International Mathematical Forum, Vol. 9, 2014, no. 30, 1441 - 1446
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.48152

Explicit Polynomial for Sums of Powers of Odd Integers

José Luis Cereceda

Distrito Telefónica, Edificio Este 1
28050 – Madrid, Spain

Abstract

In this note we derive the explicit, non-recursive form of the coefficients of the polynomial associated with the sums of powers of the first n odd integers. As a by-product, we deduce a couple of new identities involving the Bernoulli numbers.

Mathematics Subject Classification: 11B57, 11C08, 11B68

Keywords: sums of powers, odd integers, polynomial representation, Bernoulli numbers

1 Introduction

In this note we deal with the sums of powers of the first n odd integers

$$O_r(n) = 1^r + 3^r + 5^r + \cdots + (2n-1)^r,$$

where $r = 1, 2, 3, \ldots$. As we can learn from Theorem V in a paper by Witmer [4], it turns out that, for odd $r = 2p - 1$ $(p = 1, 2, 3, \ldots)$, $O_{2p-1}(n)$ can be expressed as the even polynomial in n:

$$O_{2p-1}(n) = \sum_{k=1}^{p} E_{p,k} n^{2k}, \tag{1}$$

where the coefficients $E_{p,k}$ are rational numbers independent of n that satisfy $E_{p,p} = 2^{2p-2}/p$, and the recursion formula

$$E_{p,k} = -\frac{1}{2p}\sum_{j=1}^{p-1}\binom{2p}{2j-1}E_{j,k}, \quad k < p.$$

Note that, for the case $p = 1$, from (1) we retrieve the well-known identity

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

On the other hand, from Theorem VI in [4], it follows that, for even $r = 2p$ $(p = 1, 2, 3, \dots)$, $O_{2p}(n)$ can be expressed as the odd polynomial in n:

$$O_{2p}(n) = \sum_{k=0}^{p} F_{p,k} n^{2k+1}, \tag{2}$$

where the coefficients $F_{p,k}$ are rational numbers independent of n that satisfy $F_{p,p} = 2^{2p}/(2p+1)$, and the recursion formula

$$F_{p,k} = -\frac{1}{2p+1}\sum_{j=0}^{p-1}\binom{2p+1}{2j}F_{j,k}, \quad k < p.$$

Moreover, Guo and Shen [2] gave recently the following factorized forms for $O_{2p-1}(n)$ and $O_{2p}(n)$:

$$O_{2p-1}(n) = n^2 \sum_{k=1}^{p} C_{p,k} n^{2p-2k}, \tag{3}$$

$$O_{2p}(n) = n(2n-1)(2n+1)\sum_{k=1}^{p} D_{p,k}(2n-1)^{p-k}(2n+1)^{p-k}, \tag{4}$$

together with the corresponding recurrence formulae to compute the rational coefficients $C_{p,k}$ and $D_{p,k}$ $(k = 1, 2, \dots, p)$.

However, as far as the present author is aware, there exist no explicit (recurrence-free) formulae for either $E_{p,k}$, $F_{p,k}$, $C_{p,k}$, or $D_{p,k}$ in the current literature. The purpose of this note is to provide an explicit formula for these coefficients.

2 Determination of the coefficients $E_{p,k}$ and $F_{p,k}$

To obtain $E_{p,k}$, we employ, on the one hand, the easily confirmed relation

$$O_p(n) = S_p(2n) - 2^p S_p(n), \tag{5}$$

where $S_p(n) = 1^p + 2^p + 3^p + \cdots + n^p$ is the ordinary power sum of the first n positive integers. On the other hand, we use the well-known Bernoulli formula according to which (e.g. see [3])

$$S_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} \binom{p+1}{i} B_i n^{p+1-i}, \tag{6}$$

where the B_i's are the Bernoulli numbers, namely $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, etc. Thus, using (5) and (6), we have

$$\begin{aligned} O_{2p-1}(n) &= \frac{1}{2p} \sum_{i=0}^{2p-1} \binom{2p}{i} \left(2^{2p-i} - 2^{2p-1}\right) B_i n^{2p-i} \\ &= \frac{1}{2p} \sum_{j=1}^{2p} \binom{2p}{j} \left(2^{j} - 2^{2p-1}\right) B_{2p-j} n^{j} \\ &= \frac{2^{2p-2}}{p} n^{2p} + \frac{1}{2p} \sum_{j=1}^{2p-2} \binom{2p}{j} \left(2^{j} - 2^{2p-1}\right) B_{2p-j} n^{j}. \end{aligned}$$

