Diameter of the Zero Divisor Graph of Semiring of Matrices over Boolean Semiring

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Abstract

Let $S$ be a semiring and let $Z(S)^*$ be its set of nonzero zero divisors. We denote the zero divisor graph of $S$ by $\Gamma(S)$ whose vertex set is $Z(S)^*$ and there is an edge between the vertices $x$ and $y$ ($x \neq y$) in $\Gamma(S)$ if and only if either $xy = 0$ or $yx = 0$. In this paper we study the zero divisor graph of the semiring of matrices $M_n(\mathcal{B})$, $(n > 1)$ over the Boolean semiring $\mathcal{B}$. We investigate the properties of the right zero divisors and the left zero divisors of $M_n(\mathcal{B})$ and then use these results to prove that the diameter of $\Gamma(M_n(\mathcal{B}))$ is 3.

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1 Introduction

The concept of a zero divisor graph was first introduced by Beck [5] in the study of commutative rings and later redefined by Anderson and Livingston [3]. Redmond [11] extended this concept to the non commutative case. Dolzan and Oblak [8] further extended the idea to semirings. For further results on zero divisor graphs see [1], [2], [4], [6], [7], [10].
A semiring is a nonempty set $S$ on which the operations of $+$ and $\times$ have
been defined such that the following conditions are satisfied (see [9]).
1. $(S,+)$ is a commutative monoid with identity element $0$.
2. $(S,\cdot)$ is a monoid with identity element $1$.
3. Multiplication distributes over addition from either side.
4. $0s = 0 = s0$, for all $s \in S$.

Here $1 \neq 0$, to avoid the trivial case. Again $0$ is the only absorbing zero,
for if $z \in S$ satisfy $zs = z = sz, \forall s \in S$ then $0 = 0z = z$. For any semiring
$S$, we denote the set of zero divisors by $Z(S)$, that is, $Z(S) = \{x \in S : \text{there exists } 0 \neq y \in S \text{ such that } xy = 0 \}$.
Then $Z_R(S)$ denotes the set of right zero divisors, that is, $Z_R(S) = \{x \in S : \text{there exists } 0 \neq y \in S \text{ such that } yx = 0 \}$. We associate a
zero divisor graph to the semiring $S$, denoted by $\Gamma(S)$ whose vertex set is the
set of all non zero, zero divisors of $S$. That is, the vertex set $V(\Gamma(S))$ of $\Gamma(S)$
is the set of elements in $Z(S)^* = Z(S) - \{0\}$. An unordered pair of vertices
$x, y \in V(\Gamma(S)), x \neq y$ is an edge in $\Gamma(S)$ if and only if either $xy = 0$ or $yx = 0$.
That is, we consider the graph $\Gamma(S)$ whose vertices are the elements of $Z(S)^*$
and whose edges are those pairs of distinct non zero zero divisors $x, y$ such
that either $xy = 0$ or $yx = 0$.

2 Diameter of $\Gamma(M_n(\mathfrak{B}))$

We recall that a graph is connected if there exists a path connecting any two
distinct vertices. The distance between two distinct vertices $x$ and $y$, denoted
by $d(x,y)$ is the length of the shortest path connecting them. The diameter of
a graph $\Gamma$, denoted by $\text{diam}(\Gamma)$ is equal to sup\{$d(x,y) : x, y$ distinct vertices
of $\Gamma$\}. Dolzan and Oblak [8] proved the following theorem:

Theorem 2.1 For a semiring $S$, the zero divisor graph $\Gamma(S)$ is always con-
nected and its diameter, $\text{diam}(\Gamma(S)) \leq 3$.

We denote the Boolean semiring by $(\mathfrak{B},+,\cdot)$ where $\mathfrak{B} = \{0,1\}$ and the
operations of $+$ and $\cdot$ are defined as follows: $0 + 0 = 0, 0 + 1 = 1, 1 + 0 =
1, 1 + 1 = 1, 0 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 0 = 0, 1 \cdot 1 = 1$. The semiring of all $n \times n$
matrices over $\mathfrak{B}$ is denoted by $M_n(\mathfrak{B})$, where $n > 1$. The matrix with the only
non zero entry $1$ in the $i$th row and $j$th column will be denoted by $E_{i,j}$.

