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Modules with the Closed Sum Property

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Abstract

A module *M* is said to have the closed intersection property (briefly CIP) if, the intersection of any two closed submodules of *M* is again closed [6]. In this paper we present the dual of the CIP, namely, *M* has the closed sum property (briefly CSP) for which the sum of any two closed submodules, so the submodule generated by their union, is a closed submodule, too. We investigate the concept of CSP. Basic properties and some relations of these modules are given.

Keywords: closed submodules, modules with CIP, modules with CSP, c-d-Rickart module.

0. Introduction

Throughout this paper *R* is a commutative ring with identity and every *R*-module is a unitary. Let *M* be an *R*-module and *N* be a submodule of *M*. *N* is called essential in *M* (briefly $N \leq_e M$) if, for any $X \leq M, N \cap X = 0$ implies *X=0*. *N* is called a closed submodule of *M* (briefly $N \leq^c M$) if, *N* has no proper essential extensions inside *M*; that is the only solution of the relation $N \leq_e K \leq M$ is $K = N$. In this case the submodule *K* is called closure of *N* [8]. A module *M* is said to be extending(or C_1-module) if, for every submodule *N* of *M* there exists a direct summand *K* of *M* such that $N \leq_e K$ .Equivalently, every closed submodule of *M* is a direct summand [11].An *R*-module *M* is said to have the summand intersection property (briefly SIP) if, the intersection of two summands is again a summand [5]. Dually, an *R*-module *M* is said to have the summand sum property (briefly SSP) if, the sum of two summands is again a summand [2]. This paper is structured in two sections: in the first section we introduce some general properties of modules with CSP. We prove that for an *R*-module *M*, if *M* has the CSP then *M* has the SSP. Also, we show that, if *M* has the CSP then for all *R*-homomorphism $f : S \to T$ where $S, T \le M$ and $S \oplus T \le^c M$ then Imf is closed in *M*, while in the second section we investigate some relationships between the concept of modules with CSP and other modules such as, modules with CIP and SIP.

1. Some basic properties of modules with the closed sum property.

In this section we introduce the concept of modules with CSP as a dual of modules with CIP. We investigate the basic properties of this type of modules. Before, we presented the following example.

Example 1.1. Let $M = Z_8 \oplus Z_2$, $R = Z$, and let $A = (\bar{2},\bar{1})Z$, $B = (0) \oplus Z_2$ be two submodules of *M*. Notice that *A* is closed not direct summand of *M*. Then $A + B = (\bar{2},\bar{1})Z + (0) \oplus Z_2 =$ $(\bar{2}) \oplus Z_2 \le_e Z_8 \oplus Z_2 = M$. Hence $A + B$ is not closed submodule of *M* as *Z*-module.

This leads us to introduce the following.

Definition 1.2. An *R*-module *M* is said to have the closed sum property (briefly CSP) if, the sum of any two closed submodules of *M* is again closed.

Proposition 1.3. Let *M* be an *R*-module. Then *M* has the CSP if and only if every closed submodule of *M* has the CSP.

Proof. Assume that *M* has the CSP as *R*-module and $N \le^c M$. Let A, B be two closed submodules of *N*, so A, B are closed in *M* and hence $A + B$ is closed in *M*. But $A + B \le N \le M$ implies $A + B$ is closed in *N*. Thus *N* has the CSP. Conversely, follows by taking $N = M$. □

Corollary 1.4. Let *M* be an *R*-module. Then *M* has the CSP if and only if every direct summand of *M* has the CSP.

An injective *R*-module $E(M)$ is called injective hull of a module *M* if there exists a monomorphism $\psi : M \to E(M)$ such that $\mathrm{Im}\psi \le_e E(M)$ [4] .

In the following proposition we put certain condition under which $E(M)$ has CSP, where *M* is a module with CSP. Before, we consider the following condition for an *R*-module *M*.

If $Y_1 \cap M = Y_2 \cap M$, where Y_1, Y_2 are submodules of $E(M)$ implies $Y_1 = Y_2$ …. (*)

Proposition 1.5. Let *M* be an *R*-module and let $E(M)$ be a distributive R-module. If $E(M)$ has the CSP then *M* is, too. The converse is hold, whenever $E(M)$ satisfies (∗).

