Some Simple Properties and Existence of HS-Integrals in a Locally Convex Space

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Abstract

In this paper, we define the Henstock-Stieltjes integral for functions that take values in a locally convex space. Results such as uniqueness, linearity, additivity, Cauchy Criterion, and integrability on subintervals are presented. Moreover, some conditions on the existence of the Henstock-Stieltjes integrals are also presented.

Keywords: Locally convex topological vector space, Henstock-Stieltjes integral

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1 Introduction

Many definitions of Stieltjes-type integrals have been proposed since the original one by T.J. Stieltjes in 1894 [6]. For instance, the Henstock-Stieltjes integral for real-valued functions was defined and investigated in 1998 by Lim, Yoon, and Eun [2]. In this note, we will discuss a locally convex space-valued version of the Henstock-Stieltjes integral. For Banach-valued approach, see [5]. The motivation of this study is based on the work of Marraffa [3].

Throughout this paper, (X, τ) is a Hausdorff locally convex topological vector space (briefly a locally convex space) and $\mathcal{P}(X)$ is a separating family of τ -continuous semi-norms on X that generates the topology τ . The space X is called *complete* if every Cauchy net in X converges. A family Γ of semi-norms on a vector space X has the *max-property* if for any $p_1, p_2 \in \Gamma$ there exists $p \in \Gamma$ and $c > 0$ such that $p \geq c \max\{p_1, p_2\}$.

Let $-\infty < a < b < +\infty$. Then any function $\delta : [a, b] \rightarrow (0, +\infty)$ is often called a *gauge* on $[a, b]$. For a gauge δ on $[a, b]$, a division $D = \{([u, v], t)\}$ of $[a, b]$ is called a δ -fine division if $t \in [u, v] \subset (t - \delta(t), t + \delta(t))$ for each interval-point pair $([u, v], t)$. For an arbitrary gauge δ on $[a, b]$, Cousin's lemma guarantees the existence of a δ -fine division on $[a, b]$, see [7]. Let δ be a gauge on $[a, b]$. Let D and D' be δ -fine divisions of $[a, b]$. We say that D' is *finer than* D (or D' is a *refinement* of D) if for every $([u', v'], t') \in D'$ there exists $([u, v], t) \in D$ such that $[u', v'] \subset [u, v]$ and every tag of D is a tag of D' .

Let $f : [a, b] \rightarrow X$ and $t \in (a, b)$. The function f has a *limit* at t if there exists $\alpha \in X$ such that $\forall p \in \mathcal{P}(X)$ and $\forall \epsilon > 0$, we can find $\delta > 0$ such that for any $x \in [a, b]$ with $0 < |x - t| < \delta$, we have $p(f(x) - \alpha) < \epsilon$. In this case, we write

$$\lim_{x \rightarrow t} f(x) = \alpha \quad \text{or} \quad f(x) \rightarrow \alpha \text{ as } x \rightarrow t.$$

Let $f : [a, b] \rightarrow X$ and $t \in [a, b]$ (resp. $t \in (a, b]$). The function f has a *right limit* (resp. *left limit*) at t if there exists $\alpha \in X$ such that $\forall p \in \mathcal{P}(X)$ and $\forall \epsilon > 0$, we can find $\delta > 0$ such that for any $x \in [a, b]$ with $t < x < t + \delta$ (resp. $t - \delta < x < t$), we have $p(f(x) - \alpha) < \epsilon$. The right limit (resp. left limit) of f at t , if it exists, is denoted by $f(t+)$ (resp. $f(t-)$), and we write

$$f(t+) = \lim_{x \rightarrow t^+} f(x) \quad \left(\text{resp. } f(t-) = \lim_{x \rightarrow t^-} f(x) \right).$$

A function $f : [a, b] \rightarrow X$ is said to be *continuous at* $t \in [a, b]$ if for every $p \in \mathcal{P}(X)$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in [a, b]$ with $|x - t| < \delta$, we have $p(f(x) - f(t)) < \epsilon$. The function $f : [a, b] \rightarrow X$ is said to be *continuous* on $[a, b]$ if it is continuous at every point of $[a, b]$. A function $f : [a, b] \rightarrow X$ is said to be *uniformly continuous* if for every $p \in \mathcal{P}(X)$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$p(f(x) - f(y)) < \epsilon$. A function $f : [a, b] \rightarrow X$ is said to be *bounded* if for every $p \in \mathcal{P}(X)$, there exists $M > 0$ such that $p(f(t)) \leq M$ for all $t \in [a, b]$.

