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Structure of the Indefinite Quasi-Hyperbolic Kac-Moody Algebra $QHA_4^{(2)}$

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Abstract

In this work, a class of indefinite Quasi-hyperbolic type of Kac-Moody algebras $QHA_4^{(2)}$ is considered. As a first step these algebras are realized as a graded Lie algebra of Kac-Moody type. To understand the structure of these algebras the homological and spectral sequence theory is applied. Here the components of the homology modules upto level three are computed. The structure of the components of the maximal ideal upto level four is also determined.

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1 Introduction

The theory of Kac-Moody Lie algebras is one of the modern fields of mathematical research and has got interesting connections and applications to various fields of Mathematical research, Combinatorics, Number Theory, Nonlinear differential equations, etc. A lot of work has been carried out for the finite and affine type of Kac-Moody algebras, whereas the structure of indefinite Kac-Moody algebras remains to be dealt with in detail.

Determination of the structure and multiplicities of roots of higher levels for Kac-Moody algebras is still an open problem. Feingold and Frenkel [2] computed level 2 root multiplicities for the hyperbolic Kac-Moody algebra $HA_1^{(1)}$, Kang [5,6,8] has determined the structure and obtained the root multiplicities for roots upto level 5 for $HA_1^{(1)}$ and for roots upto level 3 for $HA_2^{(2)}$. In [7], some root multiplicities are determined for the indefinite type of Kac-Moody algebra $HA_n^{(1)}$. Sthanumoorthy and Uma Maheswari [11,14,15] have computed the multiplicities of roots for a particular class of extended–hyperbolic Kac–Moody algebra $EHA_1^{(1)}$. This class of extended – hyperbolic Kac – Moody algebras was defined in Sthanumoorthy and Uma Maheswari [12]. A new class of indefinite non-hyperbolic Kac-Moody algebra called Quasi-Hyperbolic were introduced by Uma Maheswari [16]. In [17,18], another classes of indefinite non-hyperbolic Kac-Moody type QHG_2 and $QHA_2^{(1)}$ were considered. The homology modules and structure of the components of the maximal ideal upto level 4 were computed.

In this work, we are going to consider a class of a Quasi- Hyperbolic indefinite type of Kac-Moody algebra $QHA_4^{(2)}$; We first give a realization for $QHA_4^{(2)}$ whose associated with the GCM $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -1 & -a \\ 0 & -2 & 2 & -a \\ -a & -a & -a & 2 \end{pmatrix}$ where a > 2, $a \in Z^+$ as a graded Lie algebra of Kac-Moody type and then using the homological techniques developed by Benkart et al. [1] and Kang [5-8], we compute the homology module upto level three and the structure of the components of the maximal ideal upto level four.

2 Preliminaries

2.1. Kac-Moody algebras: We recall some preliminary results needed for the construction of graded Lie algebra**.** For further details on Kac-Moody algebras and root systems, one can refer to ([4], [10], & [19]).

Definition 2.1 [10]: An integer matrix $A=(a_{ij})_{i,j=1}^{n}$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(i) $a_{ii} = 2 \quad \forall \; i = 1,2,\dots,n$

(ii) $a_{ij} = 0 \;\Leftrightarrow\; a_{ji} = 0 \;\; \forall \; i, j = 1,2,\dots,n$

(iii) $a_{ij} \le 0 \;\; \forall \; i, j = 1,2,\dots,n$.

Let us denote the index set of A by N = {1,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J $\subset$ N such that I $\cup$ J = N and $a_{ij} = a_{ji} = 0 \ \forall \ i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 2.2 [4]: A realization of a matrix $A = (a_{ij})_{i,j=1}^{n}$ is a triple (H, Π, Π^{v}) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^{v} = \{\alpha_1^{v}, ..., \alpha_n^{v}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{v}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

Definition 2.3[4]:The Kac-Moody algebra g(A) associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i, $i = 1,2,...,n$ and H with the following defining relations :

$$[h, h'] = 0, \quad h, h' \in H \quad ; \quad [e_i, f_j] = \delta_{ij}\alpha_i^{v}$$

$$[h, e_j] = \alpha_j(h)e_j \quad ; \quad [h, f_j] = -\alpha_j(h)f_j \ , \ i, j \in N$$

$$(ade_i)^{1-a_{ij}} e_j = 0 \quad ; \quad (adf_i)^{1-a_{ij}} f_j = 0 \ , \forall \ i \neq j, \ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \ for \ all \ h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. Let $Q = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$, where $\alpha, \beta \in Q$.

Definition 2.4 [4]: For any $\alpha \in Q$ and $\alpha = \sum_{i=1}^{n} k_i\alpha_i$, define support of α, written as supp α , by supp $\alpha = \{i \in N / k_i \neq 0\}$. Let $\Delta (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 2.5 [4]:A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q_1,…,q_n), with $q_i > 0$ and q_i's are rational numbers.

