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A Study on the Structure of Indefinite Quasi-Affine Kac-Moody Algebras $QAC_2^{(1)}$

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Abstract

Quasi-affine Kac-Moody algebras is a special class of indefinite type of Kac-Moody algebras. In this paper, a family of quasi-affine Kac-Moody algebras $QAC_2^{(1)}$ is considered. These quasi-affine algebras are realized as a graded Lie algebra of Kac-Moody type. Using the homological and spectral sequences theory homology modules upto level three are computed and a study on the structure of these algebras is undertaken. The classification of Dynkin diagrams for a particular family of $QAC_2^{(1)}$ is also given.

Keywords: Generalized Cartan Matrix, Dynkin diagram, imaginary root, quasi hyperbolic, quasi affine, Kac-Moody algebras, spectral sequences.

1 Introduction

Kac-Moody Lie algebras is one of the rapidly growing fields of mathematical research due to the interesting connections and applications to various fields of Mathematics and Mathematical Physics, Combinatorics, Number Theory, Non-linear differential equations, etc. Among the broad classification of Kac Moody algebras into finite, affine and indefinite types, a lot of work has been carried out for the finite and affine type of Kac-Moody algebras, whereas a deeper study on the structure of indefinite Kac-Moody algebras is yet to be given completely.

Understanding the structure and determining the multiplicities of roots, explicitly for indefinite Kac-Moody algebras is still an open problem. In [2], Feingold and Frenkel computed level 2 root multiplicities for the hyperbolic Kac-Moody algebra $HA_1^{(1)}$. Kang([5]-[8]) studied the structure and obtained the multiplicities for roots upto level 5 for $HA_1^{(1)}$ and for roots upto level 3 for $HA_2^{(2)}$ and root multiplicities are determined for the indefinite type of Kac-Moody algebra $HA_n^{(1)}$. In [12] Sthanumoorthy and Uma Maheswari introduced a new class of indefinite type, namely extended – hyperbolic Kac – Moody algebras. In ([11], [13], [15]), determined the multiplicities of roots for specific classes of extended–hyperbolic Kac–Moody algebra $EHA_1^{(1)}$ and $EHA_2^{(2)}$ were determined.

Another class of indefinite non-hyperbolic Kac-Moody algebra called Quasi-Hyperbolic was introduced by Uma Maheswari [16]. In ([17], [18]), Uma Maheswari considered two specific classes of indefinite non-hyperbolic Kac-Moody type QHG_2 and $QHA_2^{(1)}$ and determined the structure of the components of the maximal ideal upto level 3. In [19], Uma Maheswari introduced another class of indefinite type, the quasi-affine Kac Moody algebras and studied about the Dynkin diagrams and properties of roots and obtained a realization for the quasi-affine family $QAG_2^{(1)}$

In this work, we are going to consider a class of a Quasi-Affine indefinite type of Kac-Moody algebra $QAC_2^{(1)}$; We give a classification of Dynkin diagrams of $QAC_2^{(1)}$. We then give a realization for a specific class of $QAC_2^{(1)}$, associated with the GCM $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -2 & -b \\ 0 & -1 & 2 & -c \\ -l & -m & -n & 2 \end{pmatrix}$ where a, b, c, l, m, n $\in Z^+$ as a graded Lie algebra of Kac-Moody type. Then using the homological techniques developed by Benkart et al. and Kang, [1] we compute the homology modules and determine the structure of the components of the maximal ideal up to level five.

2 Preliminaries

We recall some preliminary definitions and results on Kac-Moody algebras and for further details one can refer to Kac [4] and Wan [20].

Definition 2.1 [10]: An integer matrix $A=(a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(i) $a_{ii} = 2 \quad \forall \; i = 1,2,\ldots,n$; ii) $a_{ij} = 0 \iff a_{ji} = 0 \;\; \forall \; i, j = 1,2,\ldots,n$

(ii) $a_{ij} \le 0 \;\; \forall \; i, j = 1,2,\ldots,n.$

Let us denote the index set of A by N = {1,…, n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J $\subset$ N such that I $\cup$ J = N and $a_{ij} = a_{ji} = 0$ $\forall$ $i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 2.2 [4]: A realization of a matrix $A = (a_{ij})_{i,j=1}^{n}$ is a triple (H, Π, Π^{v}) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^{v} = \{\alpha_1^{v}, ..., \alpha_n^{v}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

Definition 2.3[4]: The Kac-Moody algebra g(A) associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i, $i = 1,2,...,n$ and H with the following defining relations :

$$[h, h'] = 0, \quad h, h' \in H \quad ; \quad [e_i, f_j] = \delta_{ij}\alpha_i^{v} \quad ; [h, e_j] = \alpha_j(h)e_j \quad ; \quad [h, f_j] = -\alpha_j(h)f_j \quad , i, j \in N$$

$$(ade_i)^{1-a_{ij}} e_j = 0 \quad ; \quad (adf_i)^{1-a_{ij}} f_j = 0 \quad , \forall \ i \neq j, \ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \ for \ all \ h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. Let $Q = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$ where $\alpha, \beta \in Q$.

Let $\Delta = (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 2.4 [4]: A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q_1,…,q_n), with q_i> 0 and q_i's are rational numbers.

Proposition 2.5 [4]:A GCM $A = (a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on g (A).

Definition 2.6[4]: To every GCM A is associated a Dynkin diagram S (A) defined as follows: (A) has n vertices and vertices i and j are connected by max {$|a_{ij}|$, $|a_{ji}|$} number of lines if a_{ij}. $a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If a_{ij}. a_{ji}> 4, i and j are connected by a bold faced edge, equipped with the ordered pair ($|a_{ij}|$, $|a_{ji}|$) of integers.