A function $f : [a, b] \rightarrow X$ is said to be *regulated* on $[a, b]$ if it has one sided limits at every point of $[a, b]$, that is, for every $t \in [a, b)$, $f(t+)$ exists and for every $t \in (a, b]$, $f(t-)$ exists. The set of all regulated functions $f : [a, b] \rightarrow X$ will be denoted by $G([a, b], X)$. A function $\psi : [a, b] \rightarrow X$ is said to be a *step function* on $[a, b]$ if there exists a division $D = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a, b]$ such that ψ has a constant value in X on (x_{i-1}, x_i) for every $i = 1, \dots, n$. Let (f_n) be a sequence of X -valued functions defined on $[a, b]$ and let $f : [a, b] \rightarrow X$. We say that (f_n) *converges uniformly* to f on $[a, b]$ if for every $p \in \mathcal{P}(X)$ and $\epsilon > 0$ there exists a positive integer N (independent of x) such that for all $x \in [a, b]$ and for all $n \geq N$, we have $p(f_n(x) - f(x)) < \epsilon$. With the preceding definitions, it can be shown that a function $f : [a, b] \rightarrow X$ is regulated iff there exists a sequence (ψ_n) of step functions defined on $[a, b]$ that converges uniformly to f on $[a, b]$.

A function $g : [a, b] \rightarrow X$ is said to be of *bounded variation* on $[a, b]$ if for each $p \in \mathcal{P}(X)$,

$$TV_a^b(g)_p = \sup \left\{ \sum_{i=1}^n p(g(x_i) - g(x_{i-1})) \right\} < \infty,$$

where the supremum is taken over all divisions $D = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a, b]$. The set of all functions $g : [a, b] \rightarrow X$ which are of bounded variations will be denoted by $BV([a, b], X)$.

2 Some Basic Properties

In what follows, $\mathbb{R}$ denotes the set of reals, $\mathbb{N}$ the set of all positive integers and θ is the zero vector in X .

Definition 2.1. Let $f : [a, b] \rightarrow X$ (resp. $\mathbb{R}$) and $g : [a, b] \rightarrow \mathbb{R}$ (resp. X) be any two functions. We say that the *Henstock-Stieltjes integral* of f with respect to g on $[a, b]$ exists with value $w \in X$, and write $w = (HS) \int_a^b f dg$, if given $\epsilon > 0$ and $p \in \mathcal{P}(X)$ there exists a gauge δ_p on $[a, b]$ such that for all δ_p -fine division $D = \{([u, v], t)\}$ of $[a, b]$, we have $p(S[f, g, D] - w) < \epsilon$, where $S[f, g, D] = (D) \sum f(t)[g(v) - g(u)]$ is called the *Henstock-Stieltjes sum* of f with respect to g over D . In this case w is called the *Henstock-Stieltjes integral* of f with respect to g over $[a, b]$. We also say that (f, g) is *Henstock-Stieltjes integrable* (or simply *HS-integrable*) to w on $[a, b]$. The function f is usually called the *integrand* and g the *integrator*.

Theorem 2.2. *There is at most one vector w satisfying the condition from Definition 2.1.*

Proof. Let w_1 and w_2 be two vectors satisfying Definition 2.1. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Then there exist gauges δ_p^i , $i = 1, 2$, on $[a, b]$ such that for all δ_p^i -fine division $D = \{([u, v], t)\}$ of $[a, b]$, $p(S[f, g, D] - w_i) < \frac{\epsilon}{2}$.

Define $\delta_p = \min\{\delta_p^1, \delta_p^2\}$. Then δ_p is a gauge on $[a, b]$ and hence by the Cousin's Lemma, there exists a δ_p -fine division $D = \{([u, v], t)\}$ of $[a, b]$. Clearly, D is both δ_p^1 -fine and δ_p^2 -fine. Thus,

$$p(w_1 - w_2) \leq p(S[f, g, D] - w_1) + p(S[f, g, D] - w_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

From the arbitrariness of $\epsilon > 0$, we get $p(w_1 - w_2) = 0$ for all $p \in \mathcal{P}(X)$. Since $\mathcal{P}(X)$ is separating, it follows that $w_1 - w_2 = \theta$. Hence, $w_1 = w_2$. $\square$

Theorem 2.3. *If f is HS-integrable with respect to both g_1 and g_2 on $[a, b]$, then for any $c_1, c_2 \in \mathbb{R}$, f is HS-integrable with respect to $c_1g_1 + c_2g_2$ on $[a, b]$ and*

$$(HS) \int_a^b f d(c_1g_1 + c_2g_2) = c_1 \cdot (HS) \int_a^b f dg_1 + c_2 \cdot (HS) \int_a^b f dg_2.$$

