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Some Properties About R-Polynomials[1]

Yanni Zhou, Xigou Zhang and Zhaojin Zhou

College of Mathematics and Information Science
JXNU Nanchang, P.R. China, 330022

Abstract

In this paper, we consider the special value about R-polynomials in finite coxeter groups. Knowing $R_{x,w} = (q-1)^{l(w)-l(x)}$ where $x \leq w$, if $l(w)-l(x) \leq 2$, now we give the fact that there exist some s,w such that $R_{x,w} = (q-1)^{l(w)-l(x)}$, where $x \leq w$, when $l(w) - l(x) \geq 3$. In particular, we discuss some special cases.

Keywords: the R-polynomial; coxeter groups; Hecke algebras;dihedral group

1. Introduction

A Coxeter System is a pair $(W, \mathcal{S})$ consisting of a group W and a set of generators $\mathcal{S}$, subject only to relations of the form $(ss')^{m(s,s')} = 1$, where $m(s,s) = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in $\mathcal{S}$.

An arbitrary $w \in W$ can be written as a product of elements in $\mathcal{S}$, say $w = s_1 \dots s_r$ (where $s_i \in \mathcal{S}$). Define the length $l(w)$ of w to be the smallest r for which such an expression exists, and call the expression reduced.

Lemma 1. *(see [1]) Let $s \in S$, $w \in W$ satisfy $sw < w$. Suppose $x < w$.*
(a) If $sx < x$, then $sx < sw$.
(b) If $sx > x$, then $sx \leq w$ and $x \leq sw$.
Thus, in either case, $sx \leq w$.

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We begin with a very general construction of associative algebras over a commutative ring A(with1). Such an algebra will have a free $A-$basis parameterized by the element of W, together with a multiplication law which reflects in a certain way the multiplication in W. The algebra will also depend on some parameters a_s,$b_s \in A(s \in \mathcal{S})$, subject only to the requirement that $a_s = b_t$ and $b_s = b_t$ whenever s and t are conjugate in W. The starting point for the construction is a free $A-$ module ε on the set W, with basis elements denoted $T_w(w \in W)$ which satisfy the following.

$$T_sT_w = T_{sw} \quad if\ sw > w,$$
$$T_sT_w = a_sT_w + b_sT_{sw} \quad if\ sw < w.$$

Now let$A$ be the ring $Z[q, q^{-1}]$ of Laurent polynomials over Z in the indeterminate q. With the further convention that $a_s = q-1$ and $b_s = q$ for all $s \in \mathcal{S}$, we write $\mathcal{H}$ for the resulting generic algebra and call it the Hecke algebra of W.

Proposition 2. *(see [1]) For all $w \in W$,*

$$(T_w^{-1})^{-1} = \varepsilon_w q_w \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x,$$

where $R_{x,w} \in Z[q]$ is a polynomial of degree $l(w) - l(x)$ in q, and where $R_{w,w}(q) = 1$, $\varepsilon_w = (-1)^{l(w)}$, $q_w = q^{l(w)}$.

We can know the algorithm for computing $R_{x,w}$ implied by the proof of Proposition 2. The idea is to use induction on $l(w)$, starting with the fact that $R_{w,w} = 1$ for all $w \in W$, while $R_{x,w} = 0$ unless $x \leq w$. For the induction step, we need to compute $R_{x,w}$, assuming that all polynomials $R_{y,z}$ are known for $l(z) < l(w)$. Fix $s \in S$ for which $sw < w$. Then two configurations have to be dealt with, as in Lemma 1:
(A) $x < w$, $sx < x$ (forcing $sx < sw$). Here we found that $R_{x,w} = R_{sx,sw}$, which is already known since $sw < w$.
(B) $x < w$, $x < sx$ (forcing $sx \leq w$ and $x \leq sw$). Here we found that $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$, both terms of which are already known. (Recall that the first term has degree $l(w) - l(x)$, while the second term has lower degree and might be 0.)

