Cohomology Associated to a Poisson Structure on Weil Bundles

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Abstract

Let $M$ be a paracompact smooth manifold of dimension $n$, $A$ a Weil algebra and $M^A$ the Weil bundle associated. We define and describe the notion of $\bar{d}$-Poisson cohomology and of $\bar{d}_A$-Poisson cohomology on $M^A$.

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1 Introduction

A local algebra in the sens of André Weil or simply a Weil algebra is a real, unitary, commutative algebra of finite dimension with a unique maximal ideal of codimension 1 on $\mathbb{R}$ [10].

Let $A$ be a Weil algebra and $\mathfrak{m}$ be its maximal ideal. We have

\[ A = \mathbb{R} \oplus \mathfrak{m}. \]
The first projection

\[ A = \mathbb{R} \oplus \mathfrak{m} \longrightarrow \mathbb{R} \]

is a homomorphism of algebra which is surjective, called augmentation and the unique none-zero integer \( k \in \mathbb{N} \) such that \( \mathfrak{m}^k \neq (0) \) and \( \mathfrak{m}^{k+1} = (0) \) is the height of \( A \).

If \( M \) is a smooth manifold, \( C^\infty(M) \) the algebra of differentiable functions on \( M \) and \( A \) a Weil algebra of maximal ideal \( \mathfrak{m} \), an infinitely near point to \( x \in M \) of kind \( A \) is a homomorphism of algebras

\[ \xi : C^\infty(M) \longrightarrow A \]

such that \([\xi(f) - f(x)] \in \mathfrak{m}\) for any \( f \in C^\infty(M) \).

We denote \( M_x^A \) the set of all infinitely near points to \( x \in M \) of kind \( A \) and

\[ M^A = \bigcup_{x \in M} M_x^A. \]

The set \( M^A \) is a smooth manifold of dimension \( \dim M \times \dim A \) called manifold of infinitely near points of kind \( A \)[6].

When both \( M \) and \( N \) are smooth manifolds and when

\[ h : M \longrightarrow N \]

is a differentiable application, then the application

\[ h^A : M^A \longrightarrow N^A, \, \xi \longmapsto h^A(\xi), \]

such that, for any \( g \in C^\infty(N) \),

\[ [h^A(\xi)](g) = \xi(g \circ h) \]

is also differentiable. When \( h \) is a diffeomorphism, it is the same for \( h^A \).

Moreover, if \( \varphi : A \longrightarrow B \) is a homomorphism of Weil algebras, for any smooth manifold \( M \), the application

\[ \varphi_M : M^A \longrightarrow M^B, \, \xi \longmapsto \varphi \circ \xi \]

is differentiable. In particular, the augmentation

\[ A \longrightarrow \mathbb{R} \]
defines for any smooth manifold $M$, the projection
\[ \pi_M : M^A \rightarrow M, \]
which assigns every infinitely near point to $x \in M$ to its origin $x$. Thus $(M^A, \pi_M, M)$ defines the bundle of infinitely near points or simply weil bundle \cite{7,4,10}.

If $(U, \varphi)$ is a local chart of $M$ with coordinate functions $(x_1, x_2, \ldots, x_n)$, the application
\[ U^A \rightarrow A^n, \xi \mapsto (\xi(x_1), \xi(x_2), \ldots, \xi(x_n)), \]
is a bijection from $U^A$ into an open of $A^n$. The manifold $M^A$ is a smooth manifold modeled over $A^n$, that is to say an $A$-manifold of dimension $n$ \cite{1,9}.

The set, $C^\infty(M^A, A)$ of differentiable functions on $M^A$ with values in $A$ is a commutative, unitary algebra over $A$. When one identitifies $\mathbb{R}^A$ with $A$, for $f \in C^\infty(M)$, the application
\[ f^A : M^A \rightarrow A, \xi \mapsto \xi(f) \]
is differentiable. Moreover the application
\[ C^\infty(M) \rightarrow C^\infty(M^A, A), f \mapsto f^A, \]
is an injective homomorphism of algebras and we have:
\[ (f + g)^A = f^A + g^A; (\lambda \cdot f)^A = \lambda \cdot f^A; (f \cdot g)^A = f^A \cdot g^A \]
for $\lambda \in \mathbb{R}$, $f$ and $g$ belonging to $C^\infty(M)$.