Theorem 2.7 [20]: Let A be a real n x n matrix satisfying (m1), (m2) and (m3).
(m1) A is indecomposable; (m2) $a_{ij} \le 0$ for $i \ne j$; (m3) $a_{ij} = 0$ implies $a_{ji} = 0$.Then one and only one of the following three possibilities holds for both A and ${}^tA$:
(i) det A $\ne 0$; there exists u > 0 such that A u > 0; Av $\ge$ 0 implies v > 0 or v = 0;
(ii) co rank A=1; there exists u > 0 such that Au = 0; Av $\ge$ 0 implies Av = 0;
(iii) there exists u > 0 such that Au < 0; Av $\ge$ 0, v $\ge$ 0 imply v = 0 .
Then A is of finite, affine or indefinite type iff (i), (ii) or (iii) (respectively) is satisfied.

Definition 2.8 [20]: A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.
General construction of graded Lie algebra (Benkart et al. [1], Kang [5]):
Let us start with G, the Lie algebra over a field of characteristic zero. Let V, V' be two G – modules. Let $\psi : V' \otimes V \to G$ be a G – module homomorphism. Define $G_0 = G, G_{-1} = V, G_1 = V'$; $G_+ = \sum_{n\ge1} G_n$ (resp. $G_- = \sum_{n\ge1} G_{-n}$) denote the free Lie algebra generated by V' (respectively, V); G_n (respectively, G_{-n}) for n > 1 is the space of all products of n vectors from V' (respectively V). Then $G = \sum_{n=-\infty}^{\infty} G_n$ can be given a Lie algebra structure. By extending this Lie bracket operation, $G = \sum_{n\in z} G_n$ becomes a graded Lie algebra which is generated by its local part $G_{-1} + G_0 + G_1$.

For n≥1 define the subspaces, $I_{\pm n} = \{x \in G_{\pm n} \mid (ad\, G_{\mp1})^{n-1} x = 0\}$, define $I = \oplus_{n\in z} I_n$ and $I_+ = \sum_{n>1} I_n$, $I_- = \sum_{n>1} I_{-n}$ .Then the subspaces I_+ , I_- and I are all graded ideals of G and I is the maximal graded ideal trivially intersecting the local part $G_{-1} + G_0 + G_1$.Let $L_{\pm n} = G_{\pm n} / I_{\pm n}$, for n > 1;
Consider $L = L(G,V,V',\psi) = G_- / I_- \oplus G_0 \oplus G_+ / I_+$
$= ... \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus ...,$ where $L_0 = G_0,\ L_1 = G_1,\ L_{-1} = G_{-1}$.

Then $L = \oplus_{n\in Z} L_n$ becomes a graded Lie algebra generated by its local part V ⊕ G ⊕ V* and L = G / I. By the suitable choice of V (written as the direct sum of irreducible highest weight modules), the contragradient V^* of V, the basis elements and the homomorphism $\psi : V^* \otimes V \to g^e$, form the graded Lie algebra L = L (g^e, V, V^*, ψ). For further details one can refer to ([1], [5]).
Theorem 2.9[1]: L is a Z^{n+m} –graded algebra.

Theorem 2.10[1]: Letϕ: A(C) $\to$ L be the Lie algebra homomorphism sending $E_i \to e_i$, $F_i \to f_i$, $H_i \to h_i$. Then ϕ has kernel as I(C) and I(C) is the largest graded ideal of A(C) trivially intersecting the span of $H_1,\ldots, H_{n+m}$. Also $\phi : A(C)/I(C) \to L$ is an isomorphism.

Proposition 2. 11[1]: The matrix C has rank 2n – *l* and C is symmetrizable.
We now recall the definition of homology of Lie algebra (Garland and Lepowsky) and Hochschild-Serre spectral sequence (Kang et al.).Let V be a module over a Lie algebra G. Define the space $C_q(G, V)$ for q > 0 of q – dimensional chains of the Lie algebra G with coefficients in V to be $\wedge^q (G) \otimes V$. The differential

$d_q = C_q(G,V) \to C_{q-1}(G,V)$ is defined to be

$$d_q(g_1 \wedge \ldots \wedge g_q \otimes v) = \sum_{1\le s\le t\le q} (-1)^{s+t-1}([g_s, g_t]) \wedge g_1 \wedge \ldots \wedge \hat{g}_s \wedge \ldots \wedge \hat{g}_t \wedge \ldots \wedge g_q) \otimes v$$
$$+ \sum_{1\le s\le q} (-1)^s (g_1 \wedge \ldots \wedge \hat{g}_s \wedge \ldots \wedge g_q) \otimes g_s.v,$$

for $v \in V, g_1,\ldots, g_q \in G$. For q < 0, define $C_q(G,V) = 0$ and $d_q = 0$. Then $d_q \circ d_{q-1} = 0$. The homology of the complex (C, d) = $\{C_q (G, V), dq\}$ is called the homology of the Lie algebra G with coefficients in V and is denoted by $H_q(G, V)$. When V = C, we write $H_q(G)$ for $H_q(G, C)$. Assume now that G, V, $C_q(G, V)$ are completely reducible modules in the category O over a Kac-Moody algebra g (A) with d_q having g (A)-module homomorphisms.

