phi_{12}(A)$ is a τ^*_{12}-closed or P.closed in (X, τ^*_1, τ^*_2).

4 More properties on bts's and grills

Theorem 4.1. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Then*

$$\beta(\mathcal{G}, \tau_{12}) = \{V \backslash A : V \in \tau_{12},\ A \notin \mathcal{G}\}$$

is an open base for τ_{12}^*.

Proof.

Let $U \in \tau_{12}^*$ *and* $x \in U$. *Then* $X \backslash U$ *is* P*-closed set so that* $cl_{12}^*(X \backslash U) = X \backslash U$ *and hence,* $X \backslash U = X \backslash U \cup \Phi_{12}(X \backslash U) \Rightarrow \Phi_{12}(X \backslash U) \subseteq X \backslash U$, *then* $x \notin \Phi_{12}(X \backslash U)$. *So* $\exists V \in \tau_{12}(x)$ *such that* $V \cap X \backslash U \notin \mathcal{G}$. *Let* $A = V \cap X \backslash U$. *Then* $x \notin A$ *and* $A \notin \mathcal{G}$. *Thus* $x \in V \backslash A \subseteq V \backslash (V \cap X \backslash U) = U$, *i.e.* $x \in V \backslash A \subseteq U$, *where* $V \backslash A \in \beta(\mathcal{G}, \tau_{12})$.

Note that $\beta(\mathcal{G}, \tau_{12})$ *may be not closed under finite intersection, since* τ_{12} *is allready not closed under finite intersection (since* τ_{12} *is a supra topology). Therefore,* τ_{12}^* *may be not a topology, so we only need the condition that every member of* τ_{12}^* *is a union of members of* τ_{12}.

Definition 4.1. *Let* (X, τ_1, τ_2) *be a bts and* $\mathcal{G}$ *be a grill on* X. *Then* $\beta(\mathcal{G}, \tau_{12})$ *is a base for* τ_{12}^* *if* $\forall U \in \tau_{12}^*, x \in U\ \exists V \backslash A \in \beta$ *such that* $x \in V \backslash A \subseteq U$.

Corollary 4.1. *For any grill* $\mathcal{G}$ *on a bts* (X, τ_1, τ_2), $\tau_{12} \subseteq \beta(\mathcal{G}, \tau_{12}) \subseteq \tau_{12}^*$.
Proof. Let $V \in \tau_{12}$, $\phi \notin \mathcal{G}$. *Then* $V = V \backslash \phi \in \beta(\mathcal{G}, \tau_{12}) \subseteq \tau_{12}^*$.

Example 4.1. *Let* (X, τ_1, τ_2) *be a bts. If* $\mathcal{G} = P(X) \backslash \{\emptyset\}$. *Then* $\tau_{12} = \tau_{12}^*$. *In fact, for any* τ_{12}^**-basic open set* $V = U \backslash A$, $U \in \tau_{12}$ *and* $A \notin \mathcal{G}$, *we have* $A = \emptyset$, *so that* $V = U \in \tau_{12}$, $V = U \in \beta(\mathcal{G}, \tau_{12})$ *that is* $\beta(\mathcal{G}, \tau_{12}) \subseteq \tau_{12} \subseteq \beta(\mathcal{G}, \tau_{12})$. *Hence* $\tau_{12} = \beta(\mathcal{G}, \tau_{12}) = \tau_{12}^*$.

Theorem 4.2. *Let* (X, τ_1, τ_2) *be a bts and* $\mathcal{G}$ *be a grill on* X. *If* τ_{12} *is a topology and* $U \in \tau_{12}$, *then* $U \cap \Phi_{12}(A) = U \cap \Phi_{12}(U \cap A)$.

Proof. It is similar to the proof of ([12], Theorem 2.10).

Note that if τ_{12} is a supra topology, then Theorem 4.2 above may be not satisfied as in Example 3.1, let $U = \{a, c\} \in \tau_{12}$ and take $A = \{a, b\}$. Then $\Phi_{12}(\{a, b\}) = X$, $U \cap A = \{a\}$ and $\Phi_{12}(U \cap A) = \{a\}$, then $U \cap \Phi_{12}(A) = \{a, c\}$ while, $U \cap \Phi_{12}(U \cap A) = \{a\} \neq U \cap \Phi_{12}(A)$.

Theorem 4.3. *If* $\mathcal{G}$ *is a grill on a bts* (X, τ_1, τ_2) *such that* τ_{12} *is a topology and* $\tau_{12} \backslash \{\phi\} \subseteq \mathcal{G}$. *Then for all* $U \in \tau_{12}$, $U \subseteq \Phi_{12}(U)$.