Proposition 2.6 [4]:A GCM $A = (a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on g(A).

Definition 2.7[4]: A g(A) module V is called highest weight module with highest weight $\wedge \in h^*$ if there exists a nonzero v ∊ V such that

(i) $n^+ . v = 0$
(ii) $h . v = \wedge(h)v, \ \forall h \in h$
(iii) U(g(A)) . v = V, where U(g(A)) denotes the universal enveloping algebra of g(A).

A highest weight module V with highest weight $\wedge$ has the weight space decomposition $V = \underset{\lambda \in h^*}{\oplus} V_\lambda$, where $V_\lambda = \{v \in V / h.v = \lambda(h)v, \ \forall \text{ h} \in h\}$.

Definition 2.8[4]: To every GCM A is associated a Dynkin diagram S(A) defined as follows: (A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|,|a_{ji}|\}$ number of lines if $a_{ij} . a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}. a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}| , |a_{ji}|)$ of integers.

Theorem 2.9 [19]: Let A be a real n x n matrix satisfying (m1), (m2) and (m3).

(m1) A is indecomposable;

(m2) $a_{ij} \leq 0$ for $i \neq j$;

(m3) $a_{ij} = 0$ implies $a_{ji} = 0$

Then one and only one of the following three possibilities holds for both A and ${}^tA$:

(i) det A $\neq 0$; there exists u > 0 such that A u > 0; Av ≥ 0 implies v > 0 or v $= 0$;

(ii) co rank A=1; there exists u > 0 such that Au $= 0$; Av ≥ 0 implies Av $= 0$;

(iii) there exists u > 0 such that Au < 0; Av ≥ 0, v ≥ 0 imply v $= 0$.

Then A is of finite, affine or indefinite type iff (i), (ii) or (iii) (respectively) is satisfied.

Definition 2.10[19]: A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 2.11[16]: Let A=$(a_{ij})_{i,j=1}^n$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be of Quasi Hyperbolic (QH) type if S(A) has a proper connected sub diagram of hyperbolic types with n-1 vertices. The GCM A is of QH type if S(A) is of QH type. We then say the Kac-Moody algebra g(A) is of QH type.

2.2 General construction of graded Lie algebra (Benkart et al ., [1]):

Let us start with G ,the Lie algebra over a field of characteristic zero. Let V, V' be two G – modules. Let $\psi : V' \otimes V \to G$, a G – module homomorphism.

Define $G_0 = G, G_{-1} = V, G_1 = V'$; $G_+ = \sum_{n \geq 1} G_n$ (resp. $G_- = \sum_{n \geq 1} G_{-n}$) denote the free Lie algebra generated by V' (respectively, V); G_n (respectively, G_{-n}) for n > 1 is the space of all products of n vectors from V' (respectively V). Then $G = \sum_{n=-\infty}^{\infty} G_n$ is given a Lie algebra structure by defining the Lie bracket [,] as follows: $\forall$ a, b ∈ G, $v \in V$, $w \in V'$

$[a, v] = a.v = -[v, a]$ and $[a, w] = a.w = -[w, a]$

[a, b] denote the bracket operation in G, $[w,v] = \psi(w \otimes v) = -[v,w]$

By extending this Lie bracket operation, $G = \sum_{n \in z} G_n$ becomes a graded Lie algebra which is generated by its local part $G_{-1} + G_0 + G_1$.

For n≥1 define the subspaces, $I_{\pm n} = \{x \in G_{\pm n} \mid (ad\ G_{\mp 1})^{n-1} x = 0\}$, define

$I = \underset{n \in z}{\oplus} I_n$ and $I_+ = \sum_{n>1} I_n$, $I_- = \sum_{n>1} I_{-n}$ .Then the subspaces I_+ , I_- and I are all graded ideals of G and I is the maximal graded ideal trivially intersecting the local part $G_{-1} + G_0 + G_1$.Let $L_{\pm n} = G_{\pm n} / I_{\pm n}$, for n > 1;

Consider $L = L(G, V, V', \psi) = G_- / I_- \oplus G_0 \oplus G_+ / I_+$

$= ... \oplus L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \oplus ...,$ where $L_0 = G_0$, $L_1 = G_1$, $L_{-1} = G_{-1}$.

Then $L = \oplus_{n \in Z} L_n$ becomes a graded Lie algebra generated by its local part V ⊕ G ⊕ V′ and L = G / I.

By the suitable choice of V (written as the direct sum of irreducible highest weight modules), the contragradient V^* of V, the basis elements and the homomorphism $\psi : V^* \otimes V \to g^e$, form the graded Lie algebra $L = L(g^e, V, V^*, \psi)$. For further details one can refer to ([1] , [5]).