Proposition 2.1 If $A, B \in M_n(\mathfrak{B})$ and $A$ is a non-zero matrix and $B$ is a
matrix with all rows as non-zero rows then their product $AB$ will be a non-zero
matrix.
Proof. Without loss of generality let us assume that $A = E_{p,q}$. Let $B$ be the matrix with all rows as non-zero rows. i.e., $B = [b_{jk}]$ where for each $j = 1, 2, \ldots, n$, $b_{jk} = 1$, for atleast one $k = 1, 2, \ldots, n$. Then the $p$th row in the product $AB$ is $[b_{q1} b_{q2} \ldots b_{qn}]$. Since $b_{qk} = 1$ for atleast one $k = 1, 2, \ldots, n$ the above row is a non-zero row. This shows that $AB$ is a non-zero matrix.

**Theorem 2.2** Every right zero divisor should have atleast one zero row.

Proof. Let $B$ be a right zero divisor. Then there exists a non zero matrix $A$ such that $AB = 0$. If possible, let $B$ be such that $B$ has all the rows as non-zero rows. Then by proposition 2.1, $AB \neq 0$, contradicting the fact that $AB = 0$. Hence $B$ has at least one zero row.

**Theorem 2.3** Any matrix of $M_n(\mathfrak{B})$ having atleast one zero row is a right zero divisor.

Proof. Consider a matrix $B \in M_n(\mathfrak{B})$ having only one zero row, say the $i$th row. Then there exists a matrix $E_{i,i}$ such that $E_{i,i}B = 0$. Hence $B$ is a right zero divisor. Similarly we can prove that matrices of $M_n(\mathfrak{B})$ having 2 zero rows, 3 zero rows, \ldots, $(n - 1)$ zero rows are all right zero divisors. Thus any matrix of $M_n(\mathfrak{B})$ having at least one zero row is a right zero divisor.

**Proposition 2.2** If $A, B \in M_n(\mathfrak{B})$ and $A$ is a matrix with all the columns as non-zero columns and $B$ is a non-zero matrix then their product $AB$ will be a non-zero matrix.

Proof. Without loss of generality let us assume that the non-zero matrix $B = E_{pq}$. Let $A$ be the matrix with all columns as non-zero columns. i.e., $A = [a_{ij}]$, where for each $j = 1, 2, \ldots, n$, $a_{ij} = 1$ for atleast one $i = 1, 2, \ldots, n$. Then the $q$th column in the product $AB$ is $[a_{1p} a_{2p} \ldots a_{np}]^T$. Since $a_{ip} = 1$ for atleast one $i = 1, 2, \ldots, n$ the above column is a non-zero column. This shows that $AB$ is a non-zero matrix.

**Theorem 2.4** Every left zero divisor should have at least one zero column.

Proof. Let $A$ be a left zero divisor. Then there exists a non-zero matrix $B \in M_n(\mathfrak{B})$ such that $AB = 0$. If possible let $A$ be such that, $A$ has all columns as non-zero columns. Then by proposition 2.2, $AB \neq 0$, contradicting the fact that $AB = 0$. Hence $A$ has at least one zero column.

**Theorem 2.5** Any matrix of $M_n(\mathfrak{B})$ having at least one zero column is a left zero divisor.

Proof. Consider a matrix $A \in M_n(\mathfrak{B})$ having only one zero column, say the $i$th column. Then there exists a matrix $E_{i,i}$ such that $AE_{i,i} = 0$. Hence $A$ is a left zero divisor. Similarly we can prove that matrices of $M_n(\mathfrak{B})$ having 2 zero columns, 3 zero columns, \ldots, $(n - 1)$ zero columns are all left zero divisors. Thus any matrix of $M_n(\mathfrak{B})$ having at least one zero column is a left zero divisor.
Corollary 2.6  \( E_{i,j} \) is both right zero divisor and left zero divisor.

Proof.  Follows from theorem 2.3 and theorem 2.5

Lemma 2.7  If \( x \) is a left zero divisor with \( p \)th column as the only zero column then \( xE_{p,j} = 0 \), for \( j = 1, 2, \ldots, n \).

Proof.  Let \( x = [x_{ij}] \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \). Since the \( p \)th column of \( x \) is a zero column, we have \( x_{ip} = 0 \), for \( i = 1, 2, \ldots, n \). Let \( E_{p,j} = [e_{ij}], i = 1, 2, \ldots, n, j = 1, 2, \ldots, n \), where \( e_{pj} = 1 \) and \( e_{ij} = 0 \) otherwise.