Proof.

In case $U = \emptyset$, we obviously have $\Phi_{12}(U) = \phi$. Now, if $\tau_{12} \backslash \{\emptyset\} \subseteq \mathcal{G}$, then $\Phi_{12}(X) = X$. In fact, $x \notin \Phi_{12}(X) \Rightarrow \exists V \in \tau_{12}(x)$ such that $V \cap X \notin \mathcal{G} \Rightarrow V \notin \mathcal{G}$ a contradiction. Now, by the above theorem, for any $U \in \tau_{12} \backslash \{\emptyset\}$ we have, $U \cap \Phi_{12}(X) = U \cap \Phi_{12}(U \cap X)$
$\Rightarrow U \cap X = U \cap \Phi_{12}(U)$
$\Rightarrow U = U \cap \Phi_{12}(U)$
$\Rightarrow U \subseteq \Phi_{12}(U)$.

The following example shows that if τ_{12} is not a topology on X, then the above Theorem does not satisfied.

Example 4.2. *Let* $X = \{a, b, c\}$, τ_1 *and* τ_2 *be two topologies on* X *such that* $\tau_1 = \{\emptyset, X, \{a, c\}\}$ *and* $\tau_2 = \{\emptyset, X, \{b, c\}\}$. *Let* $\mathcal{G} = \{\{a\}, \{a, b\}\{a, c\}, X\}$, $U = \{b, c\}$. *Then* $\tau_{12} = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. *Hence* $\Phi_{12}(U) = \Phi_{12}(\{b, c\}) = \emptyset$ *and consequently* $\{b, c\} \nsubseteq \Phi_{12}(\{b, c\})$.

Lemma 4.1. *For any grill* $\mathcal{G}$ *on a bts* (X, τ_1, τ_2) *and* $A, B \subseteq X$ *such that* $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$, *we have* $\Phi_{12}(A) \backslash \Phi_{12}(B) = \Phi_{12}(A \backslash B) \backslash \Phi_{12}(B)$.

Proof. It is similar to the proof of Lemma 2.12 in [12].

Note that if τ_{12} is a supra topology, then $\Phi_{12}(A \cup B) \neq \Phi_{12}(A) \cup \Phi_{12}(B)$ and consequently Lemma 4.1 is not satisfied as in the following example.

Example 4.3. *Let* $X = \{a, b, c\}$, τ_1 *and* τ_2 *be two topologies on* X *such that* $\tau_1 = \{\emptyset, X, \{a, c\}\}$ *and* $\tau_2 = \{\emptyset, X, \{b, c\}\}$. *Let* $\mathcal{G} = P(X) \backslash \{\emptyset\}$, $A = \{a\}$, $B = \{b\}$. *Then,* $\tau_{12} = \{X, \emptyset, \{a, c\}, \{b, c\}\}$,
$\Phi_{12}(A) = \{a\}$, $\Phi_{12}(B) = \{b\}$. *Also,* $\Phi_{12}(A \cup B) = \Phi_{12}(\{a, b\}) = X \neq \Phi_{12}(A) \cup \Phi_{12}(B) = \{a, b\}$. *Also, take* $A = \{a, b\}$, $B = \{b\}$. *Then* $\Phi_{12}(A) = \Phi_{12}(\{a, b\}) = X$, $\Phi_{12}(B) = \{b\}$ *and* $A \backslash B = \{a\} \Rightarrow \Phi_{12}(A \backslash B) = \{a\}$. *Hence,* $\Phi_{12}(A) \backslash \Phi_{12}(B) = \{a, c\} \neq \Phi_{12}(A \backslash B) \backslash \Phi_{12}(B) = \{a\}$.

Corollary 4.2. *Let* $\mathcal{G}$ *be a grill on a space* (X, τ_1, τ_2) *such that* $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$ *and* $B \notin \mathcal{G}$. *Then* $\Phi_{12}(A \cup B) = \Phi_{12}(A) = \Phi_{12}(A \setminus B)$

Proof.
$\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$ (by Proposition 3.1 (iii))$= \Phi_{12}(A)$.
Also, $\Phi_{12}(A \setminus B) \subseteq \Phi_{12}(A)$ (by Proposition 3.1 (i)). From the above lemma, $\Phi_{12}(A) \setminus \Phi_{12}(B) = \Phi_{12}(A \setminus B) \setminus \Phi_{12}(B) \Rightarrow \Phi_{12}(A) = \Phi_{12}(A \setminus B)$.
Finally, we have $\Phi_{12}(A \cup B) = \Phi_{12}(A) = \Phi_{12}(A \setminus B)$.