Theorem 2.12[1]: L is a Z^{n+m} –graded algebra.

Theorem 2.13[1]: Let $\phi : A(C) \to L$ be the Lie algebra homomorphism sending $E_i \to e_i$, $F_i \to f_i$, $H_i \to h_i$. Then ϕ has kernel as I(C) and I(C) is the largest graded ideal of A(C) trivially intersecting the span of $H_1, \ldots, H_{n+m}$. Also $\phi : A(C) / I(C) \to L$ is an isomorphism.

Proposition 2. 14[1]: The matrix C has rank 2n – *l* and C is symmetrizable.

We now recall the definition of homology of Lie algebra (Garland and Lepowsky, [3]) and Hochschild-Serre spectral sequence (Kang, [5]).

Let V be a module over a Lie algebra G. Define the space $C_q(G,V)$ for q > 0 of q – dimensional chains of the Lie algebra G with coefficients in V to be $\wedge^q (G) \otimes V$.

The differential $d_q = C_q(G,V) \to C_{q-1}(G,V)$ is defined to be

$$d_q(g_1 \wedge ... \wedge g_q \otimes v) = \sum_{1 \le s \le t \le q} (-1)^{s+t-1} ([g_s, g_t]) \wedge g_1 \wedge ... \wedge \hat{g}_s \wedge ... \wedge \hat{g}_t \wedge ... \wedge g_q) \otimes v$$
$$+ \sum_{1 \le s \le q} (-1)^s (g_1 \wedge ... \wedge \hat{g}_s \wedge ... \wedge g_q) \otimes g_s.v,$$

for $v \in V, g_1, ..., g_q \in G$. For q < 0, define $C_q(G,V) = 0$ and $d_q = 0$. Then $d_q \circ d_{q-1} = 0$. The homology of the complex (C, d) = $\{C_q(G, V), dq\}$ is called the homology of the Lie algebra G with coefficients in V and is denoted by $H_q(G,V)$.

When V = C, we write $H_q(G)$ for $H_q(G,C)$.

Assume now that G, V, $C_q(G,V)$ are completely reducible modules in the category O over a Kac-Moody algebra g(A) with d_q having g(A)-module homomorphisms. Let I be an ideal of G and L = G/I. Define a filtration $\{K_p = K_pC\}$ of the complex {C, d} by $K_pC_{p+q} = \{g_1 \wedge g_2 \wedge \ldots \wedge g_{p+q} \otimes v \mid g_i \in I \text{ for } p+1 \le i \le p+q\}$.

This gives rise to a spectral sequence $\{E^r_{p,q}, d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ such that

$E^2_{p,q} \cong H_p(L, H_q(I,V))$, where $E^r_{p,q}$'s are determined by $E^{r+1}_{p,q} = \mathrm{Ker}(d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})/\mathrm{Im}(d_r : E^r_{p+r,q-r+1} \to E^r_{p,q})$ with boundary homomorphisms $d_{r+1} : E^r_{p,q} \to E^r_{p-r-1,q+r}$. The modules $E^r_{p,q}$ become stable for r > max(p,q +1) for each (p,q) and is denoted by $E^\infty_{p,q}$. The spectral sequence $\{E^r_{p,q}, d_r\}$ converges to $H_n(G,V)$ in the following sense : $H_n(G,V) = \bigoplus_{p+q=n} E^\infty_{p,q}$. Then we get the following Hochschild-Serre five term exact sequence ([5]).

$$H_2(G,V) \to H_2(L,H_0(I,V) \to H_0(L,H_1(I,V)) \to H_1(G,V) \to H_1(L,H_0(I,V)) \to 0.$$

Take L = G/I, where $G = \oplus_{n\geq 1} G_n$ is the free Lie algebra generated by the subspace G_1 and $I = \oplus_{n\geq m} I_n$ the graded ideal of G generated by the subspace I_m for m ≥ 2. Then $L = \oplus_{n\geq 1} L_n$ becomes a graded Lie algebra generated by the subspace $L_1 = G_1$. Let J = I / [I, I]. J is an L-module via adjoint action generated by the subspace J_m. For m ≤ n < 2m , $J_n \cong I_n$.. If I_m and G_1 are modules over a Kac-Moody algebra g(A) then G_n has a g(A)-module structure for every $x \in g(A), v \in G, w \in G_{n-1}$, $x \cdot [v,w] = [x \cdot v, w] + [v, x \cdot w]$. I_n also has a similar module structure and we have the induced module structure of the homogeneous subspaces L_n, J_n. Then we have the following theorem proved in Kang [5].