Then the \((i, j)\)th element of \( xE_{p,j} = \sum_{k=1}^{n} x_{ik} e_{kj} = 0 \). Therefore \( xE_{p,j} = 0 \), for \( j = 1, 2, \ldots, n \).

Lemma 2.8  If \( y \) is a right zero divisor with \( q \)th row as the only zero row then \( E_{i,q}y = 0 \), for \( i = 1, 2, \ldots, n \).

Proof.  Let \( y = [y_{ij}] \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \). Since the \( q \)th row of \( y \) is a zero row, we have \( y_{qj} = 0 \), for \( j = 1, 2, \ldots, n \). Let \( E_{i,q} = [e_{ij}], i = 1, 2, \ldots, n, j = 1, 2, \ldots, n \), where \( e_{iq} = 1 \) and \( e_{ij} = 0 \) otherwise.

Then the \((i, j)\)th element of \( E_{i,q}y = \sum_{k=1}^{n} e_{ik} y_{kj} = 0 \). Therefore \( E_{i,q}y = 0 \), for \( i = 1, 2, \ldots, n \).

Let \( R \) denote the set of all right zero divisors of \( M_n(B) \) and \( L \) the set of all left zero divisors of \( M_n(B) \). Let \( R' = R - (R \cap L) \) and \( L' = L - (R \cap L) \). Then for \( d(x, y) \) where \( x, y \in V(\Gamma(M_n(B))) \) there arise three exhaustive cases. Case 1: \( x \in L, y \in R \).
Case 2: \( x \in R', y \in R' \).
Case 3: \( x \in L', y \in L' \).

Theorem 2.9  If \( x \in L, y \in R, \) then \( d(x, y) \leq 2 \).

Proof.  Case 1: For \( x \in L \), if \( y \) is that right zero divisor so that \( xy = 0 \) then \( d(x, y) = 1 \).
Case 2: Suppose that \( xy \neq 0 \). Then \( d(x, y) \neq 1 \). By theorem 2.4, since a left zero divisor should have at least one zero column, without loss of generality let us assume that the \( p \)th column of \( x \) is the only zero column. Then by lemma 2.7, \( xE_{p,j} = 0 \) for \( j = 1, 2, \ldots, n \). Therefore, \( d(x, E_{p,j}) = 1 \), for \( j = 1, 2, \ldots, n \).
By theorem 2.2, a right zero divisor should have at least one zero row. If the \( q \)th row of \( y \) is the only zero row then by lemma 2.8, \( E_{i,q}y = 0 \), for \( i = 1, 2, \ldots, n \). Therefore, \( d(E_{i,q}y) = 1 \), for \( i = 1, 2, \ldots, n \). In particular \( E_{p,q}y = 0 \). Hence \( xE_{p,q}y = 0 \). Thus \( d(x, y) = 2 \).
Lemma 2.10 If $x$ is a right zero divisor with the $p$th row as the only zero row then any left zero divisor of $x$ should have non-zero elements only in the $p$th column.

Proof. Let $x = [x_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ where $x_{pj} = 0$ for $j = 1, 2, \ldots, n$. Consider $z \in V(\Gamma(M_n(\mathbb{B})))$ with $z = [z_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$. Now the $(i, j)$th element in the product $zx$ is $\sum_{k=1}^{n} z_{ik}x_{kj}$. Suppose $z$ is a left zero divisor of $x$ with the $t$th column as a zero column. Then we know that $zx = 0$. That is, each $(i, j)$th element of $zx$ is zero. That is, the product $z_{ik}x_{kj} = 0$ for each $i, j, k = 1, 2, \ldots, n$. If possible, let $z$ contain non-zero elements in the $m$th column, where $m \neq p$. Since by assumption $z$ contains non-zero elements in the $m$th column, without loss of generality we assume that $z_{lm}$ is a non-zero element in $z$. Then in the product $zx$ there will be sums involving terms $z_{lm}x_{mj}$, $j = 1, 2, \ldots, n$. Since the $p$th row is the only zero row in $x$, $x_{mj} \neq 0$, for at least one $j = 1, 2, \ldots, n$. Hence let us suppose that $x_{mq} \neq 0$. Then $z_{lm}x_{mq} \neq 0$. Therefore, $zx \neq 0$. This is a contradiction to the hypothesis that $z$ is a left zero divisor of $x$. Hence $z$ is a left zero divisor of $x$ with non-zero elements only in the $p$th column.