Proof. Assume that A, B are closed submodules of *M*, so by [4] $A = X_1 \cap M$ and $B = X_2 \cap M$ for some direct summands X_1, X_2 of $E(M)$. Since $E(M)$ is a distributive *R*-module, then $A + B = (X_1 \cap M) + (X_2 \cap M) = (X_1 + X_2) \cap M$. But $E(M)$ has CSP , so we have $X_1 + X_2 \leq^c E(M)$ and since $E(M)$ is injective, $X_1 + X_2 \leq^{\oplus} E(M)$. Again by [4], we get $A + B$ is closed in M. Conversely, let Y_1, Y_2 be two closed submodules of $E(M)$, then $Y_1, Y_2 \leq^{\oplus} E(M)$, thus $Y_1 \cap M$ and $Y_2 \cap M$ are closed in *M*. Since *M* has CSP and $E(M)$ is a distributive, $(Y_1 + Y_2) \cap M = (Y_1 \cap M) + (Y_2 \cap M)$ is closed in *M*, so there exists a direct summand *Y* of $E(M)$ such that $Y \cap M = (Y_1 + Y_2) \cap M$ and hence $Y = (Y_1 + Y_2)$, by (∗). Thus $(Y_1 + Y_2) \leq^{\oplus} E(M)$ and so $(Y_1 + Y_2) \leq^c E(M)$. □

Definition 1.6. We say that a module M has C'_3 whenever L_1 and L_2 are closed submodules of *M* with $L_1 \cap L_2 = 0$, then the submodule $L_1 + L_2$ is also a closed submodule of *M*. Clearly, *M* has the CSP implies *M* has C'_3 .

The following lemma was appeared in [1, Th.1.4.1.], we give the details of the proof.

Lemma 1.7. Let *M* be an *R*-module. Then *M* is a quasi-continuous module if and only if *M* has the C'_3.

Proof. Let *A* and *B* are closed submodules of *M* with $A \cap B = 0$. Since *M* is an extending module, then *A* , *B* are direct summands of *M* and hence $A \oplus B \leq^{\oplus} M$, since *M* has C_3. Thus $A + B \leq^C M$. Therefore *M* has the C'_3 . Conversely, let $L \leq^c M$. Let *K* be a complement of *L*, thus *K*. But $L \cap K = 0$, so $L \oplus K \leq^c M$, by C'_3. On the other hand, $L \oplus K \leq_e M$ and hence $L \oplus K = M$. Thus $L \leq^{\oplus} M$ and so *M* satisfies C_1. Let L_1 and L_2 are direct summands of *M* with $L_1 \cap L_2 = 0$ then L_1 and L_2 are closed in *M*, but *M* has the C'_3 then $L_1 \oplus L_2 \leq^C M$ and hence $L_1 \oplus L_2 \leq^{\oplus} M$, thus *M* has the C_3 . □

Proposition 1.8. If *R*-module *M* has the CSP then *M* has the SSP.

Proof. Let *A* , *B* be two direct summands of *M*, then *A* , *B* are closed in *M*. But *M* has the *CSP*, so $A + B \leq^c M$. On the other hand, M has the C'_3 , so by previous lemma *M* is quasi-continuous, thus *M* is an extending *R*-module, $A + B \leq^{\oplus} M$ and hence *M* has the SSP. □

Corollary 1.9. Let *M* be an *R*-module has the CSP. For any decomposition $M = K_1 \oplus K_2$ and every homomorphism $\varphi : K_1 \to K_2$, $\operatorname{Im}\varphi$ is closed in *M*.

Proof. By previous proposition M has SSP. But $M = K_1 \oplus K_2$ and $\varphi : K_1 \to K_2$ is homomorphism, so by [7] $\operatorname{Im}\varphi \leq^{\oplus} M$. Thus $\operatorname{Im}\varphi$ is closed in M. □

Proposition 1.10. Let M be an R-module. If M has the CSP then for all $f : S \to T$ where S, T are submodules of M and $S \oplus T \leq^{c} M$ implies $\operatorname{Im} f$ is closed in M.

Proof. Let $M' = S \oplus T$. Since M has the *CSP* and $M' \leq^{c} M$, so by prop. 1.3 M' has the CSP. Now, $f : S \to T$ is homomorphism and $M' = S \oplus T$ has the CSP, then by cor.1.9 $\operatorname{Im} f \leq^{c} M'$ and hence $\operatorname{Im} f \leq^{c} M$. □

Proposition 1.11. Let M be an extending module. If M has the SSP then M has the CSP.