Note that, if $(X; \tau_{12})$ is a supra topological space, then $\Phi_{12}(A \cup B) \neq \Phi_{12}(A) \cup \Phi_{12}(B)$, but the converse is not true in general as in the following example.

Example 4.4. *Let* $X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a, b\}\}, \tau_2 = \{X, \emptyset, \{a, c\}\}$. *Then,* $\tau_{12} = \{X, \emptyset, \{a, b\}, \{a, c\}\}$. *Let* $\mathcal{G} = \{X, \{a\}, \{a, b\}, \{a, c\}\}$. *Then* $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B), \forall A, B \subseteq X$, *although* τ_{12} *is a supra topology.*

Theorem 4.4. *Let* $\mathcal{G}$ *be a grill on a space* (X, τ_1, τ_2) *such that* $A \subseteq \Phi_{12}(A)$, *then* $\tau^*{}_{12} - cl(A) \subseteq cl_{12}(A) = cl_{12}(\Phi_{12}(A)) = \Phi_{12}(A)$.

Proof.
Since $\tau^*{}_{12}$ is finer than τ_{12}, then $cl^*{}_{12}(A) \subseteq cl_{12}(A)$. Also, $cl_{12}(\Phi_{12}(A)) = \Phi_{12}(A) \subseteq cl_{12}(A)$(by Proposition 3.2 (ii)).
Further, $A \subseteq \Phi_{12}(A) \Rightarrow cl_{12}(A) \subseteq cl_{12}(\Phi_{12}(A)) \Rightarrow cl_{12}(A) \subseteq \Phi_{12}(A)$ Hence, $cl^*{}_{12}(A) \subseteq cl_{12}(A) = \Phi_{12}(A) = cl_{12}(\Phi_{12}(A))$.

Note that, if τ_{12} is a topology, then $\tau^*_{12} - cl(A) = cl_{12}(A) = cl_{12}(\Phi_{12}(A)) = \Phi_{12}(A)$.

Now, we introduce the notion of a semi P.open subsets of a space (X, τ_1, τ_2).

Definition 4.2. *Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. Then A is said to be semi P.open if $\exists$ a P.open set G such that $G \subseteq A \subseteq cl_{12}(G)$. The complement of semi P.open is a semi P-closed.*

Note that every P.open (resp. P.closed) set is semi P.open (P.closed) set.

Corollary 4.3. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2) and τ_{12} is a topology on X. Then the following are equivalent:-*

1. *For all P.open in $(X; \tau_1; \tau_2), A \subseteq \Phi_{12}(A)$*

2. *For any semi P.open set A in $X, A \subseteq \Phi_{12}(A)$*

Proof.
(2) $\Rightarrow$ (1)is obvious, since every P.open is semi P.open.
(1) $\Rightarrow$ (2) Let A be a semi P.open in X. Then $\exists$ P.open set G in $(X; \tau_1; \tau_2)$ such that $G \subseteq A \subseteq cl_{12}(G)$. Since $G \in \tau_{12}$, by (1), $G \subseteq \Phi_{12}(G)$. Then by Theorem (4.4) $A \subseteq cl_{12}(G) = \Phi_{12}(G) \subseteq \Phi_{12}(A) \Rightarrow A \subseteq \Phi_{12}(A)$.

5 Supra toplogy suitable for a grill

In this section we shall consider grills satisfying a certain condition and this will make the grills suitable and compatible vis-a-vis the topology of the space.

Definition 5.1. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2). Then τ_{12} is said to be suitable for $\mathcal{G}$ if $\forall A \subseteq X, A \backslash \Phi_{12}(A) \notin \mathcal{G}$.*

We now give some characterization of the notion of the suitable.

Theorem 5.1. *For a grill $\mathcal{G}$ on a bts (X, τ_1, τ_2), the following are equivalent:-*

1. *τ_{12} is suitable for $\mathcal{G}$*
2. *For any P.closed subset A in (X, τ_1^*, τ_2^*)(i.e A is τ_{12}^*-closed), $A \setminus \Phi_{12}(A) \notin \mathcal{G}$.*
3. *$A \subseteq X$ and $A \cap \Phi_{12}(A) = \emptyset \Rightarrow A \notin \mathcal{G}$.*