We denote $\mathfrak{X}(M^A)$, the set of all vector fields on $M^A$. According to \cite{1}, We have the following equivalent assertions:

1. $X : C^\infty(M^A) \rightarrow C^\infty(M^A)$ is a vector field on $M^A$;
2. $X : C^\infty(M) \rightarrow C^\infty(M^A, A)$ is a linear application which verifies
\[ X(fg) = X(f) \cdot g^A + f^A \cdot X(g) \]
for any $f, g \in C^\infty(M)$ i.e is a derivation of $C^\infty(M)$ into $C^\infty(M^A, A)$ with respect to the module structure
\[ C^\infty(M^A, A) \times C^\infty(M) \rightarrow C^\infty(M^A, A), (\varphi, f) \mapsto \varphi \cdot f^A. \]
Thus, the set $\mathfrak{X}(M^A)$ of all vector fields on $M^A$ is a $C^\infty(M^A, A)$-module.
When
\[ \theta : C^\infty(M) \longrightarrow C^\infty(M) \]
is a vector field on \( M \), the application
\[ \theta^A : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto [\theta(f)]^A, \]
is a vector field on \( M^A \). The vector field \( \theta^A \) is the prolongation to \( M^A \) of the vector field \( \theta \).

**Theorem 1** If \( X \) is a vector field on \( M^A \) considered as a derivation of \( C^\infty(M) \) into \( C^\infty(M^A, A) \), then there exists, an unique derivation
\[ \tilde{X} : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A) \]
such that such that

1. \( \tilde{X} \) is \( A \)-linear;
2. \( \tilde{X} [C^\infty(M^A)] \subset C^\infty(M^A) \);
3. \( \tilde{X}(f^A) = X(f) \) for any \( f \in C^\infty(M) \).

Thus, the application
\[ [,] : \mathfrak{X}(M^A) \times \mathfrak{X}(M^A) \longrightarrow \mathfrak{X}(M^A), (X,Y) \longmapsto \tilde{X} \circ Y - \tilde{Y} \circ X, \]
is \( A \)-bilinear and defines a structure of Lie algebra over \( A \) on \( \mathfrak{X}(M^A)[1] \).

The goal of this paper is to define and describe the notion of \( \tilde{d} \)-Poisson cohomology and of \( \tilde{d}_A \)-Poisson cohomology.

## 2 Poisson structure on Weil bundles

In this section, \( M \) is a Poisson manifold i.e there exists a bracket \( \{,\} \) on \( C^\infty(M) \) such that the pair \( (C^\infty(M), \{,\}) \) is a real Lie algebra and for any \( f \in C^\infty(M) \), the application
\[ ad(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f,g\} \]
is a derivation of commutative algebra i.e
\[ \{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\} \]
for $f, g, h \in C^\infty(M)$ [5],[8].

We denote

$$C^\infty(M) \longrightarrow \text{Der}_\mathbb{R}[C^\infty(M)], f \longmapsto \text{ad}(f),$$

the adjoint representation and $d$ the operator of cohomology associated to this representation. For any $p \in \mathbb{N}$,

$$\Lambda^p_{\text{Pois}}(M) = C^p[C^\infty(M), C^\infty(M)]$$

denotes the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree $p$ from $C^\infty(M)$ into $C^\infty(M)$. We have

$$\Lambda^0_{\text{Pois}}(M) = C^\infty(M).$$

The $A$-algebra $C^\infty(M^A, A)$ is a Poisson algebra over $A$ if there exists a bracket $\{,\}$ on $C^\infty(M^A, A)$ such that the pair $(C^\infty(M^A, A), \{,\})$ is a Lie algebra over $A$ satisfying

$$\{\varphi_1 \cdot \varphi_2, \varphi_3\} = \{\varphi_1, \varphi_3\} \cdot \varphi_2 + \varphi_1 \cdot \{\varphi_2, \varphi_3\}$$

for any $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(M^A, A)$ [3],[2].

When $M$ is a Poisson manifold with bracket $\{,\}$, for any $f, g \in C^\infty(M)$,

$$\text{ad}(fg) = \text{ad}(f) \cdot g + f \cdot \text{ad}(g).$$

For any $f \in C^\infty(M)$, let

$$[\text{ad}(f)]^A : C^\infty(M) \longrightarrow C^\infty(M^A, A), g \longmapsto \{f, g\}^A,$$

be the prolongation of the vector field $\text{ad}(f)$ and let

$$[\widehat{\text{ad}(f)}]^A : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A)$$

be the unique $A$-linear derivation such that

$$[\widehat{\text{ad}(f)}]^A(g^A) = [\text{ad}(f)]^A(g) = \{f, g\}^A$$

for any $g \in C^\infty(M)$.