Let I be an ideal of G and L = G/I. Define a filtration $\{K_p = K_pC\}$ of the complex {C, d} by $K_pC_{p+q} = \{g_1 \wedge g_2 \wedge \ldots \wedge g_{p+q} \otimes v \mid g_i \in I$ for $p+1 \le i \le p+q\}$.This gives rise to a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ such that $E^2_{p,q} \cong H_p(L, H_q(I,V))$, where $E^r_{p,q}$'s are determined by $E^{r+1}_{p,q} = \mathrm{Ker}(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})/\mathrm{Im}(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$ with boundary homomorphisms $d_{r+1} : E^r_{p,q} \to E^r_{p-r-1,q+r}$. The modules $E^r_{p,q}$ become stable for r > max(p,q +1) for each (p,q) and is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ converges to $H_n(G,V)$ in the following sense : $H_n(G,V) = \bigoplus_{p+q=n} E^\infty_{p,q}$.
We get the Hochschild-Serre five term exact sequence ([5]):

$H_2(G,V) \to H_2(L,H_0(I,V) \to H_0(L,H_1(I,V)) \to H_1(G,V) \to H_1(L,H_0(I,V)) \to 0.$
Take L = G/I, where $G = \oplus_{n\ge1} G_n$ is the free Lie algebra generated by the subspace G_1 and $I = \oplus_{n\ge m} I_n$ the graded ideal of G generated by the subspace I_m for m ≥ 2. Then $L = \oplus_{n\ge1} L_n$ becomes a graded Lie algebra generated by the subspace $L_1 = G_1$.

Let J = I / [I, I]. J is an L-module via adjoint action generated by the subspace J_m. For m ≤ n < 2m, $J_n \cong I_n$. If I_m and G_1 are modules over a Kac-Moody algebra g(A) then G_n has a g(A)-module structure for every $x \in g(A), v \in G, w \in G_{n-1}$, $x \cdot [v, w] = [x \cdot v, w] + [v, x \cdot w]$. I_n also has a similar module structure and we have the induced module structure of the homogeneous subspaces L_n, J_n. Then we have the following theorem proved by Kang.

Theorem 2.12[5]: There is an isomorphism of g (A) – modules $H_j(L, J) \cong H_{j+2}(L)$, for j ≥ 1. In particular $I_{m+1} \cong (G_1 \otimes I_m) / H_3(L)_{m+1}$.

Now, for arbitrary j ≥ m, set $I^{(j)} = \sum_{n \geq j} I_n$; then $I^{(j)}$ is an ideal of G generated by the subspace I_j. We consider the quotient algebra $L^{(j)} = G/I^{(j)}$. Let $N^{(j)} = I^{(j)} / I^{(j-1)}$. In this notation $L = L^{(m)}$. Then we have an important relation: $I_{j+1} \cong (G_1 \otimes I_j) / H_3(L^{(j)})_{j+1}$. And, there exists a spectral sequence $\{E^r_{p,q}, d_r: E^r_{p,q} \rightarrow E^r_{p-r,q+r-1}\}$ converging to $H_*(L^{(j)})$ such that and $E^2_{p,q} \cong H_p(L^{(j-1)}) \otimes \wedge^q (I_{j-1})$ and $H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}$.

Lemma 2.13[5]: In the above notation, $H_2(L) \cong I_m$.

Let us recall the Kostant's formula for symmetrizable Kac-Moody algebras [9]:
Let S={1,…,s} be a subset of N = {1,…,n} and g_s ,the subalgebra of g(A) generated by the elements e_i, f_i, i = 1, …, s and h. Let Δ_s^+ denote the set of positive roots generated by $\alpha_1, \ldots, \alpha_s$ and $\Delta_s^- = -\Delta_s^+$. Then g_s has the corresponding triangular decomposition: $g_s = n_s^- \oplus h \oplus n_s^+$, where $n_s^\pm = \bigoplus_{\alpha \in \Delta_s^\pm} g_\alpha$ and $\Delta_s = \Delta_s^+ \cup \Delta_s^-$ is the root system of g_s. Let $\Delta^\pm(s) = \Delta^\pm \setminus \Delta_s^\pm$, $n^\pm(S) = \bigoplus_{\alpha \in \Delta^\pm(S)} g_\alpha$.

Then g(A) = $n^-(S) \oplus g_s \oplus n^+(S)$. Let $W(S) = \{w \in W / w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$. For $\lambda \in h^*$ denote by $\tilde{V}(\lambda)$, the irreducible highest weight module over g(A) and $V(\lambda)$ the irreducible highest weight module over g_s.

Theorem 2.14[9]:(Kostant's formula) $H_j(n^-(S), \tilde{V}(\lambda)) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=j}} V(w(\lambda + \rho) - \rho)$.

Lemma 2.15[5]: Suppose $w = w' r_j$ and $l(w) = l(w') + 1$. Then $w \in W(S)$ if and only if w' ∈ W(s) and $w'(\alpha_j) \in \Delta^+(S)$.

3 Quasi -Affine Kac-Moody Algebra $QAC_2^{(1)}$

In this section, we first define the quasi-affine indefinite Kac Moody algebra associated with $QAC_2^{(1)}$. We give the classification of Dynkin diagrams associated with this quasi-affine family $QAC_2^{(1)}$. We then determine the structure of indefinite, quasi affine class $QAC_2^{(1)}$ obtained from the affine family $C_2^{(1)}$.

Definition 3.1 [16]: Let $A=(a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S (A) to be of Quasi affine (QA) type if S (A) has a proper connected sub diagram of affine types with n-1 vertices. The GCM A is of QA type if S (A) is of QA type. We then say the Kac-Moody algebra g(A) is of QA type.