Since $B_i = 0$ for all odd $i \geq 3$, this can be written as

$$O_{2p-1}(n) = \frac{1}{2p} \sum_{k=1}^{p} \binom{2p}{2k} \left(2^{2k} - 2^{2p-1}\right) B_{2p-2k} n^{2k}.$$

Comparing this expression with (1), we deduce that

$$E_{p,k} = \frac{1}{2p} \binom{2p}{2k} \left(2^{2k} - 2^{2p-1}\right) B_{2p-2k}, \quad k = 1, 2, \ldots, p. \tag{7}$$

Proceeding the same way with $O_{2p}(n)$, we get

$$O_{2p}(n) = \frac{1}{2p+1} \sum_{k=0}^{p} \binom{2p+1}{2k+1} \left(2^{2k+1} - 2^{2p}\right) B_{2p-2k} n^{2k+1},$$

and then, from (2), we deduce that

$$F_{p,k} = \frac{1}{2p+1} \binom{2p+1}{2k+1} \left(2^{2k+1} - 2^{2p}\right) B_{2p-2k}, \quad k = 0, 1, 2, \ldots, p. \tag{8}$$

Let us note the relationships

$$\begin{cases} F_{p,0} = \left(2 - 2^{2p}\right) B_{2p}, \\ F_{p,k} = \dfrac{4p}{2k+1} E_{p,k}, \quad \text{for } k = 1, 2, \ldots, p. \end{cases}$$

Furthermore, since $O_{2p-1}(1) = O_{2p}(1) = 1$, we have that $\sum_{k=1}^{p} E_{p,k} = \sum_{k=0}^{p} F_{p,k} = 1$, from which we obtain the identities

$$\sum_{k=1}^{p} \binom{2p}{2k} \big(2^{2k} - 2^{2p-1}\big) B_{2p-2k} = 2p,$$

and

$$\sum_{k=0}^{p} \binom{2p+1}{2k+1} \big(2^{2k+1} - 2^{2p}\big) B_{2p-2k} = 2p+1.$$

3 Determination of the coefficients $C_{p,k}$ and $D_{p,k}$

In view of equations (3) and (1), it follows that the coefficients $C_{p,k}$ and $E_{p,k}$ are related by $C_{p,k} = E_{p,p+1-k}$ $(k = 1, 2, \ldots, p)$. Thus from (7) we get

$$C_{p,k} = \frac{2^{2p-2}}{p} \binom{2p}{2k-2} \big(2^{3-2k} - 1\big) B_{2k-2}, \quad k = 1, 2, \ldots, p.$$

On the other hand, in order to determine the coefficients $D_{p,k}$, we rely on the following two pieces of information:

1. By Corollary 2.2 in [2], it holds that $D_{p,k} = 2^{2k-1} B_{p,k}$ $(k = 1, 2, \ldots, p)$, where the $B_{p,k}$'s are the (unspecified) coefficients of the polynomial for $S_{2p}(n)$:
$$S_{2p}(n) = 2(2n+1)S_1(n) \sum_{k=1}^{p} B_{p,k} \big(2S_1(n)\big)^{p-k}, \tag{9}$$
where $S_1(n) = n(n+1)/2$.

2. In [1], $S_{2p}(n)$ is expressed as
$$S_{2p}(n) = \big(n + \frac{1}{2}\big) S_1(n) \sum_{j=0}^{p-1} F_j^{(2p)} \big(S_1(n)\big)^j, \tag{10}$$
where[1]
$$F_j^{(2p)} = 8^{j+1} \sum_{m=1}^{p} \binom{2p}{2m} \binom{m}{j+1} \frac{2^{2m+1-2p} - 1}{2^{2m}(2m+1)} B_{2p-2m}. \tag{11}$$

[1]Strictly speaking, in [1] the summation appearing in the formula for $F_j^{(2p)}$ starts at $m = 0$. However, nothing changes if the summation starts at $m = 1$ because the term corresponding to $m = 0$ does vanish for all values of $j = 0, 1, 2, \ldots, p-1$.

