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Diameter of the Zero Divisor Graph of Semiring of Matrices over Boolean Semiring

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Abstract

Let S be a semiring and let $Z(S)^*$ be its set of nonzero zero divisors. We denote the zero divisor graph of S by $\Gamma(S)$ whose vertex set is $Z(S)^*$ and there is an edge between the vertices x and y $(x \neq y)$ in $\Gamma(S)$ if and only if either $xy = 0$ or $yx = 0$. In this paper we study the zero divisor graph of the semiring of matrices $M_n(\mathfrak{B})$, $(n > 1)$ over the Boolean semiring $\mathfrak{B}$. We investigate the properties of the right zero divisors and the left zero divisors of $M_n(\mathfrak{B})$ and then use these results to prove that the diameter of $\Gamma(M_n(\mathfrak{B}))$ is 3.

Mathematics Subject Classification: 5C25

Keywords: Zero divisor graph, semiring of matrices, Boolean semiring, zero divisor, diameter of a graph

1 Introduction

The concept of a zero divisor graph was first introduced by Beck [5] in the study of commutative rings and later redefined by Anderson and Livingston [3] . Redmond [11] extended this concept to the non commutative case. Dolzan and Oblak [8] further extended the idea to semirings. For further results on zero divisor graphs see [1], [2], [4], [6], [7], [10].

A semiring is a nonempty set S on which the operations of $+$ and $\times$ have been defined such that the following conditions are satisfied (see [9]).
1. $(S, +)$ is a commutative monoid with identity element 0.
2. $(S, .)$ is a monoid with identity element 1.
3. Multiplication distributes over addition from either side.
4. $0s = 0 = s0$, for all $s \in S$.

Here $1 \neq 0$, to avoid the trivial case. Again 0 is the only absorbing zero, for if $z \in S$ satisfy $zs = z = sz, \forall s \in S$ then $0 = 0z = z$. For any semiring S, we denote the set of zero divisors by $Z(S)$, that is, $Z(S) = \{x \in S :$ there exists $0 \neq y \in S$ such that $xy = 0$ or $yx = 0\}$. Then $Z_R(S)$ denotes the set of right zero divisors, that is, $Z_R(S) = \{x \in S :$ there exists $0 \neq y \in S$ such that $yx = 0\}$ and $Z_L(S)$ denotes the set of left zero divisors, that is, $Z_L(S) = \{x \in S :$ there exists $0 \neq y \in S$ such that $xy = 0$ $\}$. We associate a zero divisor graph to the semiring S, denoted by $\Gamma(S)$ whose vertex set is the set of all non zero, zero divisors of S. That is, the vertex set $V(\Gamma(S))$ of $\Gamma(S)$ is the set of elements in $Z(S)^* = Z(S) - \{0\}$. An unordered pair of vertices $x, y \in V(\Gamma(S)), x \neq y$ is an edge in $\Gamma(S)$ if and only if either $xy = 0$ or $yx = 0$. That is, we consider the graph $\Gamma(S)$ whose vertices are the elements of $Z(S)^*$ and whose edges are those pairs of distinct non zero zero divisors x, y such that either $xy = 0$ or $yx = 0$.

2 Diameter of $\Gamma(M_n(\mathfrak{B}))$

We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices x and y, denoted by $d(x, y)$ is the length of the shortest path connecting them. The diameter of a graph Γ, denoted by $diam(\Gamma)$ is equal to $sup\{d(x, y) : x, y$ distinct vertices of $\Gamma\}$. Dolzan and Oblak [8] proved the following theorem:

Theorem 2.1 *For a semiring S, the zero divisor graph $\Gamma(S)$ is always connected and its diameter, $diam(\Gamma(S)) \leq 3$.*

We denote the Boolean semiring by $(\mathfrak{B}, +, .)$ where $\mathfrak{B} = \{0, 1\}$ and the operations of $+$ and . are defined as follows: $0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 1, 0.0 = 0, 0.1 = 0, 1.0 = 0, 1.1 = 1$. The semiring of all $n \times n$ matrices over $\mathfrak{B}$ is denoted by $M_n(\mathfrak{B})$, where $n > 1$. The matrix with the only non zero entry 1 in the ith row and jth column will be denoted by $E_{i,j}$.