The general representation of the GCM of $QAC_2^{(1)}$ is $A = \begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -2 & -b \\ 0 & -1 & 2 & -c \\ -l & -m & -n & 2 \end{pmatrix}$

where a,b,c,l,m,n are positive integers. This GCM is symmetrizable when 2m/b = *l*/a = n/c. In this case we write A = DB

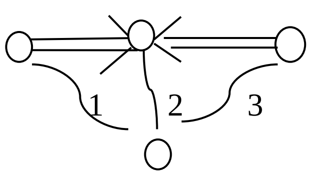
where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & l/a \end{pmatrix}$ and $B=\begin{pmatrix} 2 & -1 & 0 & -a \\ -1 & 1 & -1 & -b/2 \\ 0 & -1 & 2 & -c \\ -a & -b/2 & -c & 2c/n \end{pmatrix}$.

In the following theorem, we give the classification of the Dynkin diagrams of of indefinite quasi-affine Kac-Moody algebras $QAC_2^{(1)}$.

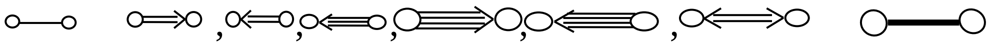
Theorem 3.2: There are 729 connected Dynkin diagrams associated with the indefinite quasi-affine Kac-Moody algebras $QAC_2^{(1)}$.

Proof: The Dynkin diagram associated with $C_2^{(1)}$ is 

By definition , the Dynkin diagram for $QAC_2^{(1)}$ is:



where  can represent any of the following 9 types of edges:



we obtain a total of 9 × 9 × 9 = 729 possible connected Dynkin diagrams associated with $QAC_2^{(1)}$.
In the next section, we are going to consider a particular family belonging to this quasi affine class and study the structure of the graded components of the maximal ideal.

By our definition, we note that the fourth vertex added must be connected to each of the three vertices in the affine diagram $C_2^{(1)}$

Fixing one possible edge for the first link i.e. 1, there are 9 possibilities each for the other two links.
We concentrate only on the connected Dynkin diagrams. Therefore, we obtain a total of 9 × 9 × 9 = 729 possible connected Dynkin diagrams associated with $QAC_2^{(1)}$.
In the next section, we are going to consider a particular family belonging to this quasi affine class and study the structure of the graded components of the maximal ideal.

4 Realization for $QAC_2^{(1)}$

In this section, we shall give a realization for a family of quasi-affine Kac Moody algebras $QAC_2^{(1)}$ as a graded Lie algebra of Kac Moody type. We begin with the affine family $C_2^{(1)}$.

Consider the Kac-Moody algebra $C_2^{(1)}$ associated with the GCM A = $\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$.

Let (h, Π, $\Pi^\vee$) be the realization of A with Π={$\alpha_1,\alpha_2,\alpha_3$}and Π^v ={$\alpha_1^v,\alpha_2^v,\alpha_3^v$} representing the set of simple roots and simple co-roots respectively.
Choose $\alpha_4^{'}$ in h^* such that $\alpha_4^{'}(\alpha_1^v)=0, \alpha_4^{'}(\alpha_2^v)=0, \alpha_4^{'}(\alpha_3^v)=1$.
Define $\lambda = a\alpha_1 + a\alpha_2 - \frac{b}{2}\alpha_3 + (a+b+c)\ \alpha_4^{'}$ ε h* … (4.1)
Set $\alpha_4 = -\lambda$.
Form the matrix $C = (<\alpha_i, \alpha_j>)_{i,j=1}^4$ where

$$<\alpha_i, \alpha_j> = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, i, j = 1,...,4.$$

Then C = $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -2 & -b \\ 0 & -1 & 2 & -c \\ -l & -m & -n & 2 \end{pmatrix}$ with 2m/b = l/a = n/c is the symmetrizable GCM of indefinite Quasi - Affine type and let the associated Kac-Moody algebra G(C) be denoted by $QAC_2^{(1)}$.

Let G be the Kac-Moody algebra associated with the GCM $\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$ of the affine family $C_2^{(1)}$. Let us take V to be the integrable highest weight irreducible module over G with the highest weight λ as defined in (4.1). Let V^* denote the contragradient module of V and ψ be the homomorphidm as defined in Section 2. As in the general construction, form the graded Lie algebra L (G^e, V, V^*, ψ).
Then $L \cong g(C)$ and L is a symmetrizable Kac-Moody algebra of Quasi- affine type associated with the GCM C (Proposition 2.11). Thus we have given the realization for this quasi affine family as a graded Lie algebra of Kac Moody type.

Next to understand the structure of these Kac Moody algebras, we apply the spectral sequences and homological techniques developed be Benkart, et.al. [1, 5]. First, we compute the homology modules of the Kac-Moody algebra for $QAC_2^{(1)}$. We note that, from the realization of L = $QAC_2^{(1)}$ as $L = L_{-1} \oplus L_0 \oplus L_+ = G/I$ and using the involutive automorphism, it is sufficient to study only about the negative part $L_- = G_-/I_-$.

