Generalized Jordan \((\sigma, \tau)\) - Higher Homomorphisms of a \(\Gamma\)-Ring \(M\) into a \(\Gamma\) Ring \(M'\)

Salah Mehdi Saleh and Fawaz Ra'ad Jarallah

Mathematics Department, Education of College
AL-Mustansirya University, Iraq

Copyright © 2014 Salah Mehdi Saleh and Fawaz Ra'ad Jarallah. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Let \(M\) and \(M'\) be two prime \(\Gamma\)-ring and \(\sigma^n, \tau^n\) be two higher homomorphism of a \(\Gamma\)-ring \(M\), for all \(n \in \mathbb{N}\) in the present paper we show that under certain conditions of \(M\), every generalized Jordan \((\sigma, \tau)\) - higher homomorphism of a \(\Gamma\)-Ring \(M\) into a prime \(\Gamma\) -Ring \(M'\) is either generalized \((\sigma, \tau)\) - higher homomorphism or \((\sigma, \tau)\) – higher anti- homomorphism.

Mathematic Subject Classification: 16 N 60, 16 U 80

Keywords: prime \(\Gamma\)-ring, homomorphis, generalized Jordan higher homomorphism

1- Introduction

Let \(M\) and \(\Gamma\) be two additive abelian groups, suppose that there is a mapping from \(M\times\Gamma\times M \to M\) (the image of \((a,\alpha,b)\) being denoted by \(a\alpha b\), \(a, b \in M\) and \(\alpha \in \Gamma\)). Satisfying for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\):

(i) \((a + b)\alpha c = a\alpha c + b\alpha c\)

\(a (\alpha + \beta) c = a\alpha c + a\beta c\)

\(a\alpha (b + c) = a\alpha b + a\alpha c\)

(ii) \((a\alpha b)\beta c = a\alpha (b\beta c)\)

Then \(M\) is called a \(\Gamma\)-ring. This definition is due to Barnes [1], [8].
A $\Gamma$-ring $M$ is called a prime if $a\Gamma M \Gamma b = (0)$ implies $a = 0$ or $b = 0$, where $a, b \in M$, this definition is due to [5].

A $\Gamma$-ring $M$ is called semiprime if $a\Gamma M a = (0)$ implies $a = 0$, such that $a \in M$, this definition is due to [7].

Let $M$ be a 2-torsion free semiprime $\Gamma$-ring and suppose that $a, b \in M$ if $a\Gamma m \Gamma b + b\Gamma m \Gamma a = 0$ for all $m \in M$, then $a\Gamma m \Gamma b = b\Gamma m \Gamma a = 0$ this definition is due to [11].

Let $M$ be a $\Gamma$-ring then $M$ is called 2-torsion free if $2a = 0$ implies $a = 0$, for every $a \in M$, this definition is due to [6].

Let $\sigma^i, \tau^i$ be two higher homomorphism of a $\Gamma$-ring $M$ then $\sigma^i, \tau^i$ are called commutative if $\sigma^i \tau^i = \tau^i \sigma^i$, for all $i \in \mathbb{N}$.

Let $M$ be a $\Gamma$-ring and $d: M \rightarrow M$ be an additive map (that is $d(a + b) = d(a) + d(b)$), then $d$ is called a derivation on $M$ if:

$$d(ab) = d(a)ab + a\alpha d(b), \text{ for all } a, b \in M \text{ and } \alpha \in \Gamma.$$ 

$d$ is called a Jordan derivation on $\Gamma$-ring if $d(aa) = d(a)\alpha a + a\alpha d(a)$, for all $a \in M$ and $\alpha \in \Gamma$, [4], [9].

Let $M$ be a $\Gamma$-ring and $f: M \rightarrow M$ be an additive map (that is $f(a + b) = f(a) + f(b)$), Then $f$ is called a generalized derivation if there exists a derivation $d: M \rightarrow M$ such that

$$f(ab) = f(a)ab + a\alpha d(b), \text{ for all } a, b \in M \text{ and } \alpha \in \Gamma.$$ 

$f$ is called a Jordan generalized derivation if there exists a Jordan derivation $d: M \rightarrow M$ such that

$$f(aa) = f(a)aa + a\alpha d(a), \text{ for all } a \in M \text{ and } \alpha \in \Gamma, [2], [3].$$

Let $\theta$ be an additive mapping of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, $\theta$ is called a homomorphism if for all $a, b \in M$ and $\alpha \in \Gamma$

$$\theta(ab) = \theta(a)\alpha \theta(b), [1], [10].$$

Let $\theta$ be an additive mapping of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, $\theta$ is called a Jordan homomorphism if for all $a, b \in M$ and $\alpha \in \Gamma$

$$\theta(aa + b\alpha a) = \theta(a)\alpha \theta(b) + \theta(b)\alpha \theta(a), [10].$$

Let $F$ be an additive mapping of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, $F$ is called a generalized homomorphism if there exists a homomorphism $\theta$ from a $\Gamma$-ring $M$ into a $\Gamma'$-ring $M'$, such that $F(ab) = F(a)\alpha \theta(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$, where $\theta$ is called a relating homomorphism.
Generalized Jordan \((\sigma, \tau)\) homomorphisms