Proposition 2.1 *If $A, B \in M_n(\mathfrak{B})$ and A is a non-zero matrix and B is a matrix with all rows as non-zero rows then their product AB will be a non-zero matrix.*

Proof. Without loss of generality let us assume that $A = E_{p,q}$. Let B be the matrix with all rows as non-zero rows. ie., $B = [b_{jk}]$ where for each $j = 1, 2, \ldots, n$, $b_{jk} = 1$, for atleast one $k = 1, 2, \ldots, n$. Then the pth row in the product AB is $[b_{q1}\ b_{q2}\ \ldots\ b_{qn}]$. Since $b_{qk} = 1$ for atleast one $k = 1, 2, \ldots, n$ the above row is a non-zero row. This shows that AB is a non-zero matrix.

Theorem 2.2 *Every right zero divisor should have atleast one zero row.*

Proof. Let B be a right zero divisor. Then there exists a non zero matrix A such that $AB = 0$. If possible, let B be such that B has all the rows as non-zero rows. Then by proposition 2.1, $AB \neq 0$, contradicting the fact that $AB = 0$. Hence B has atleast one zero row.

Theorem 2.3 *Any matrix of $M_n(\mathfrak{B})$ having atleast one zero row is a right zero divisor.*

Proof. Consider a matrix $B \in M_n(\mathfrak{B})$ having only one zero row, say the ith row. Then there exists a matrix $E_{i,i}$ such that $E_{i,i}B = 0$. Hence B is a right zero divisor. Similarly we can prove that matrices of $M_n(\mathfrak{B})$ having 2 zero rows, 3 zero rows,..., $(n-1)$ zero rows are all right zero divisors. Thus any matrix of $M_n(\mathfrak{B})$ having atleast one zero row is a right zero divisor.

Proposition 2.2 *If $A, B \in M_n(\mathfrak{B})$ and A is a matrix with all the columns as non-zero columns and B is a non-zero matrix then their product AB will be a non-zero matrix.*

Proof. Without loss of generality let us assume that the non-zero matrix $B = E_{pq}$. Let A be the matrix with all columns as non-zero columns. ie., $A = [a_{ij}]$, where for each $j = 1, 2, \ldots, n$, $a_{ij} = 1$ for atleast one $i = 1, 2, \ldots, n$. Then the qth column in the product AB is $[a_{1p}\ a_{2p}\ \ldots\ a_{np}]^T$. Since $a_{ip} = 1$ for atleast one $i = 1, 2, \ldots, n$ the above column is a non-zero column. This shows that AB is a non-zero matrix.

Theorem 2.4 *Every left zero divisor should have atleast one zero column.*

Proof. Let A be a left zero divisor. Then there exists a non-zero matrix $B \in M_n(\mathfrak{B})$ such that $AB = 0$. If possible let A be such that, A has all columns as non-zero columns. Then by proposition 2.2, $AB \neq 0$, contradicting the fact that $AB = 0$. Hence A has atleast one zero column.

Theorem 2.5 *Any matrix of $M_n(\mathfrak{B})$ having atleast one zero column is a left zero divisor.*

Proof. Consider a matrix $A \in M_n(\mathfrak{B})$ having only one zero column, say the ith column. Then there exists a matrix $E_{i,i}$ such that $AE_{i,i} = 0$. Hence A is a left zero divisor. Similarly we can prove that matrices of $M_n(\mathfrak{B})$ having 2 zero columns, 3 zero columns, $\ldots, (n-1)$ zero columns are all left zero divisors. Thus any matrix of $M_n(\mathfrak{B})$ having atleast one zero column is a left zero divisor.