It is sometimes useful to have alternate versions of (A) and (B), with s occurring on the right rather than the left. For the right-handed version, we have the relations as follows:
(C) $x < w$, $xs < x$, $ws < w$ (forcing $xs < ws$). Then $R_{x,w} = R_{xs,ws}$.
(D) $x < w$, $x < xs$ (forcing $xs \leq w$ and $x \leq ws$). Then $R_{x,w} = (q-1)R_{x,ws} + qR_{xs,ws}$.

The R-polynomials are built up, we consider the special case $l(w)-l(x) = 1$. If $w = s_1 \dots s_r$ is a reduced expression, we can obtain x by omitting a single $w = s_i$, $x = 1$. As remarked at the beginning of the proof of Proposition 2, we get $R_{x,w} = q - 1$.

To carry this a step further, consider what happens when $l(w) - l(x) = 2$. Fixing as before a reduced expression for w, we observe that (for reason of parity) x can be obtained by omitting precisely two of the factors s_i, s_j $(i < j)$. Again we can apply (A) and (C) repeatedly to reduce to the case: $w = s_i \dots s_j$, $x = s_{i+1} \dots s_{j-1}$. Taking $s = s_i$, we have the configuration: $sw < w$, $x < sx$. Therefore (B) applies and we have $R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$. The first term is known from the preceding calculation: $R_{x,w} = q - 1$. On the other hand, both sx and sw have the same length but are unequal, forcing the second term to be 0. Conclusion: $R_{x,w} = (q-1)^2$.

The intrepid reader may wish to press on with these explicit calculations. However, they rapidly become less manageable, become of the more complicated possibilities for subexpressions when more than two factors are omitted. For example, when $W = A_2$, let $x = 1$, $w = w_0$ (the longest element) in W, $R_{x,w} = (q-1)^3 + q(q-1)$.

Enumerate a simple system Δ as $\alpha_1, \dots, \alpha_n$, with corresponding simple reflections $s_1, \dots, s_n$. Then $s_1 \dots s_n$ is called a Coxeter element of W. It depends on the choice of Δ as well as on the way Δ is numbered.

As promised above, we discuss briefly some of $R_{x,w}$ which satisfy $R_{x,w} = (q-1)^{l(w)-l(x)}$. Consider that $W = A_3$ with corresponding simple reflections s_1, s_2, s_3, we can compute all the $R_{x,w}$, it will be clear to find that $R_{x,w} = (q-1)^3$, where w is any Coxeter element of W. In this paper, we wish to obtain some results about the R-polynomials in finite Coxeter groups.

2. Main results and their proofs

Proposition 3. *Let $I \subseteq S$, write $S = \{s_1, s_2, \dots, s_n\}$, $I = \{s_{i_1}, s_{i_2}, \dots, s_{i_r}\}$ and let J consist of all Coxeter elements of W_I. We have $R_{1,w} = (q-1)^{l(w)}$ for any $w \in J$.*

Proof. Proceed by induction on $l(w)$, assuming that $w = s_{i_1} s_{i_2} \dots s_{i_r}$, this is clear when $l(w) = 0$, 1 or 2. Let $3 \le l(w) < n$(or equivalently, $3 \le r < n$) such that $R_{1,w} = (q-1)^l(w)$. If $l(w)$=n(or equivalently, $r = n$). Then

$$R_{1,w} = R_{1,s_{i_1} s_{i_2} \dots s_{i_r}} = (q-1)R_{1,s_{i_2} \dots s_{i_r}} + qR_{s_{i_1}, s_{i_2} \dots s_{i_r}}.$$

Since $l(s_{i_2} \dots s_{i_r}) = n - 1 < n$, by induction

$$R_{1,s_{i_2} \dots s_{i_r}} = (q-1)^{n-1}.$$

Since $i_1 \neq i_2 \neq \dots \neq i_n$, it implies that s_{i_1} is not the subsequence of $s_{i_2} \dots s_{i_r}$, we can obtain

$$R_{s_{i_1}, s_{i_2} \dots s_{i_r}} = 0.$$

Finally

$$R_{1,w} = (q-1)(q-1)^{n-1} = (q-1)^{l(w)},$$

as requird. □

Corollary 4. *Let $I \subseteq S$, write $S = \{s_1, s_2, \ldots, s_n\}$, $I = \{s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$ and let J consist of all Coxeter elements of W_I. Finding $w_1 \in W$ and $w \in J$ such that $l(w_1w) = l(w_1) + l(w)$ (resp. $l(ww_1) = l(w) + l(w_1)$), we have $R_{w_1,w_1w} = (q-1)^{l(w)}$ (resp. $R_{w_1,ww_1} = (q-1)^{l(w)}$).*