Let \(F = (f_i)_{i \in \mathbb{N}}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) and \(\sigma, \tau\) be two homomorphism of a \(\Gamma\)-ring \(M\), \(F\) is called a generalized \((\sigma, \tau)\) - higher homomorphism if there exists a \((\sigma, \tau)\) - higher homomorphism \(\theta = (\phi_i)_{i \in \mathbb{N}}\) from a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\), such that for all \(a, b \in M\), \(\alpha \in \Gamma\), and for every \(n \in \mathbb{N}\), we have:

\[
 f_n(\alpha \alpha b) = \sum_{i=1}^{n} f_i(\alpha \alpha \phi_i(b)),
\]

(respectively \( f_n(\alpha \alpha b + b \alpha \alpha \alpha) = \sum_{i=1}^{n} f_i(\alpha \alpha \phi_i(b)) + \sum_{i=1}^{n} f_i(b) \alpha \phi_i(\alpha) \)).

For all \(a, b \in M\) and \(\alpha \in \Gamma\), [10].

Now, the main purpose of this paper is that every generalized Jordan \((\sigma, \tau)\) - higher homomorphism of a \(\Gamma\)-ring \(M\) into a prime \(\Gamma\)-ring \(M'\) is either generalized \((\sigma, \tau)\) - higher homomorphism or \((\sigma, \tau)\) - anti -higher homomorphism and every generalized Jordan \((\sigma, \tau)\)-higher homomorphism from a \(\Gamma\)-ring \(M\) into a 2-torsion free \(\Gamma\)-ring \(M'\) is a generalized Jordan triple \((\sigma, \tau)\) - higher homomorphism.

2-Generalized Jordan \((\sigma, \tau)\) - Higher Homomorphisms of a \(\Gamma\)-Rings

Definition(2.1):

Let \(F = (f_i)_{i \in \mathbb{N}}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) and \(\sigma, \tau\) be two homomorphism of a \(\Gamma\)-ring \(M\), \(F\) is called a generalized \((\sigma, \tau)\) - higher homomorphism if there exists a \((\sigma, \tau)\) - higher homomorphism \(\theta = (\phi_i)_{i \in \mathbb{N}}\) from a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\), such that for all \(a, b \in M\), \(\alpha \in \Gamma\) and for every \(n \in \mathbb{N}\), we have:

\[
 f_n(\alpha \alpha b) = \sum_{i=1}^{n} f_i(\sigma^i(\alpha)) \alpha \phi_i(\tau^i(b))
\]

Where \(\theta\) is called the relating \((\sigma, \tau)\) - higher homomorphism.

Definition (2.2):

Let \(F = (f_i)_{i \in \mathbb{N}}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) and \(\sigma, \tau\) be two homomorphism of a \(\Gamma\)-ring \(M\), \(F\) is called a Jordan generalized \((\sigma, \tau)\) - higher homomorphism if there exists a \((\sigma, \tau)\) - higher homomorphism \(\theta = \ldots\)
(ϕ)_{i \in \mathbb{N}} from a Γ-ring M into a Γ - ring M', such that for all a, b ∈ M, α ∈ Γ and for every n ∈ N, we have:

\[ f_n(a \alpha b + b \alpha a) = \sum_{i=1}^{n} f_i((\sigma^i(\alpha))\alpha \phi_i(\tau^i(b))) + \sum_{i=1}^{n} f_i((\sigma^i(b))\alpha \phi_i(\tau^i(\alpha))) \]

Where 0 is called the relating Jordan (σ, τ) - higher homomorphism.

**Definition (2.3):**
Let F = (f_i)_{i \in \mathbb{N}} be a family of additive mappings of a Γ-ring M into a Γ - ring M' and σ, τ be two homomorphism of a Γ-ring M, F is called a generalized Jordan triple (σ, τ) - higher homomorphism if there exists a Jordan triple (σ, τ) - higher homomorphism 0 = (ϕ)_{i \in \mathbb{N}} from a Γ-ring M into a Γ-ring M', such that for all a, b ∈ M, α, β ∈ Γ and for every n ∈ N:

\[ f_n(a \alpha b \beta a) = \sum_{i=1}^{n} f_i((\sigma^i(\alpha))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(\alpha))) \]

Where 0 is called the relating Jordan triple (σ, τ) - higher homomorphism.

**Definition (2.4):**
Let F = (f_i)_{i \in \mathbb{N}} be a family of additive mappings of a Γ-ring M into a Γ - ring M' and σ, τ be two homomorphism of a Γ-ring M, F is called a generalized (σ, τ) - higher anti- homomorphism if there exists a (σ, τ) - higher anti- homomorphism 0 = (ϕ)_{i \in \mathbb{N}} from a Γ-ring M into a Γ - ring M', such that for all a, b ∈ M, α ∈ Γ and for every n ∈ N we have:

\[ f_n(a \alpha b) = \sum_{i=1}^{n} f_i((\sigma^i(b))\alpha \phi_i(\tau^i(\alpha))) \]

Where 0 is called the relating (σ, τ) - higher anti- homomorphism.

Now, we present below an example of generalized higher (σ, τ) - homomorphism and it is clearly is a generalized Jordan (σ, τ) - higher homomorphism.