Proof.
(1) $\Rightarrow$ (2) Obviously by the definition.
(2) $\Rightarrow$ (3) Let (2) holds and let $A \subseteq X$ and $A \cap \Phi_{12}(A) = \emptyset$ such that $A \in \mathcal{G}$. Then $A = A \setminus \Phi_{12}(A) \in \mathcal{G}$. By (2), A is not P.closed in $(X, \tau_1^*; \tau_2^*)$. So, $\Phi_{12}(A) \nsubseteq A$ and this implies $\Phi_{12}(A) \cap A \neq \emptyset$ which a contradiction, otherwise, $A \cap \Phi_{12}(A) = \emptyset \Rightarrow A = \emptyset$ or $\Phi_{12}(A) = \emptyset$. In both cases we have $\emptyset \nsubseteq A$ which a contradiction. Hence $A \notin \mathcal{G}$.
(3) $\Rightarrow$ (1) Let $\forall A \subseteq X$ and $A \cap \Phi_{12}(A) = \emptyset \Rightarrow A \notin \mathcal{G}$. We first claim that $A \setminus \Phi_{12}(A) \cap \Phi_{12}(A \setminus \Phi_{12}(A)) = \emptyset$. In fact, $x \in A \setminus \Phi_{12}(A) \Rightarrow x \in A$ and $x \notin \Phi_{12}(A) \Rightarrow \exists O_x \in \tau_{12}(x)$ such that $O_x \cap A \notin \mathcal{G}$. Now $O_x \cap (A \setminus \Phi_{12}(A)) \subseteq O_x \cap A \notin \mathcal{G} \Rightarrow x \notin \Phi_{12}(A \setminus \Phi_{12}(A))$. Hence $(A \setminus \Phi_{12}(A)) \cap \Phi_{12}(A \setminus \Phi_{12}(A)) = \emptyset$. By (3), $A \setminus \Phi_{12}(A) \notin \mathcal{G} \forall A \subseteq X$. Hence the result.

Theorem 5.2. *Let $\mathcal{G}$ be a grill on a bts $(X; \tau_1; \tau_2)$ and $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$. The following are equivalent and each is a necessary condition for τ_{12} to be suitable for the grill $\mathcal{G}$.*

1. *For any $A \subseteq X, A \cap \Phi_{12}(A) = \emptyset \Rightarrow \Phi_{12}(A) = \emptyset$.*
2. *For any $A \subseteq X, \Phi_{12}(A \setminus \Phi_{12}(A)) = \emptyset$*
3. *For any $A \subseteq X, \Phi_{12}(A \cap \Phi_{12}(A)) = \Phi_{12}(A)$*

Proof.
(1) $\Rightarrow$ (2) It follows by noting that $(A \setminus \Phi_{12}(A)) \cap \Phi_{12}(A \setminus \Phi_{12}(A)) = \emptyset, \forall A \subseteq X$.(see Theorem 5.1)
(2) $\Rightarrow$ (3)$A = (A \setminus (A \cap \Phi_{12}(A))) \cup (A \cap \Phi_{12}(A))$
$\Rightarrow \Phi_{12}(A) = \Phi_{12}(A \setminus (A \cap \Phi_{12}(A))) \cup \Phi_{12}(A \cap \Phi_{12}(A))$
$\Rightarrow \Phi_{12}(A) = \Phi_{12}(A \setminus \Phi_{12}(A)) \cup \Phi_{12}(A \cap \Phi_{12}(A))$
$\Rightarrow \Phi_{12}(A) = \Phi_{12}(A \cap \Phi_{12}(A))(by(2))$.
(3) $\Rightarrow$ (1) Let $A \subseteq X$ and $A \cap \Phi_{12}(A) = \emptyset$. Then, $\Phi_{12}(A \cap \Phi_{12}(A)) = \Phi_{12}(A) \Rightarrow \Phi_{12}(A) = \emptyset$.

Corollary 5.1. *If τ_{12} is suitable for a grill $\mathcal{G}$ and $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$, then Φ_{12} is an idempotent, i.e $\Phi_{12}(\Phi_{12}(A)) = \Phi_{12}(A), \forall A \subseteq X$.*

Proof.
By Theorem 5.2 and Proposition 3.1(i), we get, $\Phi_{12}(A) = \Phi_{12}(A \cap \Phi_{12}(A)) \subseteq \Phi_{12}(A) \cap \Phi_{12}(\Phi_{12}(A)) = \Phi_{12}(\Phi_{12}(A))$ by Proposition 3.2 (ii). Hence $\Phi_{12}(\Phi_{12}(A)) = \Phi_{12}(A)$.