Proof. According to Proposition 3 and (A), it is clear that $R_{w_1,w_1w} = R_{1,w} = (q-1)^{l(w)}$. □

Proposition 5. *Let $I \subseteq S$, write $S = \{s_1, s_2, \ldots, s_n\}$, $I = \{s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$, where $r \geq 3$, and let J consist of all Coxeter elements of W_I. $w = s_{i_2} \ldots s_{i_1} \ldots s_{i_r}$ for $w \in J$, then $R_{s_{i_1},w} = (q-1)^{r-1}$.*

Proof. We argue by induction on $l(w)$, starting with the fact that $l(w) = 3$. Hence

$$R_{s_{i_1},s_{i_2}s_{i_1}s_{i_3}} = (q-1)R_{s_{i_1},s_{i_1}s_{i_3}} + qR_{s_{i_2}s_{i_1},s_{i_1}s_{i_3}} = (q-1)^2.$$

Consider the case $l(w) > 3$, we can find $s = s_{i_2}$ such that $sw < w$, w has two possibilities:
(a) If $w = s_{i_2}s_{i_1} \ldots s_{i_r}$, we have

$$R_{s_{i_1},w} = (q-1)R_{s_{i_1},s_{i_2}w} + qR_{s_{i_2}s_{i_1},s_{i_2}w} = (q-1)^{r-1}.$$

(b) If $w = s_{i_2}s_{i_3} \ldots s_{i_1} \ldots s_{i_r}$, we have

$$R_{s_{i_1},w} = R_{s_{i_1},s_{i_2}s_{i_3}\ldots s_{i_1}\ldots s_{i_r}} = (q-1)R_{s_{i_1},s_{i_3}\ldots s_{i_1}\ldots s_{i_r}} + qR_{s_{i_2}s_{i_1},s_{i_3}\ldots s_{i_1}\ldots s_{i_r}}$$

By the induction hypothesis, we have

$$R_{s_{i_1},s_{i_3}\ldots s_{i_1}\ldots s_{i_r}} = (q-1)^{r-2}.$$

Since $s_{i_2}s_{i_1}$ is not the subsequence of $s_{i_3} \ldots s_{i_1} \ldots s_{i_r}$, thus

$$R_{s_{i_2}s_{i_1},s_{i_3}\ldots s_{i_1}\ldots s_{i_r}} = 0.$$

Combining these, we have

$$R_{s_{i_1},w} = (q-1)^{r-1},$$

as required. □

Corollary 6. *Let $I \subseteq S$, write $S = \{s_1, s_2, \ldots, s_n\}$, $I = \{s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$ and let J consist of all Coxeter elements of W_I.*
(a) Assume that there exists $s = s_{i_j}$ such that $sw > w$ (resp. $ws > w$), where $1 \leq j \leq r$, then $R_{1,sw} \neq (q-1)^{l(w)}$ (resp. $R_{1,ws} \neq (q-1)^{l(w)}$) for $w \in J$.
(b) Let $w = s_{i_1} \ldots s_{i_r}$ for $w \in J$, assume that there exists $s = s_{i_j}$ such that $s_{i_1} \ldots s \ldots s_{i_r} > s_{i_1} \ldots s_{i_r}$, where $1 \leq j \leq r$, then $R_{1,s_{i_1}\ldots s\ldots s_{i_r}} \neq (q-1)^{r+1}$.