**Example (2.5):**
Let S_1, S_2 be two rings and f = (f_i)_{i \in \mathbb{N}} be a generalized higher (σ, τ) - homomorphism of a ring S_1 into a ring S_2 then there exist a relating higher (σ, τ) - homomorphism 0 = (ϕ)_{i \in \mathbb{N}} from a ring S_1 into S_2. Let M = M_{1 \times 2}(S_1), M' = M_{1 \times 2}(S_2) and \[ \Gamma = \left\{ \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} : m \in \mathbb{Z} \right\} \] then M and M' be tow Γ-rings. F = (F_i)_{i \in \mathbb{N}} be a family of mappings from a Γ-ring M into Γ-ring M' defined by: F_n((a \ b)) = (f_n(a) f_n(b)) then there exists a relating higher (σ, τ) - homomorphism \[ \phi = (\phi)_{i \in \mathbb{N}} from a \] Γ-ring M into Γ-ring M', such that \[ \phi_n((a \ b)) = (\theta_n(a) \ 0_n(b)) \].
Let $\sigma_1^n, \tau_1^n$ be two homomorphism of a $\Gamma$-ring $M$, such that $\sigma_1^n, \tau_1^n : M \rightarrow M \ni \sigma_1^n((a \ b)) = (\sigma^n(a) \ \sigma^n(b)), \tau_1^n((a \ b)) = (\tau^n(a) \ \tau^n(b))$ then $F_n$ is a generalized higher $(\sigma, \tau)$ - homomorphism.

**Lemma (2.6):**
Let $F = (f_i)_{i \in \mathbb{N}}$ be a generalized Jordan $(\sigma, \tau)$ - higher homomorphism of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, then for all $a, b, c \in M, \alpha, \beta \in \Gamma$ and for every $n \in \mathbb{N}$

If $\sigma^i = \sigma^i, \ \tau^i = \tau^i, \ \sigma^i \tau^i = \sigma^i \tau^{n-i}$ and $\sigma^i, \ \tau^i$ are commutative

(i) \[ f_n(a \alpha b \beta a + a \beta b \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a)) \beta \phi_i(\sigma^i \tau^{n-i}(b)) \alpha \phi_i(\tau^i(a)) \]

(ii) \[ f_n(a \alpha b \beta c + c \alpha b \beta a) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(c)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\tau^i(a)) \]

If $M'$ is a 2-torsion free commutative $\Gamma$-ring

(iii) \[ f_n(a \alpha b \alpha c) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \alpha \phi_i(\tau^i(c)) \]

(iv) \[ f_n(a \alpha b \alpha c + c \alpha b \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \alpha \phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(c)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \alpha \phi_i(\tau^i(a)) \]

**Proof:**
(i) Replacing $a \beta b + b \beta a$ for $b$ in the definition (2.2), we get:

\[ f_n(a \alpha (a \beta b + b \beta a) + (a \beta b + b \beta a) \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(a \beta b + b \beta a)) + \sum_{i=1}^{n} f_i(\sigma^i(a \beta b + b \beta a)) \alpha \phi_i(\tau^i(a)) \]

\[ = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(a) \beta \tau^i(b) + \tau^i(b) \beta \tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a) \beta \tau^i(b) + \tau^i(b) \beta \tau^i(a)) \alpha \phi_i(\tau^i(a)) \]
\[
\sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \left( \sum_{j=1}^{i} \phi_{j}(\sigma^{j-1}(a)\beta \phi_{j}(\tau^{j}(b)) + \sum_{j=1}^{i} \phi_{j}(\sigma^{j-1}(b)\beta \phi_{j}(\tau^{j}(a)) \right) + \\
\sum_{i=1}^{n} \left( \sum_{j=1}^{i} f_{j}(\sigma^{j}(a)\beta \phi_{j}(\tau^{i}(b)) + \sum_{j=1}^{i} f_{j}(\sigma^{j}(b)\beta \phi_{j}(\tau^{i}(a)) \right) \alpha \phi_{i}(\tau^{i}(a))
\]

Since \( \sigma^{i} = \sigma^{i}, \tau^{i} = \tau^{i}, \sigma^{i} \tau^{i} = \sigma^{i} \tau^{n-i} \) and \( \sigma^{i}, \tau^{i} \) are commutative

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(a))\beta \phi_{i}(\tau^{i}(b)) + \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a)) + \\
\sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\beta \phi_{i}(\sigma^{i} \tau^{n-i}(b))\alpha \phi_{i}(\tau^{i}(a)) + \sum_{i=1}^{n} f_{i}(\sigma^{i}(b))\beta \phi_{i}(\sigma^{i} \tau^{n-i}(a))\alpha \phi_{i}(\tau^{i}(a))
\]

...(1)

On the other hand:

\[
f_{n}(a \alpha b\beta b\beta a) = f_{n}(a\alpha b\beta b\beta a + \alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)
\]

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(a))\beta \phi_{i}(\tau^{i}(b)) + \sum_{i=1}^{n} f_{i}(\sigma^{i}(b))\beta \phi_{i}(\sigma^{i} \tau^{n-i}(a))\alpha \phi_{i}(\tau^{i}(a)) + \\
f_{n}(a\alpha b\beta a + a\beta b\alpha a)
\]