Proof. For $i = 1, 2$, we let $w_i = (HS) \int_a^b f dg_i$. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Then there exist gauges δ_p^i , $i = 1, 2$, on $[a, b]$ such that for all δ_p^i -fine division $D = \{([u, v], t)\}$ of $[a, b]$,

$$p(S[f, g_i, D] - w_i) < \frac{\epsilon}{2(1 + |c_i|)}.$$

Define $\delta_p = \min\{\delta_p^1, \delta_p^2\}$. Then δ_p is a gauge on $[a, b]$. Let $D = \{([u, v], t)\}$ be a δ_p -fine division of $[a, b]$. Clearly, D is both δ_p^1 -fine and δ_p^2 -fine. Thus,

$$\begin{aligned} & p(S[f, c_1g_1 + c_2g_2, D] - [c_1w_1 + c_2w_2]) \\ & \leq p(S[f, c_1g_1, D] - c_1w_1) + p(S[f, c_2g_2, D] - c_2w_2) \\ & = |c_1| \cdot p(S[f, g_1, D] - w_1) + |c_2| \cdot p(S[f, g_2, D] - w_2) \\ & < \frac{|c_1|\epsilon}{2(1 + |c_1|)} + \frac{|c_2|\epsilon}{2(1 + |c_2|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Using Definition 2.1, the result follows. $\square$

Theorem 2.4. *If f_1 and f_2 are both HS-integrable with respect to g on $[a, b]$, then for any $c_1, c_2 \in \mathbb{R}$, $c_1f_1 + c_2f_2$ is HS-integrable with respect to g on $[a, b]$ and*

$$(HS) \int_a^b (c_1f_1 + c_2f_2) dg = c_1 \cdot (HS) \int_a^b f_1 dg + c_2 \cdot (HS) \int_a^b f_2 dg.$$

Proof. Using the same arguments in the proof of Theorem 2.3 and the fact that $S[c_1 f_1 + c_2 f_2, g, D] = c_1 \cdot S[f_1, g, D] + c_2 \cdot S[f_2, g, D]$, the result follows. $\square$

Theorem 2.5. *Let $a < c < b$. If f is both HS-integrable with respect to g on $[a, c]$ and $[c, b]$, then f is HS-integrable with respect to g on $[a, b]$. Moreover,*

$$(HS) \int_a^b f dg = (HS) \int_a^c f dg + (HS) \int_c^b f dg.$$

Proof. Let $(HS) \int_a^c f dg = w_1$ and $(HS) \int_c^b f dg = w_2$. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Then there exists a gauge δ_p^1 on $[a, c]$ such that for every δ_p^1 -fine division D_1 of $[a, c]$, we have $p(S[f, g, D_1] - w_1) < \frac{\epsilon}{2}$. Also, there exists a gauge δ_p^2 on $[c, b]$ such that for every δ_p^2 -fine division D_2 of $[c, b]$, we have $p(S[f, g, D_2] - w_2) < \frac{\epsilon}{2}$. For each $t \in [a, b]$, let

$$\delta_p(t) = \begin{cases} \min\{\delta_p^1(t), c - t\}, & \text{if } t \in [a, c) \\ \min\{\delta_p^1(t), \delta_p^2(t)\}, & \text{if } t = c \\ \min\{\delta_p^2(t), t - c\}, & \text{if } t \in (c, b]. \end{cases}$$

This defines a gauge δ_p on $[a, b]$. Let $D = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_p -fine division of $[a, b]$. Then we can find $k \in \{1, 2, \dots, n\}$ such that $t_k = c$. Thus, it follows that there exists $k \in \{1, 2, \dots, n\}$ such that c is a tag of $[x_{k-1}, x_k]$. Set

$$D_1 = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{k-1} \cup \{([x_{k-1}, c], c)\}$$

and

$$D_2 = \{([c, x_k], c)\} \cup \{([x_{i-1}, x_i], t_i)\}_{i=k+1}^n.$$

Clearly, $D = D_1 \cup D_2$. Note that $\{([x_{k-1}, c], c)\}$ is δ_p^1 -fine and $\{([c, x_k], c)\}$ is δ_p^2 -fine. Thus D_1 and D_2 are δ_p^1 -fine and δ_p^2 -fine divisions of $[a, c]$ and $[c, b]$, respectively. Thus,

$$\begin{aligned} p(S[f, g, D] - [w_1 + w_2]) &= p(S[f, g, D_1] + S[f, g, D_2] - w_1 - w_2) \\ &\leq p(S[f, g, D_1] - w_1) + p(S[f, g, D_2] - w_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, by Definition 2.1, the result follows. $\square$

The next result is useful when there is no particular value that can be predicted to be the HS-integral of a function. We will see later that it has some useful consequences. The idea of the proof is borrowed from [3].