**Theorem 2** [3] For $\varphi \in C^\infty(M^A, A)$, the application

$$\tau_\varphi : C^\infty(M) \longrightarrow C^\infty(M^A, A), f \longmapsto -[\widehat{\text{ad}(f)}]^A(\varphi)$$

is a vector field on $M^A$. 
We denote 
\[ \tilde{\tau}_\varphi : C^\infty(M^A, A) \longrightarrow C^\infty(M^A, A) \]
the unique \( A \)-linear derivation such that 
\[ \tilde{\tau}_\varphi(f^A) = \tau_\varphi(f) \]
for any \( f \in C^\infty(M) \). We have for \( f \in C^\infty(M) \),
\[ \tilde{\tau}_{f^A} = [\text{ad}(f)]^A, \]
and for \( \varphi, \psi \in C^\infty(M^A, A) \) and for \( a \in A \),
\[ \tilde{\tau}_{\varphi + \psi} = \tilde{\tau}_\varphi + \tilde{\tau}_\psi; \]
\[ \tilde{\tau}_{a \cdot \varphi} = a \cdot \tilde{\tau}_\varphi; \]
\[ \tilde{\tau}_{\varphi \cdot \psi} = \varphi \cdot \tilde{\tau}_\psi + \psi \cdot \tilde{\tau}_\varphi. \]

For any \( \varphi, \psi \in C^\infty(M^A, A) \), we let
\[ \{ \varphi, \psi \}_A = \tilde{\tau}_\varphi(\psi). \]

In [3] we show that this bracket defines a structure of \( A \)-Poisson algebra on
\( C^\infty(M^A, A) \).

**Theorem 3** If \( M \) is a Poisson manifold with bracket \( \{ , \} \), then \( \{ , \}_A \) is the prolongation on \( M^A \) of the structure of Poisson on \( M \) defined by \( \{ , \} \).

## 3 \( \tilde{d} \)-Poisson cohomology

**Proposition 4** When \( M \) is a Poisson manifold with bracket \( \{ , \} \), the map
\[ C^\infty(M) \longrightarrow \text{Der}_A \left[ C^\infty(M^A, A) \right], \ f \longmapsto -[\text{ad}(f)]^A \]
is a representation of \( C^\infty(M) \) into \( C^\infty(M^A, A) \).

We denote \( \tilde{d} \) the operator of cohomology associated to this representation. For any \( p \in \mathbb{N} \),
\[ \Lambda^p_{\text{Pois}}(M^A, \sim) = C^p(C^\infty(M), C^\infty(M^A, A)) \]
denotes the $C^\infty(M^A, A)$-module of skew-symmetric multilinear forms of degree $p$ from $C^\infty(M)$ into $C^\infty(M^A, A)$. We have

$$\Lambda^0_{\text{Pois}}(M^A, \sim) = C^\infty(M^A, A).$$

We denote

$$\Lambda_{\text{Pois}}(M^A, \sim) = \bigoplus_{p=0}^n \Lambda^p_{\text{Pois}}(M^A, \sim).$$

Thus, for $\Omega \in \Lambda^p_{\text{Pois}}(M^A, \sim)$ and $f_1, ..., f_{p+1} \in C^\infty(M)$, we have

$$\tilde{d} \Omega(f_1, ..., f_{p+1}) = \sum_{i=1}^{p+1} (-1)^i [ad(f_i)]^A [\Omega(f_1, ..., \hat{f}_i, ..., f_{p+1})]$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega(\{f_i, f_j\}, f_1, ..., \hat{f}_i, ..., \hat{f}_j, ..., f_{p+1})$$

where $\hat{f}_i$ means that the term $f_i$ is omitted.

When $\eta \in \Lambda^p_{\text{Pois}}(M)$, then

$$\eta^A : C^\infty(M) \times ... \times C^\infty(M) \longrightarrow C^\infty(M^A, A), (f_1, ..., f_p) \longmapsto [\eta(f_1, ..., f_p)]^A$$

is skew-symmetric multilinear forms of degree $p$ from $C^\infty(M)$ into $C^\infty(M^A, A)$ i.e