Comparing (1) and (2), we get:

\[
f_{n}(a \alpha b\beta a + a\beta b\alpha a) = \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a)) + \\
\sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\beta \phi_{i}(\sigma^{i} \tau^{n-i}(b))\alpha \phi_{i}(\tau^{i}(a))
\]

...(2)

(ii) Replace \( a + c \) for \( a \) in the definition (2.3), we get:

\[
f_{n}((a+c)\alpha b\beta (a+c)) = \sum_{i=1}^{n} f_{i}(\sigma^{i}(a+c))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a+c))
\]

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a) + \sigma^{i}(c))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a) + \tau^{i}(c))
\]

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a)) + \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(c))
\]

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(c))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(a)) + \sum_{i=1}^{n} f_{i}(\sigma^{i}(c))\alpha \phi_{i}(\sigma^{i} \tau^{n-i}(b))\beta \phi_{i}(\tau^{i}(c))
\]

...(1)
On the other hand:

\[ f_n((a+c)\alpha b \beta (a+c)) = f_n(a \alpha b \beta a + a \alpha b \beta c + c \alpha b \beta a + c \alpha b \beta a) \]

\[ = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(c))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(c)) + f_n(a \alpha b \beta c + c \alpha b \beta a) \]

Comparing (1) and (2), we get:

\[ f_n(a \alpha b \beta c + c \alpha b \beta a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(c)) + 2 \sum_{i=1}^{n} f_i(\sigma^i(c))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(c)) \]

(iii) Replace \( \alpha \) for \( \beta \) in (ii), we get:

\[ f_n(a \alpha b \alpha c + c \alpha b \alpha a) = f_n(a \alpha b \alpha c + a \alpha b \alpha c) = 2f_n(a \alpha b \alpha c) \]

\[ = 2 \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(c)) \]

Since \( M' \) is a 2-torsion free \( \Gamma \)-ring

\[ f_n(a \alpha b \alpha c) = f_n(\alpha \sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(c)) \]

(iv) Replace \( \alpha \) for \( \beta \) in (ii), we get:

\[ f_n(a \alpha b \alpha c + c \alpha b \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(c))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(a)) \]

Definition (2.7):
Let \( F = (f_i)_{i \in \mathbb{N}} \) be a generalized Jordan triple \( (\sigma, \tau) \)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \), then for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \), we define \( \delta_n(a,b) : M \times \Gamma \times M \rightarrow M' \) by:

\[ \delta_n(a,b) = f_n(\alpha \beta b) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) \]

Lemma (2.8):
Let \( F = (f_i)_{i \in \mathbb{N}} \) be a generalized Jordan \( (\sigma, \tau) \)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \), then for all \( a, b, c \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \):

(i) \( \delta_n(a,b) = -\delta_n(b,a) \)

(ii) \( \delta_n(a + b, c) = \delta_n(a,c) + \delta_n(b,c) \alpha \)

(iii) \( \delta_n(a, b + c) = \delta_n(a,b) + \delta_n(a, c) \beta \)

(iv) \( \delta_n(a,b) + \beta = \delta_n(a,b) + \delta_n(a,b) \beta \)
Proof:
(i) \[ f_n(a \alpha b + b \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) + \sum_{i=1}^{n} f_i(\sigma^i(b))\alpha \phi_i(\tau^i(a)) \]
\[ f_n(a \alpha b) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) = -f_n(b \alpha a) - \sum_{i=1}^{n} f_i(\sigma^i(b))\alpha \phi_i(\tau^i(a)) \]
\[ \delta_n(a, b) = -\delta_n(b, a) \]

(ii) \[ \delta_n(a + b, c) = f_n((a + b)\alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(a + b))\alpha \phi_i(\tau^i(c)) \]
\[ = f_n(a \alpha c + b \alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(c)) - \sum_{i=1}^{n} f_i(\sigma^i(b))\alpha \phi_i(\tau^i(c)) \]
\[ = f_n(a \alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(c)) + f_n(b \alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(b))\alpha \phi_i(\tau^i(c)) \]
\[ = \delta_n(a, c) + \delta_n(b, c) \]

(iii) \[ \delta_n(a, b + c) = f_n(a \alpha (b + c)) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b + c)) \]
\[ = f_n(a \alpha b + a \alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(c)) \]
\[ = f_n(a \alpha b) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) + f_n(a \alpha c) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(c)) \]
\[ = \delta_n(a, b) + \delta_n(a, c) \]

(iv) \[ \delta_n(a, b + \beta) = f_n(a (\alpha + \beta) b) - \sum_{i=1}^{n} f_i(\sigma^i(a))(\alpha + \beta)\phi_i(\tau^i(b)) \]
\[ = f_n(a \alpha b) - \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha \phi_i(\tau^i(b)) + f_n(a \beta b) - \sum_{i=1}^{n} f_i(\sigma^i(a))\beta \phi_i(\tau^i(b)) \]
\[ = \delta_n(a, b) + \delta_n(a, b) \]

Remark (2.9):
Note that \( F = (f_i)_{i \in N} \) is a generalized \((\sigma, \tau)\) - higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \) if and only if \( \delta_n(a, b) = 0 \) for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in N \).