Theorem 2.15[5]: There is an isomorphism of g(A) – modules $H_j(L,J) \cong H_{j+2}(L)$, for j ≥ 1. In particular $I_{m+1} \cong (G_1 \otimes I_m) / H_3(L)_{m+1}$.

Now, for arbitrary j ≥ m, set $I^{(j)} = \sum_{n\geq j} I_n$; then $I^{(j)}$ is an ideal of G generated by the subspace I_j. We consider the quotient algebra $L^{(j)} = G/I^{(j)}$. Let $N^{(j)}=I^{(j)}/ I^{(j-1)}$. In this notation $L = L^{(m)}$.

Then we have an important relation: $I_{j+1} \cong (G_1 \otimes I_j)/H_3(L^{(j)})_{j+1}$.

And, there exists a spectral sequence $\{E^r_{p,q}, d_r: E^r_{p,q} \to E^r_{p-r,q+r-1}\}$converging to $H_*(L^{(j)})$ such that and $E^2_{p,q} \cong H_p(L^{(j-1)}) \otimes \wedge^q (I_{j-1})$ and $H_3(L^{(j)}) \cong E^\infty_{3,0} \oplus E^\infty_{2,1} \oplus E^\infty_{1,2} \oplus E^\infty_{0,3}$.

Lemma 2.16[5]: In the above notation, $H_2(L) \cong I_m$.

Let us recall the Kostant's formula for symmetrizable Kac-Moody algebras [9]:

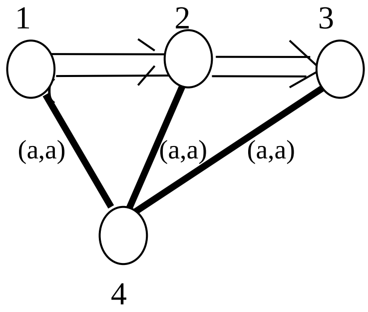
For a symmetrizable GCM A=$(a_{ij})^n_{i,j=1}$, let $\Delta \subset \mathfrak{h}^*$, Δ^+, Δ^- denote the root system of g(A), positive and negative roots, respectively, of g(A). Then we have the triangular decomposition : g(A) = n^- ⊕ h ⊕ n^+, where $n^\pm = \bigoplus_{\alpha\in\Delta^\pm} g_\alpha$. Let S={1,…,s} be a subset of N = {1,…,n} and g_s ,the subalgebra of g(A) generated by the elements e_i, f_i, i = 1, …, s and h. Let Δ_s^+ denote the set of positive roots generated by α_1, …, α_s and $\Delta_s^- = -\Delta_s^+$. Then g_s has the corresponding triangular decomposition: $g_s = n_s^- \oplus h \oplus n_s^+$, where $n_s^{\pm} = \underset{\alpha\in\Delta_s^{\pm}}{\oplus} g_\alpha$ and $\Delta_s = \Delta_s^+ \cup \Delta_s^-$ is the root system of g_s. Let $\Delta^{\pm}(s) = \Delta^{\pm} \setminus \Delta_s^{\pm}$, $n^{\pm}(S) = \underset{\alpha\in\Delta^{\pm}(S)}{\oplus} g_\alpha$. Then g(A) = $n^-(S) \oplus g_s \oplus n^+(S)$. Let $W(S) = \{w \in W / w\Delta^- \cap \Delta^+ \subset \Delta^+(S)\}$. For $\lambda \in h^*$ denote by $\tilde{V}(\lambda)$, the irreducible highest weight module over g(A) and $V(\lambda)$ the irreducible highest weight module over g_s.

Theorem 2.17[9]: (Kostant's formula) $H_j(n^-(S), \tilde{V}(\lambda)) \cong \underset{\substack{w\in W(S)\\ l(w)=j}}{\oplus} V(w(\lambda+\rho)-\rho)$.

Lemma 2.18[5]: Suppose $w = w'\ r_j$ and $l(w) = l(w')+1$. Then $w \in W(S)$ if and only if $w' \in W(s)$ and $w'(\alpha_j) \in \Delta^+(S)$.

3 Realization for $QHA_4^{(2)}$

In this section, we are going to consider a class of a Quasi- Hyperbolic indefinite type of Kac-Moody algebra $QHA_4^{(2)}$; We first give a realization for $QHA_4^{(2)}$ whose associated GCM is $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -1 & -a \\ 0 & -2 & 2 & -a \\ -a & -a & -a & 2 \end{pmatrix}$, where a > 2, $a \in Z^+$ and this GCM is symmetrizable; This algebra is obtained from the algebra $A_4^{(2)}$ associated with the GCM A = $\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$. The associated Dynkin diagram of $QHA_4^{(2)}$ is represented as



Consider the Kac-Moody algebra associated with the GCM $A_4^{(2)}$.