Proof. Suppose A and B are closed submodules of M. Since M is extending then A and B are direct summands of M, but M has the SSP, so $A + B \leq^{\oplus} M$ and hence $A + B \leq^{c} M$. □

Corollary 1.12. Let M be an injective (Or quasi-injective) R-module. Then M has the SSP if and only if M has the CSP.

Alkan and Harmanci in [9] consider the following condition for a module M.

If $M_1 \leq^{\oplus} M$, $M_2 \leq^{\oplus} M$ with $M_1 + M_2 \leq_e M$ then $M_1 + M_2 = M$ … $(\ast)$

Now we present the following condition.

If $M_1 \leq^{c} M$, $M_2 \leq^{c} M$ with $M_1 + M_2 \leq_e M$ then $M_1 + M_2 = M$ … $(\ast)_c$

The following gives a characterization for a module M with CSP.

Proposition 1.13. Let M be an R-module. Then M has the CSP if and only if every closed submodule of M satisfies $(\ast)_c$.

Proof. Let $N \leq^{c} M$. Assume that A and B are closed submodules of N with $A + B \leq_e N$, then A and B are closed in M and hence $A + B \leq^{c} M$. But $A + B \leq N$ implies $A + B \leq^{c} N$. Since $A + B \leq_e N$, so $A + B = N$. Thus N satisfies $(\ast)_c$. Conversely, let L_1 and L_2 are closed submodules of M. Suppose $L_1 + L_2 \leq_e K \leq^{c} M$, so we have L_1 and L_2 are closed in K, thus by hypothesis $L_1 + L_2 = K$, so $L_1 + L_2$ is closed in M and hence M has the CSP. □

Corollary 1.14. If M is an R-module has the CSP then M satisfies $(\ast)_c$.

Proof. Since M is closed, then the result it follows by above proposition. □

Corollary 1.15. Let M be an extending module. Then the following statements are equivalent.

(1) M has the SSP.

(2) M has the CSP.
(3) M satisfies（⁑）.
(4) M satisfies（⁑）$_c$.

Proof. (1) is equivalent to (2), by prop 1.8 and prop 1.11. (1) is equivalent to (3), by [9 , prop 14]. Obviously, (3) and (4) are coincides. □

Proposition 1.16. Let M be an R-module and let N be a closed submodule of M. If M has the CSP then M/N has the CSP.

Proof. Suppose A and B are closed submodules of M/N, then $A = L_1/N$, $B = L_2/N$ for some L_1, L_2 closed submodules in M. But M has the CSP, $L_1 + L_2 \leq^c M$ and hence $A + B = L_1/N + L_2/N = L_1 + L_2/N \leq^c M/N$. Thus M/N has the CSP. □

The converse of prop. 1.16 need not be true in general; as example: consider $M = Z_8 \oplus Z_2$ as Z-module and $N = Z_8 \oplus (0)$ is closed in $M = Z_8 \oplus Z_2$. Then $M/N = Z_8 \oplus Z_2/Z_8 \oplus (0) \cong Z_2$ which satisfies CSP. But it is well-known that $Z_8 \oplus Z_2$ does not be have the CSP as Z-module (see Ex 1.1).

Remark 1.17. Let M be an R-module. For any closed submodule K of M, if M/K has the CSP then M has the CSP.

Proof. By taking $K = (0)$. □

Remark 1.18. Clearly, the Z-modules Z_4, Z_2 are modules with the CSP, but $Z_4 \oplus Z_2$ does not be have the CSP as Z-module, to see this: Define $\phi : Z_2 \to Z_4$ by $\phi(\bar{x}) = 2\bar{x}$, for all $\bar{x} \in Z_2$, it is not hard to prove that ϕ is well-defined and homomorphism, then $\mathrm{Im}\phi = (\bar{2}) \leq_e Z_4$ and hence $\mathrm{Im}\phi$ is not closed in Z_4 as Z-module. By applying cor. 1.9 $Z_4 \oplus Z_2$ as Z-module does not be have the CSP. This example shows the direct sum of two modules with the CSP need not be have the CSP.

The next proposition giving condition under which the direct sum of modules with the CSP has the CSP, too.

Proposition 1.19. Let M_1 and M_2 be two R-modules having the CSP such that $ann_R M_1 + ann_R M_2 = R$ then $M_1 \oplus M_2$ has the CSP.