Lemma (2.10):
Let \( F = (f_i)_{i \in N} \) be a generalized Jordan \((\sigma, \tau)\) - higher homomorphism of a 2- torsion free \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \), such that \( \sigma^2 = \sigma, \tau^2 = \tau, \sigma^i \tau^a = \sigma^i, \sigma^i \tau^a = \tau^i \sigma^i \) and \( \sigma^i, \tau^i \) for all \( i \in N \) are commutative, then for all \( a, b, m \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)
Generalized Jordan ($\sigma, \tau$)

1571

(i) $\delta_n (\sigma^n (a), \sigma^n (b)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (b), \tau^n (a)) + \delta_n (\sigma^n (b), \sigma^n (a)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (a), \tau^n (b))_a = 0$

(ii) $\delta_n (\sigma^n (a), \sigma^n (b)), \alpha \phi_n (\sigma^n (m)) \alpha G_n (\tau^n (b), \tau^n (a)) + \delta_n (\sigma^n (b), \sigma^n (a)), \alpha \phi_n (\sigma^n (m)) \alpha G_n (\tau^n (a), \tau^n (b))_a = 0$

(iii) $\delta_n (\sigma^n (a), \sigma^n (b)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (b), \tau^n (a)) + \delta_n (\sigma^n (b), \sigma^n (a)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (a), \tau^n (b))_a = 0$

Proof:

(i) We prove by using the induction, we can assume that:

$\delta_n (\sigma^n (a), \sigma^n (b)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (b), \tau^n (a)) + \delta_n (\sigma^n (b), \sigma^n (a)), \beta \phi_n (\sigma^n (m)) \beta G_n (\tau^n (a), \tau^n (b))_a = 0$ for all $a, b, m \in M$, and $s, n \in N$, $s < n$.

Let $w = aabm_baa + baa \beta \beta \alpha \alpha \beta$, since $F$ is a generalized Jordan homomorphism

$f_n (w) = f_n (a \alpha (b \beta \beta \beta \beta \alpha \alpha) + b \alpha (a \beta \beta \beta \beta \beta \alpha \beta)$

$= \sum_{i=1}^{n} f_i (\sigma^i (a)) \alpha \phi_i (\sigma^i \tau^{n-i} (b \beta \beta \beta \beta \beta \alpha \alpha) + \beta \phi_i (\sigma^i \tau^{n-i} (b \beta \beta \beta \beta \beta \alpha \alpha)) \alpha \phi_i (\tau^i (a)) + \beta \phi_i (\tau^i (b))$

$= \sum_{i=1}^{n} f_i (\sigma^i (a)) \alpha \phi_i (\sigma^i \tau^{n-i} (b \beta \beta \beta \beta \beta \alpha \alpha)) \alpha \phi_i (\tau^i (a)) + \beta \phi_i (\tau^i (b))$

$= \sum_{i=1}^{n} f_i (\sigma^i (a)) \alpha \phi_i (\sigma^i \tau^{n-i} (b \beta \beta \beta \beta \beta \alpha \alpha)) \alpha \phi_i (\tau^i (a)) + \beta \phi_i (\tau^i (b))$
\[ f_n(w) = f_n((a a b) b m b (b a a) + (b a a) b m b (a a b)) \]
\[ = \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]

On the other hand:

\[ f_n(w) = f_n((a a b) b m b (b a a) + (b a a) b m b (a a b)) \]
\[ = \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]

\[ = \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]

\[ \phi_i(\tau^i(a a b)) + \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]

\[ \phi_i(\tau^i(a a b)) + \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]

\[ \phi_i(\tau^i(a a b)) + \sum_{i=1}^{n} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a a b)) \beta \phi_i(\sigma^i(\tau^{n-i}(m))) \phi_i(\tau^i(b a a)) \]
\[
= -\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b)\beta\phi_i((\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b))) - \sum_{j=1}^{i} \phi_j(\sigma^j\tau^i(a))\alpha\phi_j(\tau^j(b)) - \\
\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b))) - \sum_{j=1}^{i} \phi_j(\sigma^j\tau^i(b))\alpha\phi_j(\tau^j(a)) + \\
\sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b))) + \\
\sum_{i=1}^{n} f_i(\sigma^i(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta(\phi_i(\tau^i(a\alpha b)))
\]

\[
= -\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b))\beta\phi_i((\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a), \tau^i(b)) - \\
\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b), \tau^i(a)) + \\
\sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a\alpha b)) + \\
\sum_{i=1}^{n} f_i(\sigma^i(b))\alpha\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a\alpha b))
\]

\[
= -f_n(\sigma^n(a\alpha b))\beta\phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b)) - \\
\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(a), \tau^i(b)) - \\
f_n(\sigma^n(a\alpha b))\beta\phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a)) - \\
\sum_{i=1}^{n} f_i(\sigma^i(a\alpha b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta G_i(\tau^i(b), \tau^i(a)) + \\
f_n(\sigma^n(a))\phi_n(\tau^n\sigma^n(b))\beta\phi_n((\sigma^n(m))\beta\phi_n(\tau^n(a\alpha b)) + \\
\sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^i\sigma^i(b))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b)) + \\
f_n(\sigma^n(b))\alpha\phi_n(\tau^n\sigma^n(a))\beta\phi_n((\sigma^n(m))\beta\phi_n(\tau^n(a\alpha b)) + \\
\sum_{i=1}^{n} f_i(\sigma^i(b))\alpha\beta\phi_i(\tau^i\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(m))\beta\phi_i(\tau^i(a\alpha b))
\]

\[\ldots(2)\]