5 Computation of Homology Modules

Let S = {1, 2, 3} $\subset$ N = {1, 2, 3, 4}. Let g_s be the Kac-Moody Lie algebra of $C_2^{(1)}$, which is the subalgebra of $QAC_2{}^{(1)}$. Let Δ_S^+ denote the set of positive roots generated by {$\alpha_1, \alpha_2, \alpha_3$} and - $\Delta_S^+ = \Delta^+_S$, $\Delta^{\pm}(S) = \Delta^{\pm} \setminus \Delta^-_S$.
The only reflection of length 1 in W(S) is r_4.
$r_4(\rho) = \rho - \alpha_4$; $r_4(\rho) - \rho = -\alpha_4$ $\quad \therefore H_1(L_-) \cong V(-\alpha_4)$.
The reflections of length 2 in W(S) are r_4r_1, r_4r_2 and r_4r_3.
$r_4r_1(\rho) - \rho = -(1+l)\alpha_4 - \alpha_1$; $r_4r_2(\rho) - \rho = -(m+1)\alpha_4 - \alpha_2$;
$r_4r_3(\rho) - \rho = -(n+1)\alpha_4 - \alpha_3$.

By Kostant's formula,
$H_2(L_-) \cong V(-(l+1)\alpha_4 - \alpha_1) \oplus V(-(m+1)\alpha_4 - \alpha_2) \oplus V(-(n+1)\alpha_4 - \alpha_3)$.
The reflections of length 3 in W(S) are:
$r_4r_1r_2$, $r_4r_1r_3$, $r_4r_1r_4$ (if a.$l \neq 1$), $r_4r_3r_1$, $r_4r_3r_2$, $r_4r_3r_4$ (if n.c $\neq 1$), $r_4r_2r_1$, $r_4r_2r_3$, $r_4r_2r_4$ (if m.b $\neq 1$);

Now, $r_4r_3r_1(\rho) - \rho = -(l+n+1)\alpha_4 - \alpha_3 - \alpha_1$;
$r_4r_3r_2(\rho) - \rho = -(m+2n+1)\alpha_4 - 2\alpha_3 - \alpha_2$;
$r_4r_3r_4(\rho) - \rho = -n(c+1)\alpha_4 - (c+1)\alpha_3$ if $n.c \neq 1$;
$r_4r_1r_4(\rho) - \rho = -(a.l+l)\alpha_4 - (a+1)\alpha_1$ (if $a.l \neq 1$);
$r_4r_1r_3(\rho) - \rho = -(l+n+1)\alpha_4 - \alpha_3 - \alpha_1$;
$r_4r_1r_2(\rho) - \rho = -(2l+m+1)\alpha_4 - \alpha_2 - 2\alpha_1$;
$r_4r_2r_1(\rho) - \rho = -(3m+l+1)\alpha_4 - 3\alpha_2 - \alpha_1$;
$r_4r_2r_3(\rho) - \rho = -(3m+n+1)\alpha_4 - 3\alpha_2 - \alpha_3$;
$r_4r_2r_4(\rho) - \rho = -(m.b+m)\alpha_4 - (b+1)\alpha_2 - \alpha_1$ (if $m.b \neq 1$);

By Kostant's formula,

$$\begin{aligned} H_3(L_-) \cong\ & V(-(l+n+1)\alpha_4 - \alpha_3 - \alpha_1) \oplus V(-(m+2n+1)\alpha_4 - 2\alpha_3 - \alpha_2) \\ & \oplus V(-n(c+1)\alpha_4 - (c+1)\alpha_3) \text{ (if } n.c \neq 1) \\ & \oplus V(-(a.l+l)\alpha_4 - (a+1)\alpha_1) \text{ (if } a.l \neq 1) \\ & \oplus V(-(l+n+1)\alpha_4 - \alpha_3 - \alpha_1) \oplus V(-(2l+m+1)\alpha_4 - \alpha_2 - 2\alpha_1) \\ & \oplus V(-(3m+l+1)\alpha_4 - 3\alpha_2 - \alpha_1) \oplus V(-(3m+n+1)\alpha_4 - 3\alpha_2 - \alpha_3) \\ & \oplus V(-(m.b+m)\alpha_4 - (b+1)\alpha_2 - \alpha_1) \text{ (if } m.b \neq 1) \qquad \ldots(5.2) \end{aligned}$$

Similarly by repeated application of Kostant's formula, other homology modules $H_4(L_-)$, $H_5(L_-)$, $H_6(L_-)$ etc. can be computed.

6 Structure of the components of the Maximal Ideal in $QAC_2^{(1)}$

In this section, we study the structure of the components of maximal ideal upto level 4. Since the ideal I_- of G_- is generated by the homological subspace I_{-2}, we may write $I_- = I_-^{(2)}$. For j ≥ 2, we write $I_-^{(j)} = \sum_{n\geq j} I_{-n}$, $L_-^{(j)} = G / I_-^{(j)}$ and $N_-^{(j)} = I_-^{(j)} / I_-^{(j+1)}$. Using the homological approach and Hochschild – Serre spectral sequences theory together with the representation theory of Kac-Moody algebra, we can determine other components of the maximal ideals in $QAC_2^{(1)}$.

To determine I_{-2}:
Since $G\_$ is free and $I\_$ is generated by the subspace I_{-2} from the Hochschild –Serre five term exact sequence and using Lemma 2.13 we get, $I_{-2} \cong H_2(L_-)$; $H_2(L_-) \cong V(-(l+1)\alpha_4 - \alpha_1) \oplus V(-(m+1)\alpha_4 - \alpha_2) \oplus V(-(n+1)\alpha_4 - \alpha_3)$.
Hence $I_{-2} \cong V(-(l+1)\alpha_4 - \alpha_1) \oplus V(-(m+1)\alpha_4 - \alpha_2) \oplus V(-(n+1)\alpha_4 - \alpha_3)$.