$$\eta^A \in \Lambda^p_{\text{Pois}}(M^A, \sim).$$

Thus

**Proposition 5** For any $\eta \in \Lambda^p_{\text{Pois}}(M)$, we have $\tilde{d} \eta^A = (d \eta)^A$. 
Proof. For any \( f_1, \ldots, f_{p+1} \in C^\infty(M) \), we have
\[
(\tilde{d}\eta^A)(f_1, \ldots, f_{p+1}) = \sum_{i=1}^{p+1} (-1)^i [ad(f_i)]^A \left( \eta^A(f_1, \ldots, \hat{f}_i, \ldots, f_{p+1}) \right)
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta^A \left( \{f_i, f_j\}, f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_{p+1} \right)
= \sum_{i=1}^{p+1} (-1)^i [ad(f_i)]^A \left( \eta(f_1, \ldots, \hat{f}_i, \ldots, f_{p+1}) \right)
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} [\eta(\{f_i, f_j\}), f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_{p+1}]^A
= \sum_{i=1}^{p+1} (-1)^i \{f_i, \eta(f_1, \ldots, \hat{f}_i, \ldots, f_{p+1})\}^A
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} [\eta(\{f_i, f_j\}), f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_{p+1}]^A
= [(d\eta)(f_1, f_2, \ldots, f_{p+1})]^A.
\]

That ends the proof. ■

**Corollary 6** The 1-form \( \eta^A \) is \( \tilde{d} \)-closed i.e \( (\tilde{d}\eta^A) = 0 \), if and only if \( d\eta = 0 \). In particular when \( \eta \) is a derivation of Poisson algebra \( C^\infty(M) \).

Proof. Indeed, for \( p = 1 \), we have
\[
(\tilde{d}\eta^A)(f, g) = [ad(f)]^A[\eta^A(g)] - [ad(g)]^A[\eta^A(f)] - \eta^A(\{f, g\})
= \{f, \eta(g)\}^A - \{g, \eta(f)\}^A - [\eta(\{f, g\})]^A
= ([d\eta](f, g)]^A.
\]

for any \( f, g \in C^\infty(M) \).

Thus \( \tilde{d}\eta^A = 0 \) if and only if \( d\eta = 0 \).

When \( \eta \) is a derivation of Poisson algebra \( C^\infty(M) \), we have \( f, g \in C^\infty(M) \),
\[
\eta(\{f, g\}) = \{\eta(f), g\} + \{f, \eta(g)\}
= \{f, \eta(g)\} - \{g, \eta(f)\}
\]
i.e
\[
(\tilde{d}\eta^A)(f, g) = [d\eta(f, g)]^A
= 0.
\]

That ends the proof. ■
Proposition 7 If \( \eta \) and \( \eta' \) both are cohomologous \( d \)-closed \( p \)-forms then \( \eta^A \) and \( \eta'^A \) both are cohomologous \( \tilde{d} \)-closed \( p \)-forms.

Proof. For any \( f_1, \ldots, f_p \in C^\infty(M) \) we have
\[
[\eta^A - \eta'^A](f_1, \ldots, f_p) = \eta^A(f_1, \ldots, f_p) - \eta'^A(f_1, \ldots, f_p) \\
= \eta(f_1, \ldots, f_p)^A - \eta'(f_1, \ldots, f_p)^A \\
= \eta(f_1, \ldots, f_p) - \eta'(f_1, \ldots, f_p) \\
= [(\eta - \eta')(f_1, \ldots, f_p)]^A.
\]

If there exists \( \nu \in \Lambda^{p-1}_{Pois}(M) \) such that
\[
\eta - \eta' = d\nu
\]
then
\[
[\eta^A - \eta'^A](f_1, \ldots, f_p) = [(\eta - \eta')(f_1, \ldots, f_p)]^A \\
= [d\nu(f_1, \ldots, f_p)]^A \\
= \tilde{d}\nu^A(f_1, \ldots, f_p).
\]
i.e
\[
\eta^A - \eta'^A = \tilde{d}\nu^A.
\]

The cohomology class of the \( d \)-closed \( p \)-form \( \eta \) induces the cohomology class of the \( \tilde{d} \)-closed \( p \)-form \( \eta^A \).

Let \( Z^p_{Pois}(M^A, \sim) \) be the set of \( \tilde{d} \)-closed \( p \)-forms from \( C^\infty(M) \) into \( C^\infty(M^A, A) \) and \( B^p_{Pois}(M^A, \sim) \) be the set of \( \tilde{d} \)-exact \( p \)-forms from \( C^\infty(M) \) into \( C^\infty(M^A, A) \).