Compare (1), (2) and since \(\sigma^n = \sigma^n, \tau^n = \sigma^n, \sigma^i\tau^{n-i} = \tau^i\sigma^i\) and \(\sigma^i, \tau^i\) are commutative
\[
0 = -f_a (\sigma^n (aaba)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (a), \tau^n (b))_a -
\]
\[
f_a (\sigma^n (aaba)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (b), \tau^n (a))_a +
\]
\[
f_a (\sigma^n (a)) \alpha \phi_a (\sigma^n (b)) \beta \phi_a ((\sigma^n (m)) \beta (\phi_a (\tau^n (aaba)) -
\]
\[
\sum_{i=1}^n \phi_i (\tau^i (\sigma^n (aaba)) \alpha \phi_i (\tau^i (b))) + f_a (b) \alpha \phi_a (\sigma^n (a)) \beta (\phi_a ((\sigma^n (m)) \beta (\phi_a (\tau^n (aaba)) -
\]
\[
\sum_{i=1}^n \phi_i (\tau^i (\sigma^n (aaba)) \alpha \phi_i (\tau^i (b))) - \sum_{i=1}^n f_i (\sigma^i (aaba)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (a), \tau^i (b))_a -
\]
\[
\sum_{i=1}^n f_i (\sigma^i (aaba)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (b), \tau^i (a))_a +
\]
\[
\sum_{i=1}^n \phi_i (\tau^i (\sigma^n (aaba)) \alpha \phi_i (\tau^i (b))) - \sum_{i=1}^n f_i (\sigma^i (aaba)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (a), \tau^i (b))_a -
\]
\[
\sum_{i=1}^n \phi_i (\tau^i (\sigma^n (aaba)) \alpha \phi_i (\tau^i (b))) + \sum_{i=1}^n f_i (\sigma^i (a) \alpha \phi_i (\tau^i \sigma^i (b)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta \phi_i (\tau^i (aaba)) -
\]
\[
\sum_{i=1}^n \phi_i (\tau^i (\sigma^n (aaba)) \alpha \phi_i (\tau^i (b)))
\]

\[
= -f_a (\sigma^n (aaba)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (a), \tau^n (b))_a -
\]
\[
f_a (\sigma^n (aaba)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (b), \tau^n (a))_a +
\]
\[
f_a (\sigma^n (a)) \alpha \phi_a (\sigma^n (b)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (b), \tau^n (a))_a +
\]
\[
f_a (\sigma^n (b)) \alpha \phi_a (\sigma^n (a)) \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (a), \tau^n (b))_a -
\]
\[
\sum_{i=1}^n f_i (\sigma^i (aaba)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (a), \tau^i (b))_a -
\]
\[
\sum_{i=1}^n f_i (\sigma^i (aaba)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (b), \tau^i (a))_a +
\]
\[
\sum_{i=1}^n f_i (\sigma^i (a) \alpha \phi_i (\tau^i \sigma^i (b)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (b), \tau^i (a))_a +
\]
\[
\sum_{i=1}^n f_i (\sigma^i (b) \alpha \phi_i (\tau^i \sigma^i (a)) \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (a), \tau^i (b))_a
\]

\[
= -\delta_a (\sigma^n (b), \sigma^n (a))_a \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (a), \tau^n (b))_a -
\]
\[
\delta_a (\sigma^n (a), \sigma^n (b))_a \beta \phi_a ((\sigma^n (m)) \beta G_a (\tau^n (b), \tau^n (a))_a -
\]
\[
\sum_{i=1}^n \delta_i (\sigma^i (b), \sigma^i (a))_a \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (a), \tau^i (b))_a -
\]
\[
\sum_{i=1}^n \delta_i (\sigma^i (a), \sigma^i (b))_a \beta \phi_i ((\sigma^i \tau^n (m)) \beta G_i (\tau^i (b), \tau^i (a))_a
\]
By our hypothesis, we have:
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(b), \sigma^i(a))_\alpha \beta \Phi_n((\sigma^i \tau^{n-i}(m)) \beta G_i(\tau^i(b), \tau^i(a))_\alpha + \]
\[ \sum_{i=1}^{n-1} f_i(\sigma^i(a), \sigma^i(b))_\alpha \beta \Phi_n((\sigma^i \tau^{n-i}(m)) \beta G_i(\tau^i(b), \tau^i(a))_\alpha \]

By our hypothesis, we have:
\[ \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))_\alpha = 0 \]

(ii) Replace \( \beta \) by \( \alpha \) in (i), we get (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).

