On Reverse $\Gamma^*$-Centralizer on Semiprime $\Gamma$-Ring with Involution

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Abstract

Let $M$ be a 2-torsion free semiprime $\Gamma$–ring with involution and $S : M \rightarrow M$ be an additive mapping satisfying $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$, and $\alpha, \beta \in \Gamma$. In this paper, we will prove that $S$ is a reverse $\Gamma^*$–centralizer if it satisfies the relation $S(x\alpha y + y\alpha x) = S(x)\alpha y^* + y^*\alpha S(x) = x^*\alpha S(y) + S(y)\alpha x^*(x, y \in M$ and $\alpha \in \Gamma$).

Keywords: reverse $\Gamma^*$-centralizer, left(right)derivation, semiprime $\Gamma$-ring with involution
1 Introduction

Let $M$ and $\Gamma$ be additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$, then $M$ is called a $\Gamma$-ring if for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied:

(i) $x\beta y \in M$.

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$.

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$. (see [6], [11])

Every ring $M$ is a $\Gamma$-ring with $\Gamma = \Gamma$. However a $\Gamma$-ring need not be a ring. $\Gamma$-rings, more general than rings, were introduced by Nobusawa [4]. Bernes [2] slightly weakened the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa. Bernes [2], Luh [8] and Kyuno [7] studied the structure of $\Gamma$-rings and obtained various generalizations of corresponding properties in ring theory. Following ideas from [1], Zalar [3] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of these rings coincide. Vukman [9, 10] developed some remarkable results using centralizers on prime and semiprime rings. Let $M$ be a $\Gamma$-ring, then $M$ is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. A $\Gamma$-ring $M$ is said to be prime if $a\Gamma M \Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M \Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, $M$ is said to be a commutative $\Gamma$-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma$, $y \in M$ is called the center of the $\Gamma$-ring $M$. If $M$ is a $\Gamma$-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of $x$ and $y$ with respect to $\alpha$, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

\[
[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \quad (1)
\]

\[
[x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \quad (2)
\]

and consider the following assumption:

(A) $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$, and $\alpha, \beta \in \Gamma$.

According to assumption (A), and based on equations 1 and 2 reduce to $[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y\alpha[x, z]_\beta$.

Let us consider the following conditions:

Definition 1.1 [12] An additive mapping $D : M \rightarrow M$ is said to be a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. 

**Example 1** Let $R$ be a ring, $M = M_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \bigg| m \in \mathbb{Z} \right\}$, then $M$ is a $\Gamma$-ring. We define $D : M \to M$ by $D(a \ b) = (0 \ -b)$ for all $a, b \in R$. By using the usual addition and multiplication on matrices of $M \times \Gamma \times M$, $D$ is a derivation on $M$.

**Definition 1.2** An additive mapping $(x \alpha x) \to (x \alpha x)^*$ on a $\Gamma$-ring $M$ is called an involution if $(x \alpha y)^* = y^* \alpha x^*$ and $(x \alpha x)^{**} = x \alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. A $\Gamma$-ring $M$ equipped with an involution is called a $\Gamma$-ring $M$ with involution (also known as $\Gamma^*$-ring).

**Definition 1.3** An additive mapping $T : M \to M$ is left (right) reverse $\Gamma^*$-centralizer of a $\Gamma$-ring $M$ with involution if $T(y \alpha x) = T(y)^* \alpha x^*$ ($T(x \alpha y) = x^* \alpha T(y)$) holds for all $x, y \in M$ and $\alpha \in \Gamma$.

**Definition 1.4** An additive mapping $T : M \to M$ is left (right) Jordan $\Gamma^*$-centralizer of a $\Gamma$-ring $M$ with involution if $T(x \alpha x) = T(x)^* \alpha x^*$ ($T(x \alpha x) = x^* \alpha T(x)$) for all $x \in M$ and $\alpha \in \Gamma$.

Note:

a. A reverse $\Gamma^*$-centralizer of $\Gamma$-ring $M$ with involution is an additive mapping which is both a left and right reverse $\Gamma^*$-centralizer.

b. A Jordan $\Gamma^*$-centralizer of $\Gamma$-ring $M$ with involution is an additive mapping which is both a left and right Jordan $\Gamma^*$-centralizer.

In this paper, we prove that an additive mapping $S : M \to M$ satisfying assumption (A) and satisfies the following relation:

(B) $S(x \alpha y + y \alpha x) = S(x)^* \alpha y^* + y^* \alpha S(x) = x^* \alpha S(y) + S(y)^* \alpha x^*$ for all $x, y \in M$ and $\alpha \in \Gamma$ is a reverse $\Gamma^*$-centralizer

## 2 Reverse $\Gamma^*$-Centralizer on Semiprime $\Gamma$-Ring with Involution

The following lemma will be used in the proofs of our main results.