Theorem 5.3. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2) such that τ_{12} is suitable for the grill $\mathcal{G}$. Also, let $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$. Then $A \subseteq X$ is P.closed in $(X; \tau_1^*; \tau_2^*)$ iff it can be expressed as a union of a set which is P.closed in (X, τ_1, τ_2) and a set not in $\mathcal{G}$.*

Proof.
Let A be a P.closed subset in (X, τ_1^*, τ_2^*), then $\Phi_{12}(A) \subseteq A$. Now, $A = \Phi_{12}(A) \cup A \setminus \Phi_{12}(A)$. Since τ_{12} is suitable for $\mathcal{G}$, $A \setminus \Phi_{12}(A) \notin \mathcal{G}$ and by Proposition 3.2(ii) $\Phi_{12}(A)$ is P.closed in $(X; \tau_1; \tau_2)$.
Conversely, let $A = F \cup B$, where F is P.closed in (X, τ_1, τ_2)(i.e τ_{12}-closed) and $B \notin \mathcal{G}$. Then $\Phi_{12}(A) = \Phi_{12}(F) \cup \Phi_{12}(B) = \Phi_{12}(F)$(by Propositions 3.1 and 3.2)$\subseteq cl_{12}(F) = F \subseteq A$. Hence $\Phi_{12}(A) \subseteq A$ and consequently A is P.closed in (X, τ_1^*, τ_2^*).

Corollary 5.2. *Let (X, τ_1, τ_2) be a bts and $\mathcal{G}$ be a grill on X. Also, let τ_{12} be a suitable for $\mathcal{G}$. Then $\beta(\mathcal{G}, \tau_{12})$ is a supra topology on X and $\beta(\mathcal{G}, \tau_{12}) = \tau_{12}^*$.*

Proof.
Let U be a P.open in (X, τ_1^*, τ_2^*) (or $U \in \tau_{12}^*$). Then by Theorem 5.3, $X \setminus U = F \cup B$ where F is τ_{12}-closed and $B \notin \mathcal{G}$. Then, $U = X \setminus F \cap X \setminus B \Rightarrow U = (X \setminus F) \setminus B) = V \setminus B$, where $V \in \tau_{12}$ and $B \notin \mathcal{G}$. Thus every τ_{12}^*-open is of the form $V \setminus B$. Hence $\beta(\mathcal{G}, \tau_{12}) = \tau_{12}^*$.

Theorem 5.4. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2) such that $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$ and τ_{12} is a topology suitable for the grill $\mathcal{G}$. Let G be P.open set in (X, τ_1^*, τ_2^*) such that $G = U \setminus A$, where $U \in \tau_{12}$ and $A \notin \mathcal{G}$. Then, $cl_{12}^*(G) = cl_{12}(G) = \Phi_{12}(G) = \Phi_{12}(U) = cl_{12}(U) = cl_{12}^*(U)$.*

Proof.
Let $G = U \setminus A$, where $U \in \tau_{12}$ and $A \notin \mathcal{G}$ (note that in view of Corollary 5.2, Every τ_{12}^*-open set is of this form). Since $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$ by Theorem 4.3, we have $U \subseteq \Phi_{12}(U)$. Hence by Theorem 4.4,

$$\Phi_{12}(U) = cl_{12}(U) = cl_{12}^*(U) \tag{1}$$

Now, G being τ_1^*2-open set, we claim that $G \subseteq \Phi_{12}(G)$. In fact, $cl_{12}^*(X \setminus G) = X \setminus G \Rightarrow \Phi_{12}(X \setminus G) \subseteq X \setminus G \Rightarrow \Phi_{12}(X) \setminus \Phi_{12}(G) = \Phi_{12}(X \setminus G) \setminus \Phi_{12}(G) \subseteq \Phi_{12}(X \setminus G) \subseteq X \setminus G \Rightarrow \Phi_{12}(X) \setminus \Phi_{12}(G) \subseteq X \setminus G$. (by Lemma 4.1)$\Rightarrow X \setminus \Phi_{12}(G) \subseteq X \setminus G \Rightarrow G \subseteq \Phi_{12}(G)$. Hence, by Theorem 4.4

$$cl_{12}(G) = cl_{12}^*(G) = \Phi_{12}(G) \tag{2}$$

Again, $G \subseteq U \Rightarrow \Phi_{12}(G) \subseteq \Phi_{12}(U)$ and also, $\Phi_{12}(G) = \Phi_{12}(U \setminus A) \supseteq \Phi_{12}(U) \setminus \Phi_{12}(A) \supseteq \Phi_{12}(U)$.(by Lemma 4.1)$= \Phi_{12}(U)$(as $A \notin \mathcal{G}$). So $\Phi_{12}(\mathcal{G}) = \Phi_{12}(U)$and consequently,

$$\Phi_{12}(G) = \Phi_{12}(U) \tag{3}$$

From 1,2 and 3 we have the required result.