Let (h, Π, $\Pi^\vee$) be the realization of A with $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ Then the relations obtained from the symmetric, non degenerate bilinear form is given as follows:

$(\alpha_1,\alpha_1) = 2$, $(\alpha_1,\alpha_2) = -1$, $(\alpha_1,\alpha_3) = 0$, $(\alpha_1,\alpha_4) = 0$, $(\alpha_2,\alpha_1) = -1$, $(\alpha_2,\alpha_2) = 1$, $(\alpha_2,\alpha_3) = -1/2$, $(\alpha_3,\alpha_1) = 0$, $(\alpha_3,\alpha_2) = -1/2$, $(\alpha_3,\alpha_3) = 1/2$. Let α_4' be the element in h^*

such that $\alpha_4^{'}(\alpha_1^{v}) = 0, \alpha_4^{'}(\alpha_2^{v}) = 0, \alpha_4^{'}(\alpha_3^{v}) = 1$ and $(\alpha_4^{'}, \alpha_4^{'}) = \frac{16}{25}(5a^2 - 5a + 2)$.

Define $\lambda = \alpha_1 + (2-a)\alpha_2 + (2-3a)\alpha_3 + \frac{5a}{4}\alpha_4^{'}$. Set $\alpha_5 = -\lambda$. Form the matrix $C = (<\alpha_i, \alpha_j>)_{i,j=1}^{4}$. Then C = $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -1 & -a \\ 0 & -2 & 2 & -a \\ -a & -a & -a & 2 \end{pmatrix}$ where a > 2, $a \in Z^{+}$ is the symmetrizable GCM of Quasi- Hyperbolic type $QHA_4^{(2)}$.

Let V be the integrable highest weight irreducible module over G with the highest weight λ as defined earlier. Let V^* be the contragradient of V and ψ be the mapping as defined earlier. Let G be the Kac-Moody algebra associated with the GCM $\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$. Form the graded Lie algebra $L(G^e, V, V^*, \psi)$.

Then $L \cong g(C)$ and L is a symmetrizable Kac-Moody algebra of Quasi-hyperbolic type associated with the GCM C. Thus we have given the realization for this quasi hyperbolic family as a graded Lie algebra of Kac Moody type.
Next, we compute the homology modules of the Kac-Moody algebra for $QHA_4^{(2)}$. We note that, from the realization of $L = QHA_4^{(2)}$ as $L = L_{-1} \oplus L_0 \oplus L_{+} = G/I$ and using the involutive automorphism, it is sufficient to study only about the negative part $L_{-} = G_{-}/I_{-}$.

Computation of Homology Modules :

Let S = {1,2,3} $\subset$ N = {1,2,3,4} Let g_s is the Kac-Moody Lie algebra $A_4^{(2)}$.
Here $\Delta^{+}(S) = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4^{'} \in \Delta^{+} / k_4 \neq 0\}$. Δ_s be the root system of g_s.

The only reflection of length 1 in W(S) is r_4.
$r_4(\rho) = \rho - \alpha_4$; $r_4(\rho) - \rho = -\alpha_4$ $\therefore H_1(L_-) \cong V(-\alpha_4)$.

The reflections of length 2 in W(S) are r_4r_1, r_4r_2, r_4r_3.
$r_4r_1(\rho) - \rho = -(1+a)\alpha_4 - \alpha_1$; $r_4r_2(\rho) - \rho = -(1+a)\alpha_4 - \alpha_2$; $r_4r_3(\rho) - \rho = -(1+a)\alpha_4 - \alpha_3$.
By Kostant's formula,
$H_2(L_-) \cong \{V(-(1+a)\alpha_4 - \alpha_1) \oplus V(-(1+a)\alpha_4 - \alpha_2) \oplus (-(1+a)\alpha_4 - \alpha_3)\}$.

The reflections of length 3 in W(S) are $r_4r_1r_2$, $r_4r_1r_3$, $r_4r_1r_4$, $r_4r_2r_1$, $r_4r_2r_3$, $r_4r_2r_4$, $r_4r_3r_1$, $r_4r_3r_2$, $r_4r_3r_4$.
$r_4r_1r_2(\rho) - \rho = -(1+3a)\alpha_4 - \alpha_2 - 2\alpha_1$; $r_4r_1r_3(\rho) - \rho = -(1+2a)\alpha_4 - \alpha_3 - \alpha_1$;
$r_4r_1r_4(\rho) - \rho = -a(1+a)\alpha_4 - (1+a)\alpha_1$; $r_4r_2r_1(\rho) - \rho = -(1+4a)\alpha_4 - 3\alpha_2 - \alpha_1$;
$r_4r_2r_3(\rho) - \rho = -(1+3a)\alpha_4 - \alpha_3 - 2\alpha_2$; $r_4r_2r_4(\rho) - \rho = -a(1+a)\alpha_4 - (1+a)\alpha_2$;
$r_4r_3r_1(\rho) - \rho = -(1+2a)\alpha_4 - \alpha_3 - \alpha_1$; $r_4r_3r_2(\rho) - \rho = -(1+4a)\alpha_4 - 3\alpha_3 - \alpha_2$;