To determine I_{-3}:
We have, $I_{-(j+1)} \cong (V \otimes I_{-j}) / H_3(L_-^{(j)})_{-(j+1)}$ $j \geq 2$. $I_{-3} \cong (V \otimes I_{-j}) / H_3(L_-^{(j)})_{-3}$
When j = 2, $L_-^{(2)}$ coincides with the subspace $n^-(S)$ for S = {1, 2, 3} and

therefore we can compute $H_3(L_-^{(2)})$, using the Kostant formula.

$$\begin{aligned} H_3(L_-^{(2)}) \cong\ & V\,(-(l+n+1)\,\alpha_4-\alpha_3-\alpha_1) \oplus V\,(-(m+2n+1)\,\alpha_4-2\alpha_3-\alpha_2) \\ & \oplus V\,(-n\,(c+1)\,\alpha_4-(c+1)\,\alpha_3)\ (\text{if } n.c \neq 1) \\ & \oplus V\,(-(a.l+l)\,\alpha_4-(a+1)\,\alpha_1)\ (\text{if } a.l \neq 1) \\ & \oplus V\,(-(l+n+1)\,\alpha_4-\alpha_3-\alpha_1) \oplus V\,(-(2l+m+1)\,\alpha_4-\alpha_2-2\alpha_1) \\ & \oplus V\,(-(3m+l+1)\,\alpha_4-3\alpha_2-\alpha_1) \oplus V\,(-(3m+n+1)\,\alpha_4-3\alpha_2-\alpha_3) \\ & \oplus V\,(-(m.b+m)\,\alpha_4-(b+1)\,\alpha_2-\alpha_1)\ (\text{if } m.b \neq 1) \qquad \text{by (3.1)} \end{aligned}$$

and $H_3(L_-^{(2)})_{-3} = 0$. Hence, $I_{-3} \cong (V \otimes I_{-2})$. To determine the structure of I_{-4}: To find the structure of I_{-4}, we need to find the structure of $H_3(L_-^{(3)})_{-4}$. Consider the short exact sequence, $0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$ and the corresponding spectral sequence $\{E_{p,q}^r\}$ converging to $H_*(L_-^{(3)})$ such that $E_{p,q}^2 \cong H_p(L_-^{(2)}) \otimes \Lambda^q(I_{-2})$. We start with the sequence, $0 \to E_{2,0}^2 \xrightarrow{d_2} E_{0,1}^2 \to 0$.

Since the spectral sequence converges to $H_*(L_-^{(3)})$, we have $H_1(L_-^{(3)}) \cong E_{1,0}^\infty \oplus E_{0,1}^\infty$. But $H_1(L_-^{(3)}) \cong L_-^{(3)}/[L_-^{(3)}, L_-^{(3)}] \cong L_{-1} = V$ and $E_{1,0}^\infty = E_{1,0}^2 \cong H_1(L_-^{(2)}) \cong L_1^{(2)}/[L_-^{(2)}, L_-^{(2)}] \cong L_{-1} = V$, $E_{0,1}^\infty = E_{0,1}^3 = 0$. $\therefore$ d_2 is surjective. Since $E_{2,0}^2 = E_{0,1}^2 \cong I_{-2}$, d_2 becomes an isomorphism. Thus $E_{2,0}^\infty = E_{2,0}^3 = 0$. Now, consider the sequence $0 \to E_{3,0}^2 \xrightarrow{d_2} E_{1,1}^2 \to 0$.

By Kostant's formula, $E_{3,0}^2 \cong H_3(L_-^{(2)})$; $E_{1,1}^2 \cong V \otimes I_{-2}$, by comparing the levels of both terms, $d_2 \colon E_{3,0}^2 \to E_{1,1}^2$ is trivial. So $E_{3,0}^3 = E_{3,0}^2$ and $E_{1,1}^\infty = E_{1,1}^3 = E_{1,1}^2 \cong V \otimes I_{-2}$. $I_-^{(3)}$ is generated by I_{-3} $\therefore H_2(L_-^{(3)}) \cong I_{-3} = V \otimes I_{-2}$.

But $H_2(L_-^{(3)}) \cong E_{2,0}^\infty \oplus E_{1,1}^\infty \oplus E_{0,2}^\infty$. It follows that $E_{0,2}^\infty = E_{0,2}^4 = 0$. Therefore we find that either $E_{0,2}^3 = 0$ or $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is surjective.

In the first case, $E_{0,2}^3 = 0$, this implies that $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is trivial and that $d_2 : E_{2,1}^2 \to E_{0,2}^2$ is surjective in the sequence $0 \to E_{4,0}^2 \xrightarrow{d_2} E_{2,1}^2 \xrightarrow{d_2} E_{0,2}^2 \to 0$. Thus $E_{3,0}^\infty = E_{3,0}^4 = \mathrm{Ker}(d_3 : E_{3,0}^3 \to E_{0,2}^3)/\mathrm{Im}(d_3 : 0 \to E_{3,0}^3) = E_{3,0}^3 = E_{3,0}^2 \cong H_3(L_-^{(2)})$. By comparing levels, we see that $d_2 : E_{4,0}^2 \to E_{2,1}^2$ is trivial. Since $E_{0,2}^2 \cong \Lambda^2(I_{-2})$, $E_{4,0}^3 = E_{4,0}^2, E_{2,1}^\infty = E_{2,1}^3 = \mathrm{Ker}(d_2 : E_{2,1}^2 \to E_{0,2}^2)/\mathrm{Im}(d_2 : E_{4,0}^2 \to E_{2,1}^2) \cong$

$\mathrm{Ker}\,(d_2 : E^2_{2,1} \to E^2_{0,2})$. Since $d_2 : E^2_{2,1} \to E^2_{0,2}$ is surjective, $\Lambda^2(I_{-2}) \cong E^2_{0,2} \cong E^2_{2,1}/\mathrm{Ker}\, d_2$ $\cong (I_{-2} \otimes I_{-2})/\mathrm{Ker}\, d_2$ .Therefore $\mathrm{Ker}\, d_2 \cong S^2(I_{-2})$. Hence $E^\infty_{2,1} \cong S^2(I_{-2})$.