Theorem 5.5. *Let $\mathcal{G}$ be a grill on a bts $(X;\tau_1;\tau_2)$ such that τ_{12} is a topology suitable for the grill $\mathcal{G}$. Then $\forall G \in \tau_{12}$ and $A \subseteq X, \Phi_{12}(G \cap A) = \Phi_{12}(G \cap \Phi_{12}(A)) = cl_{12}(G \cap \Phi_{12}(A))$*

Proof. It is similar to the proof of Theorem 3.8 in [12].

Corollary 5.3. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2) such that τ_{12} is a topology suitable for the grill $\mathcal{G}$. If $G \in \tau_{12} \backslash \mathcal{G}$, then $G \subseteq X \backslash \Phi_{12}(X)$.*

Proof.
Taking $A = X$ in Theorem 5.5, we get, $cl_{12}(G \cap \Phi_{12}(X)) = \Phi_{12}(G \cap X) = \Phi_{12}(G) \forall G \in \tau_{12}$. Now, if $G \notin \mathcal{G}$, then $\Phi_{12}(G) = \emptyset$ and also, $cl_{12}(G \cap \Phi_{12}(X)) = \emptyset \Rightarrow \Phi_{12}(G \cap \Phi_{12}(X)) = \emptyset \Rightarrow G \subseteq X \setminus \Phi_{12}(X)$.

Remark 5.1. *For any grill $\mathcal{G}$ on a bts (X, τ_1, τ_2). Let A^{d^*} and A^d denote to the drived set of A with respect to τ^*_{12} and τ_{12} respectively. Then*

1. $A^{d^*}_{12} \subseteq A^d_{12}$ *and*
2. $A^{d^*}_{12} \subseteq \Phi_{12}(A)$.

In fact, (1) follows from the fact $\tau_{12} \subseteq \tau^*_{12}$ and for (2) we have $x \in A^{d^*}_{12} \Rightarrow \in cl^*_{12}(A \setminus \{x\}) = A \setminus \backslash\{x\} \cup \Phi_{12}(A\ \{x\}) \Rightarrow x \in \Phi_{12}(A \setminus \{x\}) \subseteq \Phi_{12}(A)$, i.e $A^{d^*}_{12} \subseteq \Phi_{12}(A)$.
Also, we have

Lemma 5.1. *Let $\mathcal{G}$ be a grill on a bts (X, τ_1, τ_2). Let $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B)$ for some $x \in X, \{x\} \notin \mathcal{G}$. Then, $x \in \Phi_{12}(A) \Leftrightarrow x \in A^{d^*}_{12}, \forall A \subseteq X$*

Proof.
Let $A \subseteq X$. Then $A^{d^*}_{12} \subseteq \Phi_{12}(A)$ follows from Remark 5.1. Now, suppose that $\{x\} \notin \mathcal{G}$. Then $x \in \Phi_{12}(A) \Rightarrow x \in \Phi_{12}(A \setminus \{x\})$ (by Corollary 4.2) $\Rightarrow x \in cl^*_{12}(A \setminus \{x\}) \Rightarrow x \in A^{d^*}_{12}$

Definition 5.2. *[11] A grill $\mathcal{G}$ on X is said to be a σ-grill if for any countable collection $\{A_n : n \in \mathbb{N}\}$ of subsets of X, $\cup^{\alpha}_{n=1} A_n \notin \mathcal{G}$ whenever $A_n \notin \mathcal{G} \forall n \in \mathbb{N}$*

Theorem 5.6. *Let $\mathcal{G}$ be a σ-grill on a hereditarily Lindelöf space $(X;\tau_1;\tau_2)$, then τ_{12} is a suitable for the grill $\mathcal{G}$.*

Proof.
Let$A \subseteq X$ such that $A \cap \Phi_{12}(A) = \emptyset$. Then $\forall x \in A \exists O_x \in \tau_{12}(x)$ such that $O_x \cap A \notin \mathcal{G}$. Now,$\{O_x \cap A : x \in A\}$ is a τ_{12A}- open cover of A, by Lindelöfness of A, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of A such that $A = \cup_{i=1}^{\infty}(U_{x_i} \cap A)$. As $\mathcal{G}$ is a σ-grill it follows that $A \notin \mathcal{G}$ and hence by Theorem 5.1, τ_{12} becomes suitable for $\mathcal{G}$.