Proof. Let A and B be two closed submodules of $M_1 \oplus M_2$. Since $ann_R M_1 + ann_R M_2 = R$, then by the same way of the proof of [10, prop 4.2.] $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where A_1, B_1 are submodules of M_1, A_2, B_2 are submodules of M_2. By some properties of closed submodules, we get A_1, B_1 are closed in *M₁* and A_2, B_2 are closed in M_2. Since both of M_1 and M_2 is module with the CSP, so we have $A_1 + B_1 \leq^c M_1$, $A_2 + B_2 \leq^c M_2$.Thus $A + B = (A_1 \oplus A_2) + (B_1 \oplus B_2) = (A_1 + B_1) \oplus (A_2 + B_2)$ is closed in $M_1 \oplus M_2$, by [8] . □

Proposition 1.20. Let *M* be an *R*-module, $\overline{R} = R/ann_R M$. Then *M* has the CSP as *R*-module if and only if *M* has the CSP as $\overline{R}-$module .

Proof. Obviously. □

Lemma 1.21. Let *M* be an *R*-module and let *S* be a multiplicative closed subset of *R*. If $N \leq M$, then *N* is closed in *M* as *R*-module if and only if $S^{-1}N$ is closed in $S^{-1}M$ as $S^{-1}R-$module, provided $S^{-1}A = S^{-1}B$ *iff* $A = B$.

Proof. Let *N* be a closed submodule of *M* as *R*-module and assume $S^{-1}N \leq_e S^{-1}K \leq S^{-1}M$. Suppose that *N* is not essential in *K* of *M*, so there exists $(U \neq 0) \leq K$ such that $N \cap U = 0$, then $S^{-1}N \cap S^{-1}U = S^{-1}(N \cap U) = S^{-1}(0)$ where $S^{-1}U \leq S^{-1}K$, thus $S^{-1}U = S^{-1}(0)$ and so $U = 0$ which is a contradiction. Thus *N* is essential in *K* of *M* and hence $N = K$, also $S^{-1}N = S^{-1}K$. Therefore $S^{-1}N$ is closed in $S^{-1}M$ as $S^{-1}R-$module . Conversely, suppose $S^{-1}N \leq^c S^{-1}M$. Let $N \leq_e L \leq M$, we claim that $S^{-1}N \leq_e S^{-1}L \leq S^{-1}M$. For any $W \leq S^{-1}L$, $W = S^{-1}B$ for some $B \leq L \leq M$. If $S^{-1}N \cap W = S^{-1}N \cap S^{-1}B = S^{-1}(N \cap B) = S^{-1}(0)$, then by assumption $N \cap B = 0$, so $B = 0$ and hence $W = S^{-1}(0)$ implies $S^{-1}N = S^{-1}L$, thus $N = L$. This proving that *N* is closed in *M* as *R*-module. □

We investigate the behavior of module with the CSP under localization.

Proposition 1.22. Let *M* be an *R*-module and let *S* be a multiplicative closed subset of *R*. Then *M* has the CSP as *R*-module if and only if $S^{-1}M$ has the CSP as $S^{-1}R-$module, provided $S^{-1}A = S^{-1}B$ *iff* $A = B$.

Proof. It follows directly by above lemma. □

We end this section by the following corollary.

Corollary 1.23. Let *M* be an *R*-module. Then *M* has the CSP as *R*-module if and only if M_P has the CSP as R_P- module, for every maximal ideal *P* of *R*.

2. Modules with the CSP and CIP, and related concepts.

A module *M* is said to have strongly summand intersection property (briefly SSIP) if the intersection of any number of summands of *M* is again a direct summand of *M*. A module *M* is said to have strongly summand sum property (briefly SSSP) if the sum of any number of summands of M is again a direct summand of M. (D_3) If L_1 and L_2 are summands of M with $L_1 + L_2 = M$, then $L_1 \cap L_2 \leq^{\oplus} M$. We introduce the following condition (D'_3) If K_1 and K_2 are closed submodules of M with $K_1 + K_2 = M$, then $K_1 \cap K_2 \leq^c M$. In this section we show that a module M has the CIP whenever M has the CSP and the D'_3. Also, we prove that the concepts of quasi-continuous and the CSP are coincides whenever a module has the CIP. Moreover, many properties related with CSP and other modules are given in this section.

We begin with the following remark.

Remark 2.1. Every semisimple (Or simple) R-module has the CSP. Conversely, need not be true in general, as example; the Z-module Z has the CSP but it is not semisimple nor simple.