Theorem 2.6. *Let X be a complete LCS and suppose that $\mathcal{P}(X)$ has the max-property. Then the following are equivalent.*

(i) f is HS-integrable with respect to g on $[a, b]$.

(ii) For every $\epsilon > 0$ and $p \in \mathcal{P}(X)$ there exists a gauge δ_p on $[a, b]$ such that for all δ_p -fine divisions D_1 and D_2 of $[a, b]$, $p(S[f, g, D_1] - S[f, g, D_2]) < \epsilon$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $w = (HS) \int_a^b f dg$ exists. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Then there exists a gauge δ_p such that if D_1 and D_2 are δ_p -fine divisions of $[a, b]$, then

$$p(S[f, g, D_1] - w) < \frac{\epsilon}{2} \quad \text{and} \quad p(S[f, g, D_2] - w) < \frac{\epsilon}{2}.$$

Consequently, we conclude that for such divisions D_1 and D_2 , then

$$\begin{aligned} p(S[f, g, D_1] - S[f, g, D_2]) &\leq p(S[f, g, D_1] - w) + p(S[f, g, D_2] - w) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(ii) $\Rightarrow$ (i). Assume that statement (ii) holds. Let us work with the functions $f : [a, b] \rightarrow X$ and $g : [a, b] \rightarrow \mathbb{R}$. Define the set

$$\mathbb{D} = \{(K, m) : K \text{ is a finite subset of } \mathcal{P}(X) \text{ and } m \in \mathbb{N}\}.$$

It can be shown that $\mathbb{D}$ is a directed set with respect to the partial order $\leq$ defined as follows: For all $(K, m), (L, n) \in \mathbb{D}$, $(K, m) \leq (L, n)$ iff $K \subset L$ and $m \leq n$. We shall now construct a net on the space X . Let $(K, m) \in \mathbb{D}$. Then we can write $K = \{p_1, p_2, \dots, p_k\}$ for some $p_1, p_2, \dots, p_k \in \mathcal{P}(X)$. Since $\mathcal{P}(X)$ has the max-property, we can find $p_K \in \mathcal{P}(X)$ and a constant $C_K > 0$ such that the inequality $p_i \leq C_K \cdot p_K$ for $i = 1, 2, \dots, k$ holds on X . Corresponding to m and for each $i \in \{1, 2, \dots, k\}$, we can find a gauge $\delta_{p_i}^m$ on $[a, b]$ such that for all $\delta_{p_i}^m$ -fine divisions D_1 and D_2 of $[a, b]$,

$$p_i(S[f, g, D_1] - S[f, g, D_2]) < \frac{1}{m}.$$

Also, there exists a gauge δ_K^m such that for all δ_K^m -fine divisions D_1 and D_2 of $[a, b]$,

$$p_K(S[f, g, D_1] - S[f, g, D_2]) < \frac{1}{m \cdot C_K}.$$

WLOG, assume that $\delta_K^m \leq \min\{\delta_{p_1}^m, \delta_{p_2}^m, \dots, \delta_{p_k}^m\}$. Moreover, we can assume that for each fixed finite set $K \subset \mathcal{P}(X)$, the sequence (δ_K^m) is nondecreasing. Since δ_K^m is a gauge on $[a, b]$, it follows from Cousin's Lemma that there is a δ_K^m -fine division D_K^m of $[a, b]$. Then we form the sum $S[f, g, D_K^m] \in X$. We denote this unique sum by $x_{(K, m)}$. This defines a net $(x_{(K, m)})_{(K, m) \in \mathbb{D}}$.

Claim. The net $(x_{(K, m)})_{(K, m) \in \mathbb{D}}$ is a Cauchy net.