Proof. Assume that $w = s_{i_1}s_{i_2} \ldots s_{i_r}$ which implies $s \neq s_{i_1}$, we have

$$R_{1,sw} = (q-1)R_{1,w} + qR_{s,w}.$$

By proposition 3, we can obtain $R_{1,w} = (q-1)^{l(w)}$. By proposition 5, we get $R_{s,w} = (q-1)^{l(w)-1}$. Combine these, we compute

$$R_{1,sw} = (q-1)R_{1,w} + qR_{s,w} = (q-1)^{l(w)+1} + q(q-1)^{l(w)-1},$$

this proves (a).
Suppose that s occurs in the mth position of $R_{1,s_{i_1}\dots s\dots s_{i_r}}$, where m satisfies $1 < m \leq [\frac{r+1}{2}]+1$, if r is even, $1 < m \leq \frac{r+1}{2}$, if r is odd. Proceed by induction on m, if $m = 2$, then

$$R_{1,s_{i_1}ss_{i_2}\dots s_{i_r}} = (q-1)R_{1,ss_{i_2}\dots s_{i_r}} + qR_{s_{i_1},ss_{i_2}\dots s_{i_r}}$$

Since $s_{i_1}\dots s\dots s_{i_r} > s_{i_1}\dots s_{i_r}$, it is clear that $s \neq s_{i_1}$, which implies that s_{i_1} is not the subsequence of $ss_{i_2}\dots s_{i_r}$, hence, $R_{s_{i_1},ss_{i_2}\dots s_{i_r}} = 0$, however, by the case (a), we get $R_{1,ss_{i_2}\dots s_{i_r}} \neq (q-1)^r$, thus $R_{1,s_{i_1}ss_{i_2}\dots s_{i_r}} \neq (q-1)^{r+1}$.
Assume that $m = k$, where $1 < k < [\frac{r+1}{2}]+1$, if r is even, $1 < k < \frac{r+1}{2}$, if r is odd. The case (b) holds. Now $m = k+1$, so

$$R_{1,s_{i_1}s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} = (q-1)R_{1,s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} + qR_{s_{i_1},s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}}$$

two cases are possible:
(i) Suppose that $s = s_{i_1}$, by Proposition 3 and by Proposition 5, so we have $R_{1,s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} = (q-1)^r$, while $R_{s_{i_1},s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} = (q-1)^{r-1}$. Hence

$$R_{1,s_{i_1}s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} = (q-1)^{r+1} + q(q-1)^{r-1},$$

this proves (b).
(ii) Suppose that $s \neq s_{i_1}$, note that s is not the subsequence of $s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}$, so we have $R_{s_{i_1},s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} = 0$. On the other hand, by the induction hypothesis, we have $R_{1,s_{i_2}\dots s_{i_{k-1}}ss_{i_k}\dots s_{i_r}} \neq (q-1)^{r+1}$, it can deduce $R_{1,s_{i_2}\dots s_{i_k}ss_{i_{k+1}}\dots s_{i_r}} \neq (q-1)^r$, this is to say, $R_{1,s_{i_1}\dots s\dots s_{i_r}} \neq (q-1)^{r+1}$, as required. □

Proposition 7. *Let W be a dihedral group D_m, where $m < \infty$, if $l(w)-l(x) \geq 3$, then $R_{x,w} \neq (q-1)^{l(w)-l(x)}$.*

Proof. In fact, we only consider two possibilities:
The case (a) that $x = 1$, and $w = sts\dots$ (resp. $w = tst\dots$) satisfy $l(w) \geq 3$, we have

$$R_{1,w} = (q-1)R_{1,ts\dots} + qR_{s,ts\dots}$$

It is clear that $R_{s,ts\dots} \neq 0$, forcing q is a factor of some term in $R_{1,w}$, hence $R_{1,w} \neq (q-1)^{l(w)}$.
The case (b) that $x = stst\dots s$ and $w = tst\dots t$ satisfy $l(w)-l(x) \geq 3$, we have

$$R_{x,w} = (q-1)R_{x,tw} + qR_{tx,tw}.$$

Since $l(w)-l(x) \geq 3$, we can get $l(tw)-l(tx) \geq 1$ and $tx \leq tw$, so $R_{tx,tw} \neq 0$, forcing q is a factor of some term in $R_{x,w}$, hence $R_{x,w} \neq (q-1)^{l(w)-(x)}$.
In each case, $R_{x,w} \neq (q-1)^{l(w)-(x)}$, as required. □

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