Lemma 2.2. Let M be a module has the CSP. If M has D'_3 and $L \leq^{\oplus} M$, then L has D'_3.

Proof. Let $L_1 \leq^c L$, $L_2 \leq^c L$ such that $L_1 + L_2 = L$. Since $L \leq^{\oplus} M$, so $L \leq^c M$ implies L_1 and L_2 are closed submodules of M. But $L \oplus K = M$ for some $K \leq M$, then $L_1 + (L_2 \oplus K) = (L_1 + L_2) \oplus K = M$. Since M has *CSP* then $L_2 + K \leq^c M$ and so by D'_3 on M, $L_1 \cap L_2 = L_1 \cap (L_2 \oplus K) \leq^c M$, but $L_1 \cap L_2 \leq L$ thus $L_1 \cap L_2 \leq^c L$. □

Corollary 2.3. Let M be a module has the CSP. Then M has D'_3 if and only if every direct summand of M has the D'_3.

Proposition 2.4. Suppose that every closed submodule of a module M has D'_3. If M has the CSP then M has the CIP.

Proof. Assume that A and B are two closed submodules of M. Since $A, B \leq P = A + B$ then both it are closed in P. By D'_3 on P, $A \cap B \leq^c P$ but $P \leq^c M$ implies $A \cap B \leq^c M$. □

Proposition 2.5. Let M be an R-module has D_3. If M has the CSP then M has the CIP.

Proof. By prop. 1.8 M has the SSP, but M has D_3, so by [9, lemma 19] M has the SIP. On the other hand, the CSP implies C_1, thus by [2] M has the CIP. □

Proposition 2.6. Let M be an R-module has the CSP. Then the following statements are equivalent.

(1) M has the SSIP.
(2) M has the SIP.
(3) M has the CIP.
(4) $E(M)$ has the SSIP.
(5) $E(M)$ has the SIP.
(6) $E(M)$ has the CIP.

Proof. $(1)\Leftrightarrow(2)\Leftrightarrow(4)\Leftrightarrow(5)$, since *M* has the CSP then *M* has C'_3 and hence by lemma 1.7 *M* is quasi-continuous, so by [9, prop 18] we get the result. $(2)\Leftrightarrow(3)$, since *M* has the CSP, so *M* has C_1 and hence the SIP and the CIP are coincides, by [6, prop 2.1]. $(5)\Leftrightarrow(6)$, since *E(M)* is injective, then the result is obtain by [6, cor. 2.4]. □

Proposition 2.7. Let *M* be an *R*-module has the CIP. Then *M* is a quasi-continuous module if and only if *M* has the CSP.

Proof. Since *M* is quasi-continuous then *M* has C_1, but *M* has the CIP, so by [6, prop 2.1] *M* has the *SIP*. On the other hand, *M* has C_3, thus by [9, lemma 19] *M* has the SSP and hence by extendingly, *M* has the CSP. Conversely, since *M* has CSP then *M* has C'_3, this means *M* is quasi-continuous. □

Proposition 2.8. Let *M* be an *R*-module. If the intersection of any two closed submodules of *M* is injective, then *M* has the CSP and hence the SSP.

Proof. In first, to prove that *M* has the CSP. Let *A* and *B* be two closed submodules of *M*, so by assumption $A\cap B$ is injective and hence $A\cap B$ is summand of *M*, so $M=(A\cap B)\oplus K$ for some $K\leq M$. By modular law, $A=[(A\cap B)\oplus K]\cap A=(A\cap B)\oplus(K\cap A)$. Similarly, $B=(A\cap B)\oplus(K\cap B)$, since both of *A* , *B* and *K* is closed in *M* then $K\cap A$ and $K\cap B$ are injective in *M*, so $A+B=(A\cap B)\oplus(K\cap A)\oplus(K\cap B)$ is injective, thus $A+B\leq^{\oplus} M$ and hence $A+B$ is closed submodule of *M*. Thus *M* has the CSP, also the SSP. □

Now, we introduce the following new definition.

Definition 2.9. An *R*-module *M* is called a closed simple *R*-module if the trivial submodules are only closed in *M*.

Clearly, every closed simple module has the CSP and hence every uniform module has also the CSP, but the converse is not always true , such example; Z_6 as *Z*-module has the CSP but not closed simple.

Proposition 2.10. Let $M=M_1\oplus M_2$ with M_1 is a simple *R*-module and M_2 is closed simple (not simple) *R*-module. If $Hom_R(M_1,M_2)\neq 0$ then *M* does not be have the CSP.