Let $\eta > 0$ and $p \in \mathcal{P}(X)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\eta}{2}$. Clearly, $(\{p\}, N) \in \mathbb{D}$. Assume that $(K, m), (L, n) \geq (\{p\}, N)$. Then $\{p\} \subset K, L$ and $m, n \geq N$. Consider a δ_p^N -fine division D_p^N of $[a, b]$. Because $m, n \geq N$, it follows that

$$\delta_p^N \leq \delta_p^m \quad \text{and} \quad \delta_p^N \leq \delta_p^n$$

hold on $[a, b]$. Thus, D_p^N is both δ_p^m -fine and δ_p^n -fine. Next, we consider δ_K^m -fine and δ_L^n -fine divisions D_K^m and D_L^n of $[a, b]$, respectively. Since $p \in K \cap L$, it follows that

$$\delta_K^m \leq \delta_p^m \quad \text{and} \quad \delta_L^n \leq \delta_p^n$$

hold on $[a, b]$. Hence, D_K^m is δ_p^m -fine and D_L^n is δ_p^n -fine. Thus,

$$\begin{aligned} p(x_{(K,m)} - x_{(L,n)}) &= p(S[f, g, D_K^m] - S[f, g, D_L^n]) \\ &\leq p(S[f, g, D_K^m] - S[f, g, D_p^N]) + p(S[f, g, D_p^N] - S[f, g, D_L^n]) \\ &< \frac{1}{m} + \frac{1}{n} \leq \frac{1}{N} + \frac{1}{N} < \eta. \end{aligned}$$

Since p was arbitrary, the claim follows.

By completeness of X , it follows that the net $(x_{(K,m)})_{(K,m) \in \mathbb{D}}$ converges to some $w \in X$. We show that $(HS) \int_a^b f dg = w$. Let $\epsilon > 0$ and $q \in \mathcal{P}(X)$. Then there exists $(K_1, N_1) \in \mathbb{D}$ such that for all $(K, m) \geq (K_1, N_1)$,

$$q(x_{(K,m)} - w) < \frac{\epsilon}{2}.$$

Also, we choose $N_2 \in \mathbb{N}$ such that for all $(K, m), (L, n) \geq (\{q\}, N_2)$,

$$q(x_{(K,m)} - x_{(L,n)}) < \frac{\epsilon}{2}.$$

Choose $(S, N) \in \mathbb{D}$ such that $(\{q\}, N_2) \leq (S, N)$ and $(K_1, N_1) \leq (S, N)$. Corresponding to $(S, N) \in \mathbb{D}$, a gauge δ_S^N exists. Let D be a δ_S^N -fine division of $[a, b]$. Choose any $(K, m) \in \mathbb{D}$ such that $(K, m) \geq (S, N)$. Hence,

$$\begin{aligned} q(S[f, g, D] - w) &\leq q(S[f, g, D] - S[f, g, D_K^m]) + q(S[f, g, D_K^m] - w) \\ &= q(x_{(S,N)} - x_{(K,m)}) + q(x_{(K,m)} - w) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

It follows from Definition 2.1 that f is HS-integrable with respect to g on $[a, b]$. $\square$

We now consider one application of the Cauchy Criterion. This result tells us that the HS-integral can be considered as a function of intervals.

Theorem 2.7. *Let X be a complete LCS and suppose that $\mathcal{P}(X)$ has the max-property. If f is HS-integrable with respect to g on $[a, b]$, then for every $[c, d] \subset [a, b]$, f is HS-integrable with respect to g on $[c, d]$.*

Proof. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Then using Theorem 2.6, there exists a gauge δ_p on $[a, b]$ such that for all δ_p -fine divisions D_1 and D_2 of $[a, b]$, $p(S[f, g, D_1] - S[f, g, D_2]) < \epsilon$. Let Q_1 and Q_2 be δ_p -fine divisions of $[c, d]$. We consider the following cases:

Case 1. $a = c$ and $b = d$. This is obvious.

Case 2. $a = c$ and $d < b$. Let Q be a δ_p -fine division of $[d, b]$. Clearly, $D_1 = Q_1 \cup Q$ and $D_2 = Q_2 \cup Q$ are δ_p -fine divisions of $[a, b]$. Now observe that

$$\begin{aligned} S[f, g, D_1] - S[f, g, D_2] &= S[f, g, Q_1] + S[f, g, Q] \\ &\quad - (S[f, g, Q_2] + S[f, g, Q]) \\ &= S[f, g, Q_1] - S[f, g, Q_2]. \end{aligned}$$

Hence, $p(S[f, g, Q_1] - S[f, g, Q_2]) < \epsilon$.

Case 3. $a < c$ and $d = b$. This is similar to case 2.