Theorem 5.7. *Let $\mathcal{G}$ be a grill on a space (X, τ_1, τ_2) such that τ_{12} is a suitable for the grill $\mathcal{G}$ and for each $x \in X, \{x\} \notin \mathcal{G}$. Also, let $\Phi_{12}(A \cup B) = \Phi_{12}(A) \cup \Phi_{12}(B) \forall A, B \subseteq X$. If $A \subseteq X$ is P.closed in $(X; \tau_1^*; \tau_2^*)$, then A can be written as a union of a P.perfect set in (X, τ_1, τ_2) and a set $\notin \mathcal{G}$.*

Proof.
Let $A \subseteq X$ be P.closed in (X, τ_1^*, τ_2^*). Then by Theorem 5.3, $A = \Phi_{12}(A) \cup B$, where $\Phi_{12}(A)$ is P.closed in (X, τ_1, τ_2) and $B \notin \mathcal{G}$. Since τ_{12} is suitable for $\mathcal{G}$, by Corollary 5.1, $\Phi_{12}(A) = \Phi_{12}(\Phi_{12}(A))$. Now, $\Phi_{12}(A)$ being τ_{12}-closed,

$$(\Phi_{12}(A))_{12}^{d} \subseteq \Phi_{12}(A) \tag{4}$$

Again as $\{x\} \notin \mathcal{G} \forall x \in X, \Phi_{12}(B) \subseteq B_{12}^{d^*} \forall B \subseteq X$(Using Lemma 5.1). So $\Phi_{12}(\Phi_{12}(A)) \subseteq (\Phi_{12}(A))_{12}^{d^*} \subseteq (\Phi_{12}(A))_{12}^{d}$. So

$$\Phi_{12}(A) \subseteq (\Phi_{12}(A))_{12}^{d} \tag{5}$$

from 4 and 5, $\Phi_{12}(A) = (\Phi_{12}(A))_{12}^{d}$, Showing that $\Phi_{12}(A)$ is a P.perfect in (X, τ_1, τ_2).

We now take up a few topological properties, specially certain separation axioms, and study them briefly in respect of the families τ_{12} and τ_{12}^*. The first part of the following theorem is similar to the first part of Theorem 3.18 in [12] and we shall prove the second part. At the first we introduce the following definitions which are found in [7].

Definition 5.3. *Let (X, τ_1, τ_2) be a bts. Then it is called:*

1. *P^*T_2(or P^*-Hausdorff) if $\forall x, y \in X, x \neq y, \exists$ P.open sets G, H such that $x \in G, y \in H$ and $G \cap H = \emptyset$*
2. *P^*-Urysohn or $P^*T_{2\frac{1}{2}}$ if $\forall x, y \in X, x \neq y, \exists$ P.open sets G, H such that $x \in G, y \in H$ and $cl_{12}(G) \cap cl_{12}(H) = \emptyset$*
3. *P^*-regular if $\forall x \in X, \forall$ P.open set G such that $x \in G, \exists$ P.open sets H such that $x \in H \subseteq cl_{12}(H) \subseteq G$.*

Theorem 5.8. *Let $\mathcal{G}$ be a grill on a space (X, τ_1, τ_2) such that τ_{12} a topology on X, suitable for the grill $\mathcal{G}$ and $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$. Then*

1. (X, τ_1, τ_2) *is* P^*T_2 *or* P^**-Urysohn iff* (X, τ_1^*, τ_2^*) *is respectively so.*

2. *If* (X, τ_1^*, τ_2^*) *is* P^**-regular, then* $\tau_{12} = \tau_{12}^*$

Proof.As mentioned, we shall prove part (2) and for part (1) see [12]. so, for any $A \subseteq X$, we clearly have $cl_{12}^*(A) \subseteq cl_{12}(A)$ since $\tau_{12} \subseteq \tau_{12}^*$. Now, let $x \notin cl_{12}^*(A)$. Then for some P.open set G in (X, τ_1^*, τ_2^*) containing x we have $G \cap A = \emptyset$. By P^*-regularity of $(X, \tau_1^*, \tau_2^*), \exists$ P.open set H in (X, τ_1^*, τ_2^*) such that $H = U \setminus B$, where $U \in \tau_{12}$ and $B \notin \mathcal{G}$ and also, $x \in H \subseteq cl_{12}^*(H) \subseteq G$. Now,$U \cap A \subseteq cl_{12}(U) \cap A \subseteq cl_{12}^*(H) \cap A$ (by Theorem 5.4)$\subseteq G \cap A = \emptyset$, where $x \in U \in \tau_{12} \Rightarrow x \notin cl_{12}(A)$. Hence $cl_{12}(A) \subseteq cl_{12}^*(A)$ and consequently, $cl_{12}(A) = cl_{12}^*(A) \forall A \subseteq X$, which proves that $\tau_{12} = \tau_{12}^*$