Proof. Let $\phi\in Hom_R(M_1,M_2)$, $\phi\neq 0$. Suppose *M* has the CSP, we claim that $\mathrm{Im}\phi\neq M_2$. If we assume $\mathrm{Im}\phi=M_2$, then ($\phi\neq 0$) is an epimorphism, but M_1 is simple implies $Ker\phi=0$ and hence ϕ is an isomorphism, that is $M_1\cong M_2$, and so M_2 is simple which is a contradiction. On the other hand, *M* has the CSP implies $\mathrm{Im}\phi\leq^c M_2$ and hence $\mathrm{Im}\phi=0$, so $\phi=0$, this is a contradiction. □

Proposition 2.11. Let M_1 be an *R*-module and let M_2 be a closed simple *R*-module. If $M_1\oplus M_2$ has the CSP then either :

(1) $Hom_R(M_1,M_2)=0$, or
(2) for all $\phi\in Hom_R(M_1,M_2)$, $\phi\neq 0$ implies ϕ is an epimorphism.

Proof. Assume that $Hom_R(M_1,M_2)\neq 0$. Let $\phi\in Hom_R(M_1,M_2)$, $\phi\neq 0$, then by cor. 1.9, $\mathrm{Im}\phi$ is closed in M_2, but M_2 is closed simple and $\phi\neq 0$ imply $\mathrm{Im}\phi=M_2$ and hence ϕ is an epimorphism. □

Recall that an R-module M is a coquasi-Dedekind Module if, for any $f\in End_R(M)$, $f\neq 0$ implies f is an epimorphism [12].

Proposition 2.12. Let M_1 be an R-module and let M_2 be a closed simple R-module such that $Hom_R(M_1,M_2)\neq 0$. If $M_1\oplus M_2$ has the CSP then M_2 is a coquasi-Dedekind Module.

Proof. Assume that $Hom_R(M_1,M_2)\neq 0$, so by prop 2.10 R-homomorphism $\phi: M_1\to M_2$ and $\phi\neq 0$ implies $\mathrm{Im}\phi=M_2$. Suppose that M_2 is not a coquasi-Dedekind Module, so there exists $\psi\in End_R(M)$, $\psi\neq 0$ with $\mathrm{Im}\psi\neq M_2$. But $\mathrm{Im}(\psi\circ\phi)=\psi(\phi(M_1))=\psi(M_2)=\mathrm{Im}\psi\neq M_2$, so $(\psi\circ\phi\neq 0)\in Hom_R(M_1,M_2)$ and $\mathrm{Im}(\psi\circ\phi)\neq M_2$, that is $\psi\circ\phi$ not an epimorphism, this is a contradiction with prop. 2.11. □

Corollary 2.13. Let M be a closed simple R-module and $M^*=M\oplus M$. If M^* has the CSP then M is a coquasi-Dedekind Module.

Proof. Obviously. □

Corollary 2.14. Let M be a closed simple R-module and $M^*=M\oplus M$. If M^* has both the CIP and the CSP then $S=End_R(M)$ is a division ring.

Proof. It follows by [6, cor. 2.25] and cor. 2.13. □

Proposition 2.15. Let M_i be a closed simple R-module, for $i=1,2$. If M_2 is a projective and $M_1\oplus M_2$ has the CSP then either $Hom_R(M_1,M_2)=0$ or $M_1\cong M_2$.

Proof. By prop. 2.11. we have, either $Hom_R(M_1,M_2)=0$ or ϕ is an isomorphism, for all $\phi\in Hom_R(M_1,M_2)$ and $\phi\neq 0$. Since M_2 is projective then $Ker\phi$ is a direct summand of M_1 and hence $Ker\phi\leq^c M_1$. But M_1 is closed simple and $\phi\neq 0$, then $Ker\phi=0$, so ϕ is a monomorphism and hence $M_1\cong M_2$. □

A ring R is called hereditary if every factor module of every injective module is also injective. Lam T.Y. in [13, Ex.10] giving the following result :

Proposition 2.16. A ring *R* is hereditary if and only if the sum of two injective submodules of any *R*-module is injective.

Valcan D. and Napoca C. in [2] presented theorem 2.11, we shall prove this theorem by another proof with equivalence third as following.

Theorem 2.17. The following statements are equivalent for a ring *R*.