Therefore, by equating the polynomials in (9) and (10), it is readily seen that $B_{p,k} = F_{p-k}^{(2p)}/2^{p+2-k}$, and then from (11) we obtain

$$B_{p,k} = \frac{1}{2^{2k-1}} \sum_{m=1}^{p} \binom{2p}{2m} \binom{m}{p+1-k} \frac{2-2^{2p-2m}}{2m+1} B_{2p-2m}.$$

To get the desired coefficients $D_{p,k}$, we simply have to multiply this expression by 2^{2k-1}:

$$D_{p,k} = \sum_{m=1}^{p} \binom{2p}{2m} \binom{m}{p+1-k} \frac{2-2^{2p-2m}}{2m+1} B_{2p-2m}, \quad k = 1, 2, \ldots, p. \qquad (12)$$

From formula (12) we quickly find that $D_{p,1} = 1/(2p+1)$, a result already given in [2].

However, by using (12) we can directly evaluate the factorized form (4) of $O_{2p}(n)$ without resorting to any recurrence formulae. As a simple example, next we give explicitly the polynomial for $O_{14}(n)$ whose coefficients are obtained from (12):

$$\begin{aligned} &1^{14} + 3^{14} + 5^{14} + \cdots + (2n-1)^{14} \\ &= nN \left(\frac{1}{15}N^6 - \frac{28}{15}N^5 + \frac{448}{15}N^4 - \frac{2816}{9}N^3 + \frac{91904}{45}N^2 - 7168N + \frac{28672}{3} \right), \end{aligned}$$

where N is shorthand notation for $(2n-1)(2n+1)$. Note that the obtained coefficients $D_{7,k}$, $k = 1, 2, \ldots, 7$, alternate in sign in accordance with the general rule $\operatorname{sgn} D_{p,k} = (-1)^{k-1}$, $k = 1, 2, \ldots, p$, conjectured in [2].

Further, we observe that $O_{2p}(n)$ can alternatively be expressed as

$$O_{2p}(n) = n \sum_{k=1}^{p} D'_{p,k} N^k,$$

where

$$D'_{p,k} = \sum_{m=1}^{p} \binom{2p}{2m} \binom{m}{k} \frac{2-2^{2p-2m}}{2m+1} B_{2p-2m}, \; k = 1, 2, \ldots, p. \qquad (13)$$

Finally, we note the relationships between the coefficients $F_{p,k}$ of the polynomial (2) and the coefficients $D'_{p,k}$:

$$\begin{cases} F_{p,0} = \displaystyle\sum_{k=1}^{p} (-1)^k D'_{p,k}, \\ F_{p,k} = 2^{2k} \displaystyle\sum_{s=k}^{p} (-1)^{s-k} \binom{s}{k} D'_{p,s}, \quad \text{for } k = 1, 2, \ldots, p. \end{cases}$$

From the above relationships, we can deduce a couple of new identities involving the Bernoulli numbers. Indeed, using (13), and recalling that $F_{p,0} = (2-2^{2p})B_{2p}$, from the first relationship we obtain

$$\sum_{k=1}^{p}(-1)^k \sum_{m=1}^{p} \binom{2p}{2m}\binom{m}{k} \frac{2-2^{2p-2m}}{2m+1} B_{2p-2m} = \left(2-2^{2p}\right)B_{2p}.$$

Likewise, using (8) and (13), from the second relationship we obtain

$$\sum_{s=k}^{p}(-1)^{s-k}\binom{s}{k} \sum_{m=1}^{p} \binom{2p}{2m}\binom{m}{s} \frac{2-2^{2p-2m}}{2m+1} B_{2p-2m} = \frac{1}{2p+1}\binom{2p+1}{2k+1}\left(2-2^{2p-2k}\right)B_{2p-2k},$$

which holds for $k = 1, 2, \ldots, p$.

References

[1] J.L. Cereceda, Explicit form of the Faulhaber polynomials, to appear in *The College Mathematics Journal*.

[2] S. Guo and Y. Shen, On sums of powers of odd integers, *Journal of Mathematical Research with Applications* **33** (2013), 666-672.

[3] H. Sherwood, Sums of power of integers and Bernoulli numbers, *The Mathematical Gazette* **54** (1970), 272-274.

[4] E.E. Witmer, The sums of powers of integers, *The American Mathematical Monthly* **42** (1935), 540-548.

Received: September 3, 2014