Theorem 2.11 Let $x, y \in R'$
1. If the positions of zero rows of $x$ and $y$ are non-overlapping then $d(x, y) = 3$.
2. If the positions of zero rows of $x$ and $y$ are overlapping then $d(x, y) = 2$.

Proof. Let $x, y \in R'$. Then $d(x, y) \neq 1$.
1. Suppose the positions of zero rows of $x$ and $y$ are non-overlapping. Without loss of generality assume that the $p$th row of $x$ is the zero row and the $q$th row of $y$ is the zero row. Then, by lemma 2.10, any left zero divisor of $x$ has non-zero elements only in the $p$th column and any left zero divisor of $y$ has non-zero elements only in the $q$th column. i.e., $x$ and $y$ do not have any common left zero divisors. Thus $d(x, y) \neq 2$. Hence by Theorem 2.1, $d(x, y) = 3$.
2. Suppose the positions of zero rows of $x$ and $y$ are overlapping. Without loss of generality assume that the $p$th row of $x$ and the $p$th row of $y$ are the zero rows. Then by lemma 2.10, they have left zero divisors with non-zero elements only in the $p$th column. That is, they have common left zero divisors. Thus $d(x, y) = 2$.

Lemma 2.12 If $x$ is a left zero divisor with the $q$th column as the only zero column then any right zero divisor of $x$ should have non-zero elements only in the $q$th row.

Let $x = [x_{ij}]$, for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ where $x_{iq} = 0$ for $i = 1, 2, \ldots, n$. Consider $z \in V(\Gamma(M_n(\mathbb{B})))$ with $z = [z_{ij}]$, for $i = 1, 2, \ldots, n, j =$
1, 2, . . . , n. Now the (i, j)th element in the product xz is \( \sum_{k=1}^{n} x_{ik} z_{kj} \). Suppose z is a right zero divisor of x with the tth row as a zero row. Then we know that \( xz = 0 \). That is, each (i, j)th element of xz is zero. That is, the product \( x_{ik} z_{kj} = 0 \) for each \( i, j, k = 1, 2, \ldots, n \). If possible let z contain non-zero elements in the mth row, where \( m \neq q \). Since by assumption z contains non-zero elements in the mth row, without loss of generality we assume that \( z_{ml} \) is a non-zero element in z. Then in the product xz there will be sums involving terms \( x_{im} z_{ml}, i = 1, 2, \ldots, n \). Since the qth column is the only zero column in x, \( x_{im} \neq 0 \), for atleast one \( i = 1, 2, \ldots, n \). Hence let us suppose that \( x_{qm} \neq 0 \). Then \( x_{qm} z_{ml} \neq 0 \). Therefore, \( xz \neq 0 \). This is a contradiction to the hypothesis that z is a right zero divisor of x. Hence z is a right zero divisor of x with non-zero elements only in the qth row.

**Theorem 2.13** Let \( x, y \in L' \).
1. If the positions of zero columns of x and y are non-overlapping then \( d(x, y) = 3 \).
2. If the positions of zero columns of x and y are overlapping then \( d(x, y) = 2 \).

**Proof.** Let \( x, y \in L' \). Then \( d(x, y) \neq 1 \).
1. Suppose the positions of zero columns of x and y are non-overlapping. Without loss of generality assume that the pth column of x is the zero column and the qth column of y is the zero column. Then by lemma 2.12 any right zero divisor of x has non-zero elements only in the pth row and any right zero divisor of y has non-zero elements only in the qth row. ie., x and y do not have any common right zero divisors. Thus \( d(x, y) \neq 2 \). Hence by Theorem 2.1, \( d(x, y) = 3 \).
2. Suppose the positions of zero columns of x and y are overlapping. Without loss of generality assume that the pth column of x and the pth column of y are the zero columns. Then by lemma 2.12, they have right zero divisors with non-zero elements only in the pth row. That is, they have common right zero divisors. Thus \( d(x, y) = 2 \).

**Theorem 2.14** The diameter of \( \Gamma(M_n(\mathfrak{B})) \) is 3.

**Proof.** Follows from theorem 2.9, theorem 2.11 and theorem 2.13.

**References**


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