(1) *R* is hereditary.
(2) All injective *R*-modules have the SSP.
(3) All injective *R*-modules have the CSP.

Proof. $(1)\Rightarrow(2)$, let *M* be an injective *R*-module. Assume that *A* and *B* are two direct summands of *M*, so by prop. 5.16, $A+B$ is injective in *M* and hence $A+B\leq^{\oplus} M$ .Thus *M* has the *SSP*.
$(2)\Rightarrow(1)$, let L_1, L_2 be an injective submodules of an *R*-module *M*, then L_1, L_2 are direct summands of *M*. Thus, $L_1\oplus K_1=M$ and $L_2\oplus K_1=M$ for some K_1, K_2 submodules of *M*, this is implies $E(L_1)\oplus E(K_1)=E(M)$, $E(L_2)\oplus E(K_2)=E(M)$. But $E(L_1)=L_1$ and $E(L_2)=L_2$, this implies L_1, L_2 are direct summands of $E(M)$,and hence by SSP on $E(M)$, we get $L_1+L_2\leq^{\oplus} E(M)$, thus L_1+L_2 is an injective submodules of $E(M)$, this implies that L_1+L_2 is an injective submodules of *M*. Therefore by above proposition *R* is hereditary.
$(2)\Leftrightarrow(3)$, it follows by cor. 1.12. □

Proposition 2.18. Let *M* be a faithful and closed simple module over integral domain *R*. If $M\oplus M$ has the CSP then *M* is divisible.

Proof. Let $(r\neq 0)\in R$, define *R*-homomorphism $\psi : M\to M$ by $\psi(m)=rm$, for all $m\in M$. By cor. 1.9, $\mathrm{Im}\psi = rM$ is closed in *M*. Since *M* is faithful, then $\mathrm{Im}\psi = rM\neq 0$ but *M* is closed simple, so $rM=\mathrm{Im}\psi = M$ and hence *M* is divisible. □

Corollary 2.19. Let *M* be a torsion free and closed simple module over integral domain *R*. If $M\oplus M$ has the CSP then *M* is injective.

Proof. It follows by above proposition and [3, prop 2.7]. □
We shall consider the following definition.

Definition 2.20. Let *M* and *N* be an *R*-modules. *M* is called closed dual-Rickart relative to *N* (briefly c-d-Rickart) if, for any $\phi\in Hom_R(M,N)$, $\mathrm{Im}\phi\leq^{c} N$. In specially , *M* is called c-d-Rickart if, for any $\phi\in End_R(M)$, $\mathrm{Im}\phi\leq^{c} M$.

Theorem 2.21. Let *M* be an *R*-module with CSP. If $M^{*}=M_1\oplus M_2$ is closed submodule of *M* then M_1 is c-d-Rickart relative to M_2.

Proof. Assume $\phi \in Hom_R(M_1, M_2)$. Let $W = \{m + \phi(m) : m \in M_1\}$, it is not hard to prove that W is a direct summand of M^*, so W is closed in M^* and hence W is a closed submodule of M. But $M_1 \oplus \phi(M_1) = M_1 + W \leq^c M$ implies $\phi(M_1) \leq^c M$. Since $\phi(M_1) \leq M_2 \leq M$, hence $\mathrm{Im}\phi$ is closed in M_2. □

Corollary 2.22. Let M be an R-module. If $M^* = M \oplus M$ has the CSP then M is c-d-Rickart.

Proof. Since M^* has the CSP and $M \leq^c M^*$, so that M has the CSP. Since $M^* \leq^c M^*$, thus by last theorem M is c-d-Rickart relative to M; that is M is c-d-Rickart. □

Before to introduce the following theorem, we need the following lemma.

Lemma 2.23. Let M and N be an R-modules. Then M is c-d-Rickart relative to N if and only if for all $M' \leq^{\oplus} M$ and for any $N' \leq N$, M' is c-d-Rickart relative to N'.

Proof. Suppose that M is c-d-Rickart relative to N. Let $\psi : M' \to N'$ be an R-homomorphism. Since $M = M' \oplus T$ for some $T \leq M$, we consider $M' \oplus T = M \xrightarrow{\rho} M' \xrightarrow{\psi} N' \xrightarrow{i} N$ then $i \circ \psi \circ \rho(M) \leq^c N$ implies $\mathrm{Im}\psi \leq^c N$, but $\mathrm{Im}\psi \leq N'$, thus $\mathrm{Im}\psi \leq^c N'$. Conversely, it is clear. □

Theorem 2.24. Let $\{M_i\}_{i\in I}$ be a class of R-modules and N be an R-module. Then N has the CSP implies $\bigoplus_{i\in I} M_i$ is c-d-Rickart relative to N if and only if M_i is c-d-Rickart relative to N, where I is a finite set.