Corollary 2.6 *$E_{i,j}$ is both right zero divisor and left zero divisor.*

Proof. Follows from theorem 2.3 and theorem 2.5

Lemma 2.7 *If x is a left zero divisor with pth column as the only zero column then $xE_{p,j} = 0$, for $j = 1, 2, \ldots, n$.*

Proof. Let $x = [x_{ij}]$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Since the pth column of x is a zero column, we have $x_{ip} = 0$, for $i = 1, 2, \ldots, n$. Let $E_{p,j} = [e_{ij}], i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, where $e_{pj} = 1$ and $e_{ij} = 0$ otherwise. Then the (i, j)th element of $xE_{p,j} = \sum_{k=1}^{n} x_{ik}e_{kj} = x_{ip} = 0$. Therefore $xE_{p,j} = 0$, for $j = 1, 2, \ldots, n$.

Lemma 2.8 *If y is a right zero divisor with qth row as the only zero row then $E_{i,q}y = 0$, for $i = 1, 2, \ldots, n$.*

Proof. Let $y = [y_{ij}]$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Since the qth row of y is a zero row, we have $y_{qj} = 0$, for $j = 1, 2, \ldots, n$. Let $E_{i,q} = [e_{ij}], i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, where $e_{iq} = 1$ and $e_{ij} = 0$ otherwise. Then the (i, j)th element of $E_{i,q}y = \sum_{k=1}^{n} e_{ik}y_{kj} = y_{qj} = 0$. Therefore $E_{i,q}y = 0$, for $i = 1, 2, \ldots, n$.

Let R denote the set of all right zero divisors of $M_n(\mathfrak{B})$ and L the set of all left zero divisors of $M_n(\mathfrak{B})$. Let $R' = R - (R \cap L)$ and $L' = L - (R \cap L)$. Then for $d(x, y)$ where $x, y \in V(\Gamma(M_n(\mathfrak{B})))$ there arise three exhaustive cases.
Case 1: $x \in L, y \in R$.
Case 2: $x \in R', y \in R'$.
Case 3: $x \in L', y \in L'$.

Theorem 2.9 *If $x \in L, y \in R$, then $d(x, y) \leq 2$.*

Proof. Case 1: For $x \in L$, if y is that right zero divisor so that $xy = 0$ then $d(x, y) = 1$.
Case 2: Suppose that $xy \neq 0$. Then $d(x, y) \neq 1$. By theorem 2.4, since a left zero divisor should have atleast one zero column, without loss of generality let us assume that the pth column of x is the only zero column. Then by lemma 2.7, $xE_{p,j} = 0$ for $j = 1, 2, \ldots, n$. Therefore, $d(x, E_{p,j}) = 1$, for $j = 1, 2, \ldots, n$. By theorem 2.2, a right zero divisor should have atleast one zero row. If the q th row of y is the only zero row then by lemma 2.8, $E_{i,q}y = 0$, for $i = 1, 2, \ldots, n$. Therefore, $d(E_{i,q}y) = 1$, for $i = 1, 2, \ldots, n$. In particular $E_{p,q}y = 0$. Hence $xE_{p,q}y = 0$. Thus $d(x, y) = 2$.

Lemma 2.10 *If x is a right zero divisor with the pth row as the only zero row then any left zero divisor of x should have non-zero elements only in the pth column.*

Proof. Let $x = [x_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ where $x_{pj} = 0$ for $j = 1, 2, \ldots, n$. Consider $z \in V(\Gamma(M_n(\mathfrak{B})))$ with $z = [z_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$. Now the (i, j)th element in the product zx is $\sum_{k=1}^{n} z_{ik}x_{kj}$. Suppose z is a left zero divisor of x with the tth column as a zero column. Then we know that $zx = 0$. That is, each (i, j)th element of zx is zero. That is, the product $z_{ik}x_{kj} = 0$ for each $i, j, k = 1, 2, \ldots, n$. If possible, let z contain non-zero elements in the mth column, where $m \neq p$. Since by assumption z contains non-zero elements in the mth column, without loss of generality we assume that z_{lm} is a non-zero element in z. Then in the product zx there will be sums involving terms $z_{lm}x_{mj}, j = 1, 2, \ldots, n$. Since the pth row is the only zero row in x, $x_{mj} \neq 0$, for atleast one $j = 1, 2, \ldots, n$. Hence let us suppose that $x_{mq} \neq 0$. Then $z_{lm}x_{mq} \neq 0$. Therefore, $zx \neq 0$. This is a contradiction to the hypothesis that z is a left zero divisor of x. Hence z is a left zero divisor of x with non-zero elements only in the pth column.