If $E^3_{0,2}$ is nonzero and $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective, since $E^3_{3,0} = E^2_{3,0}$ is irreducible, $d_3 : E^3_{3,0} \to E^3_{0,2}$ is an isomorphism. Thus $E^\infty_{3,0} = E^4_{3,0} = 0$ and $H_3(L^{(2)}_-) \cong E^3_{3,0} \cong E^3_{0,2} \cong E^2_{0,2} / \mathrm{Im}(d_2 : E^2_{2,1} \to E^2_{0,2}) \cong \Lambda^2(I_{-2})/\mathrm{Im}(d_2 : E^2_{2,1} \to E^2_{0,2})$.

Since all the modules, here are completely reducible over $C^{(1)}_2$,
$\mathrm{Im}(d_2 : E^2_{2,1} \to E^2_{0,2}) \cong \Lambda^2(I_{-2}) / H_3(L^{(2)}_-)$. We get, $d_2 : E^2_{4,0} \to E^2_{2,1}$ is trivial.
Thus $E^\infty_{2,1} = E^3_{2,1} = \mathrm{Ker}(d_2 : E^2_{2,1} \to E^2_{0,2}) / \mathrm{Im}(d_2 : E^2_{4,0} \to E^2_{2,1}) = Ker(d_2 : E^2_{2,1} \to E^2_{0,2})$.
Since $\mathrm{Im}\, d_2 \cong \Lambda^2(I_{-2}) / H_3(L^{(2)}_-) \cong E^2_{2,1} / \mathrm{Ker}\; d_2 \cong (I_{-2} \otimes I_{-2}) / \mathrm{Ker}\; d_2$,
$\Rightarrow$ $\mathrm{Ker}\, d_2 \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-)$ $\therefore E^\infty_{3,0} \oplus E^\infty_{2,1} \cong S^2(I_{-2}) \oplus H_3(L^{(2)}_-)$
Consider $0 \to E^2_{5,0} \xrightarrow{d_2} E^2_{3,1} \to 0$. By comparing levels, we see that $d_2 : E^2_{3,1} \to E^2_{1,2}$ is trivial. Thus $E^3_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2})$. By comparing the levels of the terms in the sequence $0 \to E^3_{4,0} \xrightarrow{d_3} E^3_{1,2} \to 0$, we get $d_3 = 0$. Therefore $E^\infty_{1,2} = E^4_{1,2} = E^2_{1,2} \cong V \otimes \Lambda^2(I_{-2})$. Since $E^\infty_{0,3}$ is a sub module of $E^2_{0,3} \cong \Lambda^3(I_{-2})$. we see that $H_3(L^{(3)}_-) \cong H_3(L^{(2)}_-) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus M$, where M is a direct sum of level >6 irreducible representations of $C^{(1)}_2$. Therefore $H_3(L^{(3)}_-)_{-4} \cong 0$.Thus $I_{-4} \cong (V \otimes I_{-3}) / H_3(L^{(3)}_-)_{-4} \cong (V \otimes I_{-3})$

To determine the structure of I_{-5}:

Consider the short exact sequence, $0 \to N^{(3)}_- \to L^{(4)}_- \to L^{(3)}_- \to 0$ and the corresponding spectral sequence $\{E^r_{p,q}\}$ converging to $H_*(L^{(4)}_-)$ such that $E^2_{p,q} \cong H_p(L^{(3)}_-) \otimes \Lambda^q(I_{-3})$. Clearly $d_2 : E^2_{2,0} \to E^2_{0,1}$ is an isomorphism and $E^\infty_{2,0} = 0$. We now consider the sequence, $0 \to E^2_{3,0} \xrightarrow{d_2} E^2_{1,1} \to 0$. $E^2_{3,0} \cong H_3(L^{(3)}_-)$, $E^2_{1,1} \cong V \otimes I_{-3}$. $E^2_{3,0} \cong \mathrm{Ker}\, d_2 \oplus \mathrm{Im}\, d_2$. By comparing the levels of both terms, we get d_2: $E^2_{3,0} \to E^2_{1,1}$ is trivial. $H_2(L^{(4)}_-) \cong I_{-4} = V \otimes I_{-3}$. $E^\infty_{1,1} = E^3_{1,1} = E^2_{1,1}/\mathrm{Im}\; d_2 \cong (V \otimes I_{-3})/\mathrm{Im}\, d_2$. But Im d_2 = 0; Hence $E^3_{3,0} \cong E^2_{3,0}$ *and* $E^\infty_{1,1} = E^3_{1,1} \cong I_{-4}$. $E^\infty_{0,2} = E^4_{0,2} = 0$.. Therefore $d_3 : E^3_{3,0} \to E^3_{0,2}$ is surjective. $E^3_{0,2}$ is a sub module of $E^2_{0,2} \cong \wedge^2(I_{-3})$, by comparing the levels we get Ker d_2 must contain $V \otimes \wedge^2(I_{-2})$. Thus $E^{\infty}_{3,0} = E^4_{3,0} \cong V \otimes \wedge^2(I_{-2}) \oplus M'$, where M' is a direct sum of level >6 irreducible highest weight representations of $C_2^{(1)}$. Hence $(E^{\infty}_{3,0})_{-5} = 0$. Thus $(E^{\infty}_{2,1})_{-5} = (E^{\infty}_{1,2})_{-5} = (E^{\infty}_{0,3})_{-5} = 0$. Therefore $H_3(L^{(4)}_{-})_{-5} \cong 0)$ and $I_{-5} \cong (V \otimes I_{-4})$. Thus we have proved the following structure theorem.