Case 4. $a < c$ and $d < b$. Let R_1 and R_2 be δ_p -fine divisions of $[a, c]$ and $[d, b]$, respectively. Clearly, $D_1 = R_1 \cup Q_1 \cup R_2$ and $D_2 = R_1 \cup Q_2 \cup R_2$ are δ_p -fine divisions of $[a, b]$. Again observe that

$$\begin{aligned} S[f, g, D_1] - S[f, g, D_2] &= S[f, g, R_1] + S[f, g, Q_1] + S[f, g, R_2] \\ &\quad - (S[f, g, R_1] + S[f, g, Q_2] + S[f, g, R_2]) \\ &= S[f, g, Q_1] - S[f, g, Q_2]. \end{aligned}$$

Hence, in any case $p(S[f, g, Q_1] - S[f, g, Q_2]) < \epsilon$. Thus, by Theorem 2.6, f is HS-integrable with respect to g on $[c, d]$. $\square$

3 Some Existence Theorems

This section presents some conditions for both the integrand and the integrator in order for the HS-integrals to exist. In what follows, χ_A denotes the *characteristic function* of $A \subset [a, b]$. Our first existence theorem depends on the following lemmas.

Lemma 3.1. *Assume that $g \in G([a, b], X)$. Then $\chi_{\{a\}}$ is HS-integrable wrt g on $[a, b]$. Moreover,*

$$(HS) \int_a^b \chi_{\{a\}} dg = g(a+) - g(a).$$

Proof. Let $\epsilon > 0$ and $p \in \mathcal{P}(X)$. Since $g(a+)$ exists, there exists $\delta' > 0$ such that $t \in (a, a + \delta') \Rightarrow p(g(t) - g(a+)) < \epsilon$. Define a gauge δ on $[a, b]$ as follows. For each $t \in [a, b]$, let

$$\delta(t) = \begin{cases} \frac{t-a}{2}, & \text{if } t \in (a, b] \\ \frac{\delta'}{2}, & \text{if } t = a. \end{cases}$$

Let $D = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ -fine division of $[a, b]$. We know that

$$t_1 \in [a, x_1] \subset (t_1 - \delta(t_1), t_1 + \delta(t_1)).$$

If $t_1 \neq a$, then $0 < x_1 - a < 2\delta(t_1) = t_1 - a \leq x_1 - a$ and we get a contradiction. Hence, $t_1 = a$. Thus, $\chi_{\{a\}}(t_1) = 1$ and $\chi_{\{a\}}(t_2) = \cdots = \chi_{\{a\}}(t_n) = 0$. Observe that $a < x_1 < t_1 + \delta(t_1) = a + \delta(a) = a + \frac{\delta'}{2} < a + \delta'$. This implies that $x_1 \in (a, a + \delta')$. Thus,

$$\begin{aligned} p(S[\chi_{\{a\}}, g, D] - [g(a+) - g(a)]) &= p(\chi_{\{a\}}(t_1)[g(x_1) - g(a)] - [g(a+) - g(a)]) \\ &= p([g(x_1) - g(a)] - [g(a+) - g(a)]) \\ &= p(g(x_1) - g(a+)) < \epsilon. \end{aligned} \quad \square$$

A similar result leads to the following lemma.

Lemma 3.2. Assume that $g \in G([a, b], X)$. Then $\chi_{\{b\}}$ is HS-integrable wrt g on $[a, b]$. Moreover,

$$(HS) \int_a^b \chi_{\{b\}} dg = g(b) - g(b-).$$

Lemma 3.3. Assume that $g \in G([a, b], X)$ and let $c \in (a, b)$. Then $\chi_{\{c\}}$ is HS-integrable wrt g on $[a, b]$. Moreover,

$$(HS) \int_a^b \chi_{\{c\}} dg = g(c+) - g(c-).$$

Proof. Clearly, g is both regulated on $[a, c]$ and $[c, b]$. By Lemma 3.2,

$$(HS) \int_a^c \chi_{\{c\}} dg = g(c) - g(c-).$$

By Lemma 3.1,

$$(HS) \int_c^b \chi_{\{c\}} dg = g(c+) - g(c).$$

Hence,

$$\begin{aligned} (HS) \int_a^b \chi_{\{c\}} dg &= (HS) \int_a^c \chi_{\{c\}} dg + (HS) \int_c^b \chi_{\{c\}} dg \\ &= [g(c) - g(c-)] + [g(c+) - g(c)] \\ &= g(c+) - g(c-). \end{aligned} \quad \square$$

If we agree to set $g(a-) = g(a)$ and $g(b+) = g(b)$, then Lemmas 3.1, 3.2, and 3.3 yield