$r_4 r_3 r_4(\rho) - \rho = -a(1+a)\alpha_4 - (1+a)\alpha_3$;

Hence, by Kostant formula,

$$\begin{aligned} H_3(L_-) \cong \{ & V(-(1+3a)\alpha_4 - \alpha_2 - 2\alpha_1) \oplus V(-(1+2a)\alpha_4 - \alpha_3 - \alpha_1) \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_1) \\ & \oplus V(-(1+4a)\alpha_4 - 3\alpha_2 - \alpha_1) \oplus V(-(1+4a)\alpha_4 - 3\alpha_2 - \alpha_1) \oplus V(-(1+3a)\alpha_4 - \alpha_3 - 2\alpha_2) \\ & \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_2) \oplus V(-(1+2a)\alpha_4 - \alpha_3 - \alpha_1) \\ & \oplus V(-(1+4a)\alpha_4 - 3\alpha_3 - \alpha_2) \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_3) \} \end{aligned} \tag{3.1}$$

The other homology modules $H_4(L_-)$, $H_5(L_-)$, $H_6(L_-)$ etc. can be computed in a similar manner.

4 Structure of the Maximal Ideal in $QHA_4^{(2)}$

In this section, we study the structure of the components of maximal ideal upto level 4. Since the ideal I_- of G_- is generated by the homological subspace I_{-2}, we may write $I_- = I_-^{(2)}$. For j ≥ 2, we write $I_-^{(j)} = \sum_{n \geq j} I_{-n}$, $L_-^{(j)} = G / I_-^{(j)}$ and $N_-^{(j)} = I_-^{(j)} / I_-^{(j+1)}$. Using the homological approach and Hochschild – Serre spectral sequences theory together with the representation theory of Kac-Moody algebra, we can determine other components of the maximal ideals in $QHA_4^{(2)}$.

To determine I_{-2}:

Since G_- is free and I_- is generated by the subspace I_{-2} from the Hochschild –Serre five term exact sequence and using Lemma 2.15 we get, $I_{-2} \cong H_2(L_-)$;

$H_2(L_-) \cong \{V(-(1+a)\alpha_4 - \alpha_1) \oplus V(-(1+a)\alpha_4 - \alpha_2) \oplus (-(1+a)\alpha_4 - \alpha_3)\}$.

$\therefore \quad I_{-2} \cong \{V(-(1+a)\alpha_4 - \alpha_1) \oplus V(-(1+a)\alpha_4 - \alpha_2) \oplus (-(1+a)\alpha_4 - \alpha_3)\}$.

To determine I_{-3}:

We have, $I_{-(j+1)} \cong (V \otimes I_{-j}) / H_3(L_-^{(j)})_{-(j+1)} \quad j \geq 2$.

When j = 2, $L_-^{(2)}$ coincides with the subspace $n^-(S)$ for S = {1, 2, 3} and therefore we can compute $H_3(L_-^{(2)})$, using the Kostant formula.

$$\begin{aligned} H_3(L_-^{(2)}) \cong \{ & V(-(1+3a)\alpha_4 - \alpha_2 - 2\alpha_1) \oplus V(-(1+2a)\alpha_4 - \alpha_3 - \alpha_1) \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_1) \\ & \oplus V(-(1+4a)\alpha_4 - 3\alpha_2 - \alpha_1) \oplus V(-(1+4a)\alpha_4 - 3\alpha_2 - \alpha_1) \oplus V(-(1+3a)\alpha_4 - \alpha_3 - 2\alpha_2) \\ & \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_2) \oplus V(-(1+2a)\alpha_4 - \alpha_3 - \alpha_1) \\ & \oplus V(-(1+4a)\alpha_4 - 3\alpha_3 - \alpha_2) \oplus V(-a(1+a)\alpha_4 - (1+a)\alpha_3) \}, \end{aligned}$$ by equation (3.1)

Since a > 2, $H_3(L_-^{(2)})_{-3} = 0$ and we obtain $I_{-3} \cong (V \otimes I_{-2}) / H_3(L_-^{(2)})_{-3} \cong V \otimes I_{-2}$.

To determine the structure of I_{-4}:

To find the structure of I_{-4}, we need to find the structure of $H_3(L_-^{(3)})_{-4}$.