Theorem 6.1: With the usual notations, let $L = \oplus_{n \in Z} L_n$ be the realization of $QAC_2^{(1)}$ associated with the symmetrizable GCM $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -2 & -b \\ 0 & -1 & 2 & -c \\ -l & -m & -n & 2 \end{pmatrix}$ with 2m/b = *l*/a = n/c.. Then the structure of the components of the maximal ideal upto level 5 are given by:

i) $I_{-2} \cong V(-(l+1)\alpha_4 - \alpha_1) \oplus V(-(m+1)\alpha_4 - \alpha_2) \oplus V(-(n+1)\alpha_4 - \alpha_3)$.

ii) $I_{-3} \cong (V \otimes I_{-2})$

iii) $I_{-4} \cong (V \otimes I_{-3})$

iv) $I_{-5} \cong (V \otimes I_{-4})$

7 Conclusion

In this work, we have considered a class of quasi affine Kac-Moody algebra $QAC_2^{(1)}$ and determined the structure of the components in the graded ideals upto level five. The components of higher level can be computed in a similar manner. This work gives further scope for understanding the complete structure of this indefinite quasi affine algebra and also will help in the determination of the multiplicities of root spaces and weight spaces.

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References

[1] G M Benkart, S J Kang and K C Misra, Graded Lie algebras of Kac-Moody type, Adv. Math. **97(**1993), 154-190.

[2] A J Feingold and I B Frenkel, A hyperbolic Kac-Moody algebra and the theory of siegal modular forms of genus 2, Math. Ann. **263**(1983), 87-144.

[3] G Garland and J Lepowsky, Lie algebra homology and the Macdonald-Kac formula, *Invent. Math.* **34** (1976), 37-76.

[4] V G Kac, Infinite Dimensional Lie Algebra, Cambridge University Press 3rd ed. Cambridge, 1990

[5] S J Kang, Kac-Moody Lie algebras, Spectral Sequences and the Witt formula, Trans. Amer. Math Soc. **339**(1993), 463-495

[6] S J Kang, Root multiplicities of the hyperbolic Kac-Moody algebra $HA_1^{(1)}$ J. Algebra **160** (1993), 492-593

[7] S J Kang, Root multiplicities of the hyperbolic Kac-Moody Lie algebra $HA_n^{(1)}$ J. Algebra **170** (1994), 277-299

[8] S J Kang, On the hyperbolic Kac-Moody Lie algebra $HA_1^{(1)}$ Trans. Amer. Math. Soc. **341** (1994), 623-638

[9] L S Liu, Kostant's formula for Kac-Moody Lie algebras J. Algebra **149** (1992), 155-178

[10] R V Moody, A new class of Lie algebras, J. Algebra **10** (1968), 211-230

[11] N Sthanumoorthy and A Uma Maheswari, Root multiplicities of extended hyperbolic Kac-Moody algebras, Comm. Algebra **24**(14) (1996), 4495-4512

[12] N Sthanumoorthy and A Uma Maheswari, Purely imaginary roots of Kac-Moody algebras Comm. Algebra **24**(2)(1996), 677-693

[13] N Sthanumoorthy, P.L. Lilly and A. Uma Maheswari, Root multiplicities of some classes of extended-hyperbolic Kac-Moody and extended – hyperbolic generalized Kac-Moody algebras, Contemporary Mathematics **343**(2004), 315-347

[14] N Sthanumoorthy, A. Uma Maheswari and P.L. Lilly, Extended – Hyperbolic Kac-Moody $EHA_2^{(2)}$ Algebras :Structure and Root Multiplicities Comm. Algebra, **32**(6)(2004), 2457-2476

[15] N Sthanumoorthy and A. Uma Maheswari, Structure and Root Multiplicities for two classes of Extended Hyberbolic Kac-Moody Algebras $EHA_1^{(1)}$ and $EHA_2^{(2)}$ for all cases ,Communications in Algebra **40 (**2012), 632-665

[16] A. Uma Maheswari, Imaginary Roots and Dynkin Diagrams of Quasi Hyberbolic Kac-Moody Algebras, *Trans Stellar* **4**(2)(2014), 19-28

[17] A. Uma Maheswari and S. Krishnaveni , A study on the structure of a class of indefinite non-hyperbolic Kac-Moody algebras, QHG_2 International Journal of Mathematics and Computer Applications Research (IJMCAR) 4(4)(2014), 97-110

[18] A. Uma Maheswari and S. Krishnaveni, On the structure of indefinite quasi hyperbolic Kac-Moody algebras $QHA_2^{(1)}$ Int.J.ofAlgebra,Vol 8 no. 11 (2014), 517-528

[19] A.Uma Maheswari, Quasi-Affine Kac-Moody Algebras: Classification of Dynkin Diagrams & Realization for a Class of $QAG_2^{(1)}$ -to appear in International Journal of Mathematical Sciences, 2014

[20] Wan Zhe-Xian, Introduction to Kac-Moody Algebra, Singapore: World Scientific Publishing Co.Pvt.Ltd.(1991)

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