We denote
\[
H^p_{Pois}(M^A, \sim) = Z^p_{Pois}(M^A, \sim)/B^p_{Pois}(M^A, \sim).
\]

For \( p = 0 \), \( \Lambda_0^{Pois}(M^A, \sim) = C^\infty(M^A, A) \). It is obvious that \( H^0(M^A, \sim) \) is the center of \( C^\infty(M^A, A) \) i.e the set
\[
\left\{ \phi \in (M^A, A); [\text{ad}(f)]^A(\phi) = 0 \text{ for every } f \in C^\infty(M) \right\}.
\]

For \( p = 1 \), we have
\[
H^1_{Pois}(M^A, \sim) = 0.
\]
4 \( \widetilde{d}_A \)-Poisson cohomology

The map

\[
C^\infty(M^A, A) \longrightarrow \text{Der}_A[C^\infty(M^A, A)], \varphi \mapsto \widetilde{\tau}_\varphi,
\]

is a representation of \( C^\infty(M^A, A) \) into \( C^\infty(M^A, A) \). We denote \( \widetilde{d}_A \) the cohomology operator associated to this representation.

For any \( p \in \mathbb{N} \), \( \Lambda^p_{\text{Pois}}(M^A, \sim_A) = C^p[C^\infty(M^A, A), C^\infty(M^A, A)] \) denotes the \( C^\infty(M^A, A) \)-module of skew-symmetric multilinear forms of degree \( p \) on \( C^\infty(M^A, A) \) into \( C^\infty(M^A, A) \). We have

\[
\Lambda^0_{\text{Pois}}(M^A, \sim_A) = C^\infty(M^A, A).
\]

We denote

\[
\Lambda_{\text{Pois}}(M^A, \sim_A) = \bigoplus_{p=0}^n \Lambda^p_{\text{Pois}}(M^A, \sim_A).
\]

For \( \Omega \in \Lambda^p_{\text{Pois}}(M^A, \sim_A) \) and \( \varphi_1, \varphi_2, ..., \varphi_{p+1} \in C^\infty(M^A, A) \), we have

\[
\widetilde{d}_A \Omega(\varphi_1, ..., \varphi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \widetilde{\tau}_{\varphi_i}[\Omega(\varphi_1, ..., \widehat{\varphi_i}, ..., \varphi_{p+1})]
\]

\[
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega(\{\varphi_i, \varphi_j\}_A, \varphi_1, ..., \widehat{\varphi_i}, ..., \widehat{\varphi_j}, ..., \varphi_{p+1})
\]

i.e

\[
\widetilde{d}_A \Omega(\varphi_1, \varphi_2, ..., \varphi_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \{\varphi_i, \Omega(\varphi_1, ..., \widehat{\varphi_i}, ..., \varphi_{p+1})\}_A
\]

\[
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega(\{\varphi_i, \varphi_j\}_A, \varphi_1, ..., \widehat{\varphi_i}, ..., \widehat{\varphi_j}, ..., \varphi_{p+1}).
\]

For \( p = 1 \), we have

\[
\widetilde{d}_A \Omega(\varphi, \psi) = \{\Omega(\varphi), \psi\}_A + \{\varphi, \Omega(\psi)\}_A - \Omega(\{\varphi, \psi\}_A)
\]

for any \( \varphi, \psi \in C^\infty(M^A, A) \). Thus

**Corollary 8** The 1-form \( \Omega \) is \( \widetilde{d}_A \)-closed i.e \( \widetilde{d}_A \Omega = 0 \) if, and only if,

\[
\Omega(\{\varphi, \psi\}_A = \{\Omega(\varphi), \psi\}_A + \{\varphi, \Omega(\psi)\}_A
\]

i.e \( \Omega \) is a derivation of the algebra \( C^\infty(M^A, A) \).
Let $Z^p_{\text{Pois}}(M^A, \sim_A)$ be the set of $\tilde{d}_A$-closed $p$-forms from $C^\infty(M^A, A)$ into $C^\infty(M^A, A)$ and $B^p_{\text{Pois}}(M^A, \sim_A)$ be the set of $\tilde{d}_A$-exact $p$-forms from $C^\infty(M)$ into $C^\infty(M^A, A)$. We denote

$$H^p_{\text{Pois}}(M^A, \sim_A) = Z^p_{\text{Pois}}(M^A, \sim_A) / B^p_{\text{Pois}}(M^A, \sim_A).$$

For $p = 0$, $H^0_{\text{Pois}}(M^A, \sim_A) = C^\infty(M^A, A)$. It is obvious that $H^0_{\text{Pois}}(M^A, \sim_A)$ is the center of $C^\infty(M^A, A)$ i.e the set

$$\{\varphi \in C^\infty(M^A, A); \{\varphi, \phi\}_A = 0 \text{ for every } \phi \in C^\infty(M^A, A)\}.$$

For $p = 1$, we have

$$H^1_{\text{Pois}}(M^A, \sim_A) = 0.$$

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