Consider the short exact sequence, $0 \to N_-^{(2)} \to L_-^{(3)} \to L_-^{(2)} \to 0$ and the corresponding spectral sequence $\{E_{p,q}^r\}$ converging to $H_*(L_-^{(3)})$ such that $E_{p,q}^2 \cong H_p(L_-^{(2)}) \otimes \Lambda^q(I_{-2})$. We start with the sequence, $0 \to E_{2,0}^2 \xrightarrow{d_2} E_{0,1}^2 \to 0.$

Since the spectral sequence converges to $H_*(L_-^{(3)})$, we have $H_1(L_-^{(3)}) \cong E_{1,0}^\infty \oplus E_{0,1}^\infty$. But $H_1(L_-^{(3)}) \cong L_-^{(3)}/[L_-^{(3)}, L_-^{(3)}] \cong L_{-1} = V$ and $E_{1,0}^\infty = E_{1,0}^2 \cong H_1(L_-^{(2)}) \cong L_1^{(2)}/[L_-^{(2)}, L_-^{(2)}] \cong L_{-1} = V$, $E_{0,1}^\infty = E_{0,1}^3 = 0$. $\therefore$ d_2 is surjective.

Since $E_{2,0}^2 = E_{0,1}^2 \cong I_{-2}$, d_2 becomes an isomorphism. Thus $E_{2,0}^\infty = E_{2,0}^3 = 0$.

Now, consider the sequence $0 \to E_{3,0}^2 \xrightarrow{d_2} E_{1,1}^2 \to 0.$

By Kostant formula, $E_{3,0}^2 \cong H_3(L_-^{(2)})$; $E_{1,1}^2 \cong V \otimes I_{-2}$ and since $V \otimes I_{-2}$ is a direct sum of irreducible highest weight modules over $A_4^{(2)}$ of level 3, by comparing the levels of both terms, d_2: $E_{3,0}^2 \to E_{1,1}^2$ is trivial. So $E_{3,0}^3 = E_{3,0}^2$ and $E_{1,1}^\infty = E_{1,1}^3 = E_{1,1}^2 \cong V \otimes I_{-2}$.

$I_-^{(3)}$ is generated by I_{-3} $\therefore$ $H_2(L_-^{(3)}) \cong I_{-3} = V \otimes I_{-2}$.

But $H_2(L_-^{(3)}) \cong E_{2,0}^\infty \oplus E_{1,1}^\infty \oplus E_{0,2}^\infty$. It follows that $E_{0,2}^\infty = E_{0,2}^4 = 0$. Therefore we find that either $E_{0,2}^3 = 0$ or $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is surjective.

In the first case, $E_{0,2}^3 = 0$, this implies that $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is trivial and that $d_2 : E_{2,1}^2 \to E_{0,2}^2$ is surjective in the sequence $0 \to E_{4,0}^2 \xrightarrow{d_2} E_{2,1}^2 \xrightarrow{d_2} E_{0,2}^2 \to 0.$

Thus $E_{3,0}^\infty = E_{3,0}^4 = \mathrm{Ker}(d_3 : E_{3,0}^3 \to E_{0,2}^3)/\mathrm{Im}(d_3 : 0 \to E_{3,0}^3)$

$$= E_{3,0}^3 = E_{3,0}^2 \cong H_3(L_-^{(2)})$$

By comparing levels, we see that $d_2 : E_{4,0}^2 \to E_{2,1}^2$ is trivial. Since $E_{0,2}^2 \cong \Lambda^2(I_{-2})$, $E_{4,0}^3 = E_{4,0}^2$ and $E_{2,1}^\infty = E_{2,1}^3 = \mathrm{Ker}(d_2 : E_{2,1}^2 \to E_{0,2}^2)/\mathrm{Im}(d_2 : E_{4,0}^2 \to E_{2,1}^2) \cong \mathrm{Ker}\,(d_2 : E_{2,1}^2 \to E_{0,2}^2)$. Since $d_2 : E_{2,1}^2 \to E_{0,2}^2$ is surjective, $\Lambda^2(I_{-2}) \cong E_{0,2}^2 \cong E_{2,1}^2/\mathrm{Ker}\,d_2 \cong (I_{-2} \otimes I_{-2})/\mathrm{Ker}\,d_2$. Therefore $\mathrm{Ker}\,d_2 \cong S^2(I_{-2})$. Hence $E_{2,1}^\infty \cong S^2(I_{-2})$.

If $E_{0,2}^3$ is nonzero and $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is surjective, since $E_{3,0}^3 = E_{3,0}^2$ is irreducible, $d_3 : E_{3,0}^3 \to E_{0,2}^3$ is an isomorphism. Thus $E_{3,0}^\infty = E_{3,0}^4 = 0$ and $H_3(L_-^{(2)}) \cong E_{3,0}^3 \cong E_{0,2}^3 \cong E_{0,2}^2 / \mathrm{Im}(d_2 : E_{2,1}^2 \to E_{0,2}^2)$
$\cong \Lambda^2(I_{-2})/\mathrm{Im}(d_2 : E_{2,1}^2 \to E_{0,2}^2)$.