Remark 5.2. *It follows from the above theorem that if* $\mathcal{G}$ *is a grill on* X *such that* $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$ *and* τ_{12} *is a suitable for the grill* $\mathcal{G}$*, then* (X, τ_1, τ_2) *is* P^**-regular whenever* (X, τ_1^*, τ_2^*) *is* P^**-regular. But the converse of this theorem may fail as is shown in the following example.*

Example 5.1. *Let* $X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}\}$ *and* $\tau_2 = \{X, \emptyset, \{b, c\}\}$*. Then* $(X; \tau_1; \tau_2)$ *is* P^**-regular. If* $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$*. Then* $\mathcal{G}$ *is a grill on* X *such that* $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$*. Also,* $A \subseteq X, A \cap \Phi_{12}(A) = \emptyset \Rightarrow A \notin \mathcal{G}$*, i.e* τ_{12} *is a suitable for the grill* $\mathcal{G}$*. Also, we have* $\tau_{12}^* = \{X, \emptyset, \{a\}, \{b, c\}, \{b\}, \{a, b\}\}$*. Hence* (X, τ_{12}^*) *or* (X, τ_1^*, τ_2^*) *is not* P^**-regular. Since,* $b \notin \{a, c\}, \{a, c\}$ *is P.closed in* $(X; \tau_1^*; \tau_2^*)$ *and all P.open sets containing* b *intersect* X *which is the only P.open set containing* $\{a, c\}$

Our next example shows that under the stated conditions of Theorem 5.8 τ_{12} may coincide with τ_{12}^* even (X, τ_1^*, τ_2^*) is not P^*-regular.

Example 5.2. *Let* X *be an uncountable set and* (X, τ_1, τ_2) *be a bts such that and* τ_1 *is the indiscrete topology and* $\tau_2 = \tau_{co}$*, the co-countable topology. Let* $\mathcal{G}$ *be the grill of all uncountable subsets of* X*. Then* $\tau_{12} = \tau_{co}$ *and* (X, τ_{12}) *is a hereditarily Lindelöf space and* $\mathcal{G}$ *is* σ*-grill and by the Theorem 5.6* τ_{12} *is a suitable for the grill* $\mathcal{G}$*. Also, clearly,* $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$*. We show that* $\tau_{12} = \tau_{co} = \tau_{12}^*$ *or* $\tau_{co} = \tau_{co}^*$*. Indeed, for* $V \in \tau_{co}^*$ *with* $V = U \setminus A$*, where* $U \in \tau_{co}$ *and* $A \notin \mathcal{G}$*.* U' *and* A *are countable. Now,*
$V' = X \cap (U \cap A')' = X \cap (U' \cup A) = U' \cup A$*, which is countable. Then* $V \in \tau_{co}$*. It follows that* $\tau_{co} = \tau_{co}^*$ *and clearly* (X, τ_{co}) *and* (X, τ_{co}^*) *are not* P^**-regular.*

Theorem 5.9. *Let* $\mathcal{G}$ *be a grill on a bts* (X, τ_1, τ_2) *such that* $\tau_{12} \setminus \{\emptyset\} \subseteq \mathcal{G}$ *and that* τ_{12} *is a topology suitable for the grill* $\mathcal{G}$*. Then* (X, τ_1, τ_2) *is* P^**-connected* $\Leftrightarrow (X, \tau_1^*, \tau_2^*)$ *is* P^**-connected*

Proof. It is similar to the proof of Theorem 3.22 in [12].

The following example shows that if τ_{12} is a supra topology, then $(X;\tau_1;\tau_2)$ is connected while $(X;\tau_1^*;\tau_2^*)$is not connected. In fact, Let $X=\{a,b,c\},\tau_1=\{X,\emptyset,\{a,b\}\}$ and $\tau_2=\{X,\emptyset,\{b,c\}\}$. Let $\mathcal{G}=P(X)\setminus\{\emptyset,\{b\}\}$. Then $\tau_{12}=\{X,\emptyset,\{a,b\},\{b,c\}\}$ and (X,τ_{12}) is connected. Clearly, $\tau_{12}\setminus\{\emptyset\}\subseteq\mathcal{G}$ and $\forall A\subseteq X, A\cap\Phi_{12}(A)=\emptyset\Rightarrow A\notin\mathcal{G}$, i.e τ_{12} is a suitable for the grill $\mathcal{G}$. It is easy to see that:
$\tau_{12}^*=\{X,\emptyset,\{a\},\{c\},\{a,b\},\{b,c\},\{a,c\}\}$.
Hence τ_{12}^* is a supra topology and (X,τ_{12}^*) is not P^* is connected, since $X=\{a\}\cup\{b,c\}$

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