**Lemma 2.1** [5] Let $M$ be a semiprime $\Gamma$-ring satisfying assumption (A) and $D : M \to M$ be a derivation of $M$ and $a \in M$ be a fixed element, then we have:
(i) If \( D(x)\alpha D(y) = 0 \) for all \( x, y \in M \) and \( \alpha \in \Gamma \), then \( D = 0 \).

(ii) If \( a\alpha x - x\alpha a \in Z(M) \) for all \( x \in M \) and \( \alpha \in \Gamma \), then \( a \in Z(M) \).

**Lemma 2.2** Let \( M \) be a semiprime \( \Gamma \)-ring with involution satisfying assumption (A). Let \( a \in M \) be a fixed element and let \( S(x) = a\alpha x^* + x^*\alpha a \) satisfying relation (B). Then \( a \in Z(M) \).

**Proof.** We have

\[
S(x\beta y + y\beta x) = S(x)\beta y^* + y^*\beta S(x)
= a\alpha(x\beta y + y\beta x)^* + (x\beta y + y\beta x)^*\alpha a
= (a\alpha x^* + x^*\alpha a)\beta y^* + y^*\beta(a\alpha x^* + x^*\alpha a)
= a\alpha y^*\beta x^* - y^*\alpha a\alpha x^* - x^*\alpha a\beta y^* + x^*\beta y^*\alpha a \text{ by assumption (A)}
= [a, y^*]_\alpha \beta x^* - x^*\beta[a, y^*]_\alpha
= [a, y^*]_\alpha \beta x^* = x^*\beta[a, y^*]_\alpha
\]

and then from part (ii) of Lemma 2.1, we get \( a \in Z(M) \).

**Lemma 2.3** Let \( M \) be a semiprime \( \Gamma \)-ring with involution satisfying assumption (A). Then every mapping \( T \) of \( M \) satisfying relation (B) maps \( Z(M) \) into \( Z(M) \).

**Proof.** Take any \( c \in Z(M) \) and denote \( a = T(c) \). Then \( 2T(c\alpha x) = T(c\alpha x + x\alpha c) = T(c)\alpha x^* + x^*\alpha T(c) = a\alpha x^* + x^*\alpha a \).

Now, we will show that \( S(x) = 2T(c\alpha x) \) satisfies relation (B):

\[
S(x\alpha y + y\alpha x) = 2T(c\beta(x\alpha y + y\alpha x)) = 2T((c\beta x)\alpha y + y\alpha(c\beta x))
= 2T((c\alpha y)\beta x + x\beta(c\alpha y)) = S(x)\alpha y^* + y^*\alpha S(x)
= S(y)\beta x^* + x^*\beta S(y)
\]

Hence, by Lemma 2.2, we get \( a \in Z(M) \). Thus, we prove the following main theorem

**Theorem 2.4** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring with involution and \( S : M \to M \) be an additive mapping which satisfying assumption (A) and satisfying relation (B), then \( S \) is a reverse \( \Gamma^* \)-centralizer of \( M \).

**Proof:** We have \( S(x\alpha y + y\alpha x) = S(x)\alpha y^* + y^*\alpha S(x) = x^*\alpha S(y) + S(y)\alpha x^* \).

Replacing \( y \) by \( x\beta y + y\beta x \), we get

\[
S(x\alpha(x\beta y + y\beta x)) + (x\beta y + y\beta x)\alpha x) = S(x)\alpha(x\beta y + y\beta x)^* + (x\beta y + y\beta x)^*\alpha S(x)
\]
Now it follows that \([S(x), x^*]_\alpha \beta y^* = y^* \beta [S(x), x^*]_\alpha\) holds for all \(x, y \in M\) and \(\alpha, \beta \in \Gamma\) and so we get \([S(x), x^*]_\alpha \in Z(M)\). The next goal is to show that \([S(x), x^*]_\alpha = 0\) holds.

Take any \(c \in Z(M)\), then

\[
2S(cax) = S(cax + xac) = S(c)ax^* + x^*aS(c) = 2S(x)ac^*.
\]

Using Lemma 2.3, we get

\[
S(cax) = S(x)ac^* = S(c)ax^*.
\]

Therefore,

\[
[S(x), x^*]_\alpha \beta c^* = S(x)ax^* \beta c^* - x^*aS(x)\beta c^* = S(c)bx^*ax^* - x^*aS(c)bx^* = [S(c), x^*]_\alpha \beta x^* = 0
\]

Since \([S(x), x^*]_\alpha\) itself is a central element, then

\[
2S(xax) = S(xax + xax) = S(x)ax^* + x^*aS(x) = 2S(x)ax^* = 2x^*aS(x)
\]

Hence, our goal is achieved.

Acknowledgements. This work is supported by the School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia.

References


Received: October 15, 2014; Published: November 27, 2014