Lemma 3.4. *Assume that $g \in G([a, b], X)$ and let $t \in [a, b]$. Then $\chi_{\{t\}}$ is HS-integrable wrt g on $[a, b]$. Moreover,*

$$(HS) \int_a^b \chi_{\{t\}} dg = g(t+) - g(t-).$$

Lemma 3.5. *Assume that $g \in G([a, b], X)$. Then $\chi_{(a,b)}$ is HS-integrable wrt g on $[a, b]$. Moreover,*

$$(HS) \int_a^b \chi_{(a,b)} dg = g(b-) - g(a+).$$

Proof. We have $\chi_{(a,b)} = 1 - \chi_{\{a\}} - \chi_{\{b\}}$. Thus,

$$\begin{aligned} (HS) \int_a^b \chi_{(a,b)} dg &= (HS) \int_a^b 1 \cdot dg - (HS) \int_a^b \chi_{\{a\}} dg - (HS) \int_a^b \chi_{\{b\}} dg \\ &= [g(b) - g(a)] - [g(a+) - g(a)] - [g(b) - g(b-)] \\ &= g(b-) - g(a+). \end{aligned} \quad \square$$

Lemma 3.6. *Assume that $g \in G([a, b], X)$. Then for every $[c, d] \subset [a, b]$, $\chi_{(c,d)}$ is HS-integrable wrt g on $[a, b]$. Moreover,*

$$(HS) \int_a^b \chi_{(c,d)} dg = g(d-) - g(c+).$$

Proof. We consider the following cases:

Case 1. $c = a$ and $d = b$. Using Lemma 3.5, we get

$$(HS) \int_a^b \chi_{(c,d)} dg = (HS) \int_c^d \chi_{(c,d)} dg = g(d-) - g(c+).$$

Case 2. $c = a$ and $d < b$. Then

$$\begin{aligned} (HS) \int_a^b \chi_{(c,d)} dg &= (HS) \int_a^d \chi_{(c,d)} dg + (HS) \int_d^b \chi_{(c,d)} dg \\ &= (HS) \int_c^d \chi_{(c,d)} dg + (HS) \int_d^b 0 dg \\ &= g(d-) - g(c+). \end{aligned}$$

Case 3. $a < c$ and $d = b$. Then

$$\begin{aligned} (HS) \int_a^b \chi_{(c,d)} dg &= (HS) \int_a^c \chi_{(c,d)} dg + (HS) \int_c^b \chi_{(c,d)} dg \\ &= (HS) \int_c^d 0 dg + (HS) \int_c^d \chi_{(c,d)} dg \\ &= g(d-) - g(c+). \end{aligned}$$

Case 4. $a < c$ and $d < b$. Then

$$\begin{aligned} (HS) \int_a^b \chi_{(c,d)} dg &= (HS) \int_a^c \chi_{(c,d)} dg + (HS) \int_c^d \chi_{(c,d)} dg + (HS) \int_d^b \chi_{(c,d)} dg \\ &= (HS) \int_a^c 0 dg + (HS) \int_c^d \chi_{(c,d)} dg + (HS) \int_d^b 0 dg \\ &= g(d-) - g(c+). \end{aligned} \quad \square$$

We now state and prove the first existence theorem of this section.

Theorem 3.7. *Assume that $g \in G([a, b], X)$. Then for every step function $\psi : [a, b] \rightarrow \mathbb{R}$, ψ is HS-integrable wrt g on $[a, b]$. Moreover, the value of $(HS) \int_a^b \psi dg$ can be calculated.*

Proof. Write $\psi = \sum_{i=0}^n \hat{c}_i \chi_{\{x_i\}} + \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)}$ where $\{[x_{i-1}, x_i]\}_{i=1}^n$ is some division of $[a, b]$ and $\hat{c}_i, c_i \in \mathbb{R}$ for $i = 1, \dots, n$. Applying Lemma 3.4, Lemma 3.6, and Theorem 2.4, we conclude that ψ is HS-integrable wrt g on $[a, b]$. Moreover,

$$(HS) \int_a^b \psi dg = \sum_{i=0}^n \hat{c}_i [g(x_i+) - g(x_i-)] + \sum_{i=1}^n c_i [g(x_i+) - g(x_{i-1}-)]$$

where we agreed that $g(a-) = g(a)$ and $g(b+) = g(b)$. $\square$

Theorem 3.8. *Assume that X is a complete LCS. If $f : [a, b] \rightarrow X$ is continuous on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is of bounded variation on $[a, b]$, then f is HS-integrable with respect to g on $[a, b]$. Moreover, for each $p \in \mathcal{P}(X)$,*

$$p \left((HS) \int_a^b f dg \right) \leq \sup_{t \in [a, b]} p(f(t)) \cdot TV_a^b(g).$$

Proof. Let $q \in \mathcal{P}(X)$ and let $\epsilon > 0$. Let $M = TV_a^b(g)$. Since g is of bounded variation, M is a nonnegative real number. Since f is continuous, it follows that f is uniformly continuous on $[a, b]$. Thus, there exists $\delta' > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta'$, we have $q(f(x) - f(y)) < \frac{\epsilon}{2(M+1)}$.