Proof. $\Rightarrow$) It follows directly by previous lemma.
$\Leftarrow$) Suppose M_i is c-d-Rickart relative to N, for all $i \in I(\mathit{finite})$. Let $\phi \in Hom_R(\bigoplus_{i\in\Lambda} M_i, N)$, $\phi = \{\phi_i\}_{i\in I}$ and $\phi_i = \phi|_{M_i}$, so $\mathrm{Im}\phi = \sum_{i\in I} \mathrm{Im}\phi_i$, I is finite. But $\mathrm{Im}\phi_i \leq^c N$ and N has the *CSP*, thus $\sum_{i\in I} \mathrm{Im}\phi_i \leq^c N$ and hence $\mathrm{Im}\phi \leq^c N$. □

Consider the following definition.

Definition 2.25. An R-module M is said to have strongly closed sum property (briefly SCSP) if, the sum of any number of closed submodules of M is again closed in M.

Proposition 2.26. Let R be an extending ring. Then the following statements are equivalent.

(1) R is semisimple.
(2) All R-modules has the SCSP.
(3) All R-modules has the CSP.
(4) All projective R-modules has the CSP.

Proof. It is obvious that, 1) implies 2) implies 3) implies 4). Assume that (4) holds. Let K be a submodule of R_R. Choose a free R-module F and an epimorphism $\rho : F \to K$. Since $F \oplus R$ is a projective R-module then by assumption $F \oplus R$ has the CSP. Consider the injection map $i : K \to R$ and $g = i \circ \rho : F \to R$ is R-homomorphism, so by cor. 1.9 $\operatorname{Im} g \leq^c R_R$, but $\operatorname{Im} g = \operatorname{Im}(i \circ \rho) = (i \circ \rho)(F) = \rho(F) = K$, thus $K \leq^c R_R$ and hence by extendingly of R, $K \leq^{\oplus} R_R$. Therefore R is semisimple. □

Corollary 2.27. The following statements are equivalent for an extending ring R.

(1) R is semisimple.
(2) All R-modules has the SCSP.
(3) All R-modules has the CSP.
(4) All projective R-modules has the CSP.
(5) All R-modules has the CIP.
(6) All injective R-modules has the CIP.
(7) All injective R-modules has the SIP.
(8) All injective R-modules is semisimple.
(9) All quasi-injective R-modules has the CIP.
(10) All quasi-injective R-modules has the SIP.
(11) All quasi-injective R-modules is semisimple.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$, it follows by above proposition.
$(1) \Leftrightarrow (5) \Leftrightarrow (6)$, it follows by [6, prop 2.6].
$(1) \Leftrightarrow (6) \Leftrightarrow (8) \Leftrightarrow (9) \Leftrightarrow (11)$, it follows by [6, cor. 2.7].
$(6) \Leftrightarrow (7)$ and $(9) \Leftrightarrow (10)$, it follows by [6, cor. 2.4]. □

Proposition 2.28. Let M be an extending R-module, $S = End_R(M)$. If S has the CSP then M has the CSP and the CIP.

Proof. Since $S = End_R(M)$ has the CSP, so by prop. 1.8 S has the SSP. Thus by [7 , Th 2.3] M has both the SSP and the SIP. But M is extending, thus by prop 1.11, M has the CSP and the CIP. □

Corollary 2.29. Let M be an R-module and let $S = End_R(M)$ has C_1. If M has both the SSP and the SIP then S has the CSP.

Proof. By [7, Th. 2.3], $S = End_R(M)$ has the SSP, but S has C_1, so by prop 1.11 S has the CSP. □

We finish this the paper by the following corollary.

Corollary 2.30. Let M be an R-module has D_3 and let $S = End_R(M)$ has C_1. Then S has the CSP if and only if M has the SSP.

Proof. $\Rightarrow$), it follows by some the way of proof of prop. 2.28.
$\Leftarrow$), let M has the SSP. Since M has D_3, then by [3, lemma 19] M has the SIP but $S = End_R(M)$ has C_1, thus by above corollary the result is obtained. □

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