Lemma (2.11):
Let \( F = (f_i)_{i \in N} \) be a generalized Jordan \( (\sigma, \tau) \) - higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime \( \Gamma \)-ring \( M' \), then for all \( a, b, m \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)

(i) \[ \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha = \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))_\alpha = 0 \]

(ii) \[ \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \Phi_n((\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))_\alpha = \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \alpha \Phi_n((\sigma^n(m)) \alpha G_n(\tau^n(a), \tau^n(b))_\alpha = 0 \]

(iii) \[ \delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \Phi_n((\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))_\beta = \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\beta \alpha \Phi_n((\sigma^n(m)) \alpha G_n(\tau^n(a), \tau^n(b))_\beta = 0 \]

Proof:
(i) By lemma (2.10) (i), we have:
\[ \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))_\alpha = 0 \]

Since by lemma (let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring and suppose that \( a, b \in M \) if \( a \Gamma m \Gamma b + b \Gamma m \Gamma a = 0 \) for all \( m \in M \), then \( a \Gamma m \Gamma b = b \Gamma m \Gamma a = 0 \),
we get:
\[ \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha = \]
\[ \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \Phi_n((\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))_\alpha = 0 \]

(ii) Replace \( \alpha \) for \( \beta \) in (i), we obtain (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we obtain (iii).
Theorem (2.12):
Let $F = \{ f \}_{i \in N}$ be a generalized Jordan $(\sigma, \tau)$ - higher homomorphism from a $\Gamma$-ring $M$ into a prime $\Gamma$-ring $M'$, then for all $a, b, c, d, m \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$
(i) $\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$
(ii) $\delta_n(\sigma^n(a), \sigma^n(b)),\alpha \phi_b(\sigma^n(m))\alpha G_a(\tau^n(d), \tau^n(c)) = 0$
(iii) $\delta_n(\sigma^n(a), \sigma^n(b)),\alpha \phi_b(\sigma^n(m))\alpha G_a(\tau^n(b), \tau^n(c)) = 0$

Proof:
(i) Replacing $a + c$ for $a$ in lemma (2.11) (i), we get:
$\delta_n(\sigma^n(a + c), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a + c)) = 0$
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) +$
$\delta_n(\sigma^n(c), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(c)) = 0$
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

By lemma (2.11) (i), we get:
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(c)) +$
$\delta_n(\sigma^n(c), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

Therefore, we get:
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(c))\beta $
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

$= -\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(c))\beta$
$\delta_n(\sigma^n(c), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

Since $M'$ is a prime $\Gamma$-ring and therefore:
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(c)) = 0$ ... (1)

Replacing $b + d$ for $b$ in lemma (2.11) (i), we get:
$\delta_n(\sigma^n(a), \sigma^n(b + d)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b + d), \tau^n(a)) = 0$
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) +$
$\delta_n(\sigma^n(a), \sigma^n(d)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(d), \tau^n(a)) = 0$
$\delta_n(\sigma^n(a), \sigma^n(d)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

By lemma (2.11) (i), we get:
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(d), \tau^n(a)) +$
$\delta_n(\sigma^n(a), \sigma^n(d)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$

Therefore, we get:
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(d), \tau^n(a))\beta$
$\delta_n(\sigma^n(a), \sigma^n(b)),\beta \phi_b(\sigma^n(m))\beta G_a(\tau^n(b), \tau^n(a)) = 0$
= -\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha \beta \phi_n(\sigma^n(m)) \beta
\delta_n(\sigma^n(a), \sigma^n(d))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha = 0

Since $M'$ is a prime $\Gamma$-ring and therefore:
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha = 0 \quad \ldots (2)\]

Now,
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b + d), \tau^n(a + c))_\alpha = 0\]
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(c))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(c))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(c))_\alpha = 0\]

Since by lemma (2.11) and (1) and (2), we get:
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(c))_\alpha = 0\]

(ii) Replace $\alpha$ for $\beta$ in (1), we obtain (ii).

(iii) Replacing $\alpha + \beta$ for $\alpha$ in (ii), we get:
\[\delta_n(\sigma^n(a), \sigma^n(b))_{\alpha+\beta} \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_{\alpha+\beta} = 0\]
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta + \delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta = 0\]

By (i) and (ii), we get:
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha + \delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha = 0\]

Therefore, we have:
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta \alpha \phi_n(\sigma^n(m)) \alpha \delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta = 0\]

= -\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta \alpha \phi_n(\sigma^n(m)) \alpha
\delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta = 0\]

Since $M'$ is a prime $\Gamma$-ring, then:
\[\delta_n(\sigma^n(a), \sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta = 0\]

3- The Main Results

Theorem (3.1):
Every generalized Jordan \((\sigma, \tau)\) - higher homomorphism from a \(\Gamma\)-ring \(M\) into a prime \(\Gamma\)-ring \(M'\) is either generalized \((\sigma, \tau)\) - higher homomorphism or \((\sigma, \tau)\) - higher anti-homomorphism.