Define a gauge δ on $[a, b]$ by $\delta(t) = \frac{\delta'}{2}$ for all $t \in [a, b]$. Let D_1 and D_2 be any two δ -fine divisions of $[a, b]$. Then we can choose a third δ -fine division D_3 such that it is both finer than D_1 and D_2 . Let $([u, v], t) \in D_1$ and suppose

that $D_u^v = \{([z_{i-1}, z_i], t_i)\}_{i=1}^r$ is the refinement of $([u, v], t)$ in D_3 . Consider the difference $\Delta(u, v) = f(t)[g(v) - g(u)] - S[f, g, D_u^v]$. Now, we get

$$\Delta(u, v) = \sum_{i=1}^r \{[f(t) - f(t_i)] \cdot [g(z_i) - g(z_{i-1})]\}.$$

Since $t \in [u, v]$ and $t_i \in [z_{i-1}, z_i] \subset [u, v] \subset (t - \delta(t), t + \delta(t))$ for all $i \in \{1, \dots, r\}$ we have $|t - t_i| \leq v - u < 2\delta(t) = \delta'$. Thus for all $i \in \{1, \dots, r\}$ we get $q(f(t) - f(t_i)) < \frac{\epsilon}{2(M+1)}$. Hence,

$$\begin{aligned} q(\Delta(u, v)) &\leq \sum_{i=1}^r q([f(t) - f(t_i)] \cdot [g(z_i) - g(z_{i-1})]) \\ &= \sum_{i=1}^r \{|g(z_i) - g(z_{i-1})| \cdot q([f(t) - f(t_i)])\} \\ &< \sum_{i=1}^r \left\{ |g(z_i) - g(z_{i-1})| \cdot \frac{\epsilon}{2(M+1)} \right\} \\ &= \frac{\epsilon}{2(M+1)} \cdot \sum_{i=1}^r |g(z_i) - g(z_{i-1})| \leq \frac{\epsilon TV_u^v(g)}{2(M+1)}. \end{aligned}$$

Note that since D_3 is finer than D_1 , it follows that

$$\sum_{([u,v],t) \in D_1} \Delta(u, v) = S[f, g, D_1] - S[f, g, D_3].$$

Thus,

$$\begin{aligned} q(S[f, g, D_1] - S[f, g, D_3]) &= q\left(\sum_{([u,v],t) \in D_1} \Delta(u, v)\right) \\ &\leq \sum_{([u,v],t) \in D_1} q(\Delta(u, v)) \\ &\leq \frac{\epsilon}{2(M+1)} \cdot \sum_{([u,v],t) \in D_1} TV_u^v(g) \\ &= \frac{\epsilon}{2(M+1)} \cdot TV_a^b(g) = \frac{\epsilon M}{2(M+1)} < \frac{\epsilon}{2}. \end{aligned}$$

By similar argument, $q(S[f, g, D_3] - S[f, g, D_2]) < \frac{\epsilon}{2}$. Hence,

$$\begin{aligned} q(S[f, g, D_1] - S[f, g, D_2]) &\leq q(S[f, g, D_1] - S[f, g, D_3]) \\ &\quad + q(S[f, g, D_3] - S[f, g, D_2]) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, by Theorem 2.6, the first part of the result follows.

For the last part, let $p \in \mathcal{P}(X)$ and $\eta > 0$. Choose a gauge δ on $[a, b]$ such that $p\left(S[f, g, D] - (HS) \int_a^b f dg\right) < \eta$ whenever D is a δ -fine division of $[a, b]$. By hypothesis, $TV_a^b(g) < +\infty$. Fix a δ -fine division $D = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ of $[a, b]$. Observe that

$$\begin{aligned} p(S[f, g, D]) &\leq \sum_{i=1}^n p(f(t_i)[g(x_i) - g(x_{i-1})]) \\ &\leq \sum_{i=1}^n \sup_{t \in [a, b]} p(f(t)) \cdot |g(x_i) - g(x_{i-1})| \\ &\leq \sup_{t \in [a, b]} p(f(t)) \cdot TV_a^b(g). \end{aligned}$$

Hence,

$$\begin{aligned} p\left((HS) \int_a^b f dg\right) &\leq p\left((HS) \int_a^b f dg - S[f, g, D]\right) + p(S[f, g, D]) \\ &< \eta + \sup_{t \in [a, b]} p(f(t)) \cdot TV_a^b(g). \end{aligned}$$

Because $\eta > 0$ was arbitrary, the result follows. $\square$

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