Since all the modules, here are completely reducible over $A_4^{(2)}$,
$\mathrm{Im}(d_2 : E_{2,1}^2 \to E_{0,2}^2) \cong \Lambda^2(I_{-2}) / H_3(L_-^{(2)})$. We get, $d_2 : E_{4,0}^2 \to E_{2,1}^2$ is trivial.

Thus $E_{2,1}^\infty = E_{2,1}^3 = \mathrm{Ker}(d_2 : E_{2,1}^2 \to E_{0,2}^2) / \mathrm{Im}(d_2 : E_{4,0}^2 \to E_{2,1}^2) = Ker(d_2 : E_{2,1}^2 \to E_{0,2}^2)$.

Since $\mathrm{Im}\,d_2 \cong \Lambda^2(I_{-2}) / H_3(L_-^{(2)}) \cong E_{2,1}^2 / \mathrm{Ker}\ d_2 \cong (I_{-2} \otimes I_{-2}) / \mathrm{Ker}\ d_2$,

$\Rightarrow$ $\operatorname{Ker} d_2 \cong S^2(I_{-2}) \oplus H_3(L_-^{(2)})$ $\therefore E_{3,0}^{\infty} \oplus E_{2,1}^{\infty} \cong S^2(I_{-2}) \oplus H_3(L_-^{(2)})$

Consider $0 \to \mathrm{E}_{5,0}^2 \xrightarrow{\mathrm{d}_2} \mathrm{E}_{3,1}^2 \to 0$. By comparing levels, we see that $\mathrm{d}_2 : \mathrm{E}_{3,1}^2 \to \mathrm{E}_{1,2}^2$ is trivial. Thus $\mathrm{E}_{1,2}^3 = \mathrm{E}_{1,2}^2 \cong \mathrm{V} \otimes \Lambda^2(\mathrm{I}_{-2})$. By comparing the levels of the terms in the sequence $0 \to \mathrm{E}_{4,0}^3 \xrightarrow{\mathrm{d}_3} \mathrm{E}_{1,2}^3 \to 0$, we get $\mathrm{d}_3 = 0$. Therefore $\mathrm{E}_{1,2}^{\infty} = \mathrm{E}_{1,2}^4 = \mathrm{E}_{1,2}^2 \cong \mathrm{V} \otimes \Lambda^2(\mathrm{I}_{-2})$. Since $\mathrm{E}_{0,3}^{\infty}$ is a sub module of $\mathrm{E}_{0,3}^2 \cong \Lambda^3(\mathrm{I}_{-2})$. $\therefore$ $H_3(L_-^{(3)}) \cong H_3(L_-^{(2)}) \oplus S^2(I_{-2}) \oplus (V \otimes \Lambda^2(I_{-2})) \oplus \mathrm{M}$, where M is a direct sum of level 6 irreducible representations of $\mathrm{A}_4^{(2)}$. Therefore $\mathrm{H}_3(\mathrm{L}_-^{(3)})_{-4} \cong S^2(I_{-2})$ and $I_{-4} \cong (V \otimes I_{-3}) / H_3(L_-^{(3)})_{-4} \cong (V \otimes I_{-3}) / S^2(I_{-2})$.

From the above results, we get the structure of the components of the maximal ideal I- (upto level 4) in the Quasi – hyperbolic Kac-Moody algebra $\mathrm{QHA}_4^{(2)}$.

Thus we have proved the following structure theorem.

Theorem 4.1: With the usual notations, let $L = \oplus_{n \in Z} L_n$ be the realization of $\mathrm{QHA}_4^{(2)}$ associated with the GCM $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -1 & -a \\ 0 & -2 & 2 & -a \\ -a & -a & -a & 2 \end{pmatrix}$ where a > 2, $a \in Z^+$. Then we have following :

i) $I_{-2} \cong \{V(-(1+a)\alpha_4 - \alpha_1) \oplus V(-(1+a)\alpha_4 - \alpha_2) \oplus (-(1+a)\alpha_4 - \alpha_3)\}$.

ii) $I_{-3} \cong V \otimes I_{-2}$.

iii) $I_{-4} \cong (V \otimes I_{-3}) / S^2(I_{-2})$.

5 Conclusion

In this work, we have considered a class of quasi hyperbolic Kac-Moody algebra $\mathrm{QHA}_4^{(2)}$ and determined the structure of the components in the graded ideals upto level four. This work gives further scope for understanding the complete structure of this indefinite, quasi hyperbolic algebra.

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