Proof:
Let \(F = (f_i)_{i \in \mathbb{N}}\) be a generalized Jordan \((\sigma, \tau)\) - higher homomorphism of a \(\Gamma\)-ring \(M\) into a prime \(\Gamma\)-ring \(M'\), then by theorem (2.12) (i);

\[ \delta_n((\sigma^n(a), \sigma^n(b), \beta \phi_n(\sigma^n(m))) \beta G_n(\tau^n(d), \tau^n(c)))_\alpha = 0. \]

Since \(M'\) is a prime \(\Gamma\)-ring therefore either \(\delta_n((\sigma^n(a), \sigma^n(b)))_\alpha = 0\) or \(G_n(\tau^n(d), \tau^n(c))_\alpha = 0\), for all \(a, b, c, d \in M, \alpha, \beta \in \Gamma\) and \(n \in \mathbb{N}\). If \(G_n(\tau^n(d), \tau^n(c))_\alpha = 0\) for all \(c, d \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\) then \(\delta_n((\sigma^n(a), \sigma^n(b)))_\alpha = 0\). Hence, we get \(F\) is a generalized \((\sigma, \tau)\) - higher homomorphism.

But if \(G_n(\tau^n(d), \tau^n(c))_\alpha \neq 0\) for all \(c, d \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\) then we get \(F\) is a \((\sigma, \tau)\) - higher anti-homomorphism.

Proposition (3.2):
Let \(F = (f_i)_{i \in \mathbb{N}}\) be a generalized Jordan \((\sigma, \tau)\) - higher homomorphism from a \(\Gamma\)-ring \(M\) into 2-torsion free \(\Gamma\)-ring \(M'\), such that \(a \alpha b \beta a = a \beta b \alpha a\), for all \(a, b \in M\) and \(\alpha, \beta \in \Gamma\), \(a' \alpha b' \beta a' = a' \beta b' \alpha a'\), for all \(a', b' \in M'\) and \(\alpha, \beta \in \Gamma\), \(\sigma^2 = \sigma^1, \tau^2 = \tau^1, \sigma^i \tau^i = \sigma^i \tau^{n-i}\) and \(\sigma^i, \tau^i\) are commutative for all \(i \in \mathbb{N}\), then \(F\) is a generalized Jordan triple \((\sigma, \tau)\) - higher homomorphism.

Proof:
Replace \(a \beta b + b \beta a\) for \(b\) in the definition (2.2), we get:

\[
\begin{align*}
& f_n(a \alpha (a \beta b + b \beta a) + (a \beta b + b \beta a) \alpha a) \\
& = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(a \beta b + b \beta a)) + \sum_{i=1}^{n} f_i(\sigma^i(a \beta b + b \beta a)) \alpha \phi_i(\tau^i(a)) \\
& = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(a) \beta + \tau^i(b) \beta + \tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a \beta \beta a) + \sigma^i(b \beta a - \sigma^i(a)) \alpha \phi_i(\tau^i(a)) \\
& = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \left( \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(a) \beta \phi_j(\tau^j(b)) + \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(b) \beta \phi_j(\tau^j(a))) \right) + \\
& \sum_{i=1}^{n} \left( \sum_{j=1}^{i} f_j(\sigma^j \beta \phi_j(\tau^j \sigma^j(a) + \sum_{j=1}^{i} f_j(\sigma^j \beta \phi_j(\tau^j \sigma^j(a))) \right) \alpha \phi_i(\tau^i(a)).
\end{align*}
\]
Since $a'ab'\beta a' = a'\beta b'\alpha a'$, for all $a', b' \in M'$ and $\alpha, \beta \in \Gamma$, $\sigma'^{i} = \sigma^{i}$, $\tau'^{i} = \tau^{i}$, $\sigma^{i} \tau^{j} = \sigma^{i} \tau^{n-j}$ and $\sigma^{i}, \tau^{i}$ are commutative.

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha_{i}(\sigma^{i} \tau^{n-i}(a)\beta_{i}(\tau^{i}(b)) + 2 \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha_{i}(\sigma^{i} \tau^{n-i}(b)\beta_{i}(\tau^{i}(a)) + \\
\sum_{i=1}^{n} f_{i}(\sigma^{i}(b))\beta_{i}(\sigma^{i} \tau^{n-i}(a)\alpha_{i}(\tau^{i}(a))
\]

...(1)

On the other hand:

\[
= \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha_{i}(\sigma^{i} \tau^{n-i}(a)\beta_{i}(\tau^{i}(b)) + 2 \sum_{i=1}^{n} f_{i}(\sigma^{i}(b))\beta_{i}(\sigma^{i} \tau^{n-i}(a)\alpha_{i}(\tau^{i}(a)) + 2f_{n}(aab\beta a)
\]

...(2)

Compare (1) and (2), we get:

\[2f_{n}(aab\beta a) = 2 \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha_{i}(\sigma^{i} \tau^{n-i}(b)\beta_{i}(\tau^{i}(a))
\]

Since $M'$ is a 2-torsion free $\Gamma$-ring, we get:

\[f_{n}(aab\beta a) = \sum_{i=1}^{n} f_{i}(\sigma^{i}(a))\alpha_{i}(\sigma^{i} \tau^{n-i}(b)\beta_{i}(\tau^{i}(a))
\]

Hence, $F$ is a generalized Jordan triple $(\sigma, \tau)$ - higher homomorphism.

References


Received: September 11, 2014; Published: October 27, 2014