Theorem 2.11 *Let $x, y \in R'$*
1.If the positions of zero rows of x and y are non-overlapping then $d(x, y) = 3$.
2.If the positions of zero rows of x and y are overlapping then $d(x, y) = 2$.

Proof. Let $x, y \in R'$. Then $d(x, y) \neq 1$.
1. Suppose the positions of zero rows of x and y are non-overlapping. Without loss of generality assume that the pth row of x is the zero row and the qth row of y is the zero row. Then, by lemma 2.10, any left zero divisor of x has non-zero elements only in the pth column and any left zero divisor of y has non-zero elements only in the qth column. ie., x and y do not have any common left zero divisors. Thus $d(x, y) \neq 2$. Hence by Theorem 2.1, $d(x, y) = 3$.
2. Suppose the positions of zero rows of x and y are overlapping. Without loss of generality assume that the pth row of x and the pth row of y are the zero rows. Then by lemma 2.10, they have left zero divisors with non-zero elements only in the pth column. That is, they have common left zero divisors. Thus $d(x, y) = 2$.

Lemma 2.12 *If x is a left zero divisor with the qth column as the only zero column then any right zero divisor of x should have non-zero elements only in the qth row.*

Let $x = [x_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ where $x_{iq} = 0$ for $i = 1, 2, \ldots, n$. Consider $z \in V(\Gamma(M_n(\mathfrak{B})))$ with $z = [z_{ij}]$, for $i = 1, 2, \ldots, n, j =$

$1, 2, \ldots, n$. Now the (i, j)th element in the product xz is $\sum_{k=1}^{n} x_{ik}z_{kj}$. Suppose z is a right zero divisor of x with the tth row as a zero row. Then we know that $xz = 0$. That is, each (i, j)th element of xz is zero. That is, the product $x_{ik}z_{kj} = 0$ for each $i, j, k = 1, 2, \ldots, n$. If possible let z contain non-zero elements in the mth row, where $m \neq q$. Since by assumption z contains non-zero elements in the mth row, without loss of generality we assume that z_{ml} is a non-zero element in z. Then in the product xz there will be sums involving terms $x_{im}z_{ml}, i = 1, 2, \ldots, n$. Since the qth column is the only zero column in x, $x_{im} \neq 0$, for atleast one $i = 1, 2, \ldots, n$. Hence let us suppose that $x_{qm} \neq 0$. Then $x_{qm}z_{ml} \neq 0$. Therefore, $xz \neq 0$. This is a contradiction to the hypothesis that z is a right zero divisor of x.Hence z is a right zero divisor of x with non-zero elements only in the qth row.

Theorem 2.13 *Let $x, y \in L'$.*
1.If the positions of zero columns of x and y are non-overlapping then $d(x, y) = 3$.
2.If the positions of zero columns of x and y are overlapping then $d(x, y) = 2$.

Proof. Let $x, y \in L'$. Then $d(x, y) \neq 1$.
1. Suppose the positions of zero columns of x and y are non-overlapping. Without loss of generality assume that the pth column of x is the zero column and the qth column of y is the zero column. Then by lemma 2.12 any right zero divisor of x has non-zero elements only in the pth row and any right zero divisor of y has non-zero elements only in the qth row. ie., x and y do not have any common right zero divisors. Thus $d(x, y) \neq 2$. Hence by Theorem 2.1, $d(x, y) = 3$.
2. Suppose the positions of zero columns of x and y are overlapping. Without loss of generality assume that the pth column of x and the pth column of y are the zero columns. Then by lemma 2.12, they have right zero divisors with non-zero elements only in the pth row. That is, they have common right zero divisors. Thus $d(x, y) = 2$.

Theorem 2.14 *The diameter of $\Gamma(M_n(\mathfrak{B}))$ is 3.*

Proof. Follows from theorem 2.9, theorem 2.11 and theorem 2.13.

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