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On Tensor Products of Bornological Modules

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Abstract

We present a study of tensor products of bornological modules over bornological commutative rings by means of an elementary approach. We also present some applications to the study of modules of bounded multilinear mappings and modules of bounded homogeneous polynomials.

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1 Introduction

The notion of a bounded subset of a (real or complex) topological vector space was introduced by Kolmogoroff [11] and von Neumann [17]. It played such a fundamental role in functional analysis and its applications that motivated the definition and the study of more general and abstract classes of bounded sets, the so called *bornologies*. A *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X (called *bounded sets*) that contains the singletons and is stable under the passage to subsets and the formation of finite unions. In addition to functional analysis [1, 9, 25], bornologies have shown to be very useful in several contexts, such as topology [13], topological algebra [2, 25], and noncommutative geometry and cyclic homology [16, 24].

In the present paper we are going to work in the setting of modules over a commutative ring. In that setting bornologies were studied mainly by Pombo during the last twenty years, who considered linearly bornologized modules in [19, 20], bornological modules over topological rings in [21], and vectorially bornologized modules in [18]; additionally, (not necessarily abelian) bornological groups have been discussed recently in [22]. Moreover, Bernardes and Pombo studied bornological topological modules in [2, 4, 5]. Our main goal here is to complement these works by presenting a study of tensor products of bornological modules by means of an elementary approach, which avoids the use of the language and of the methods of category theory [14, 23].

We apply the results presented here to the study of modules of bounded multilinear mappings and modules of bounded homogeneous polynomials. In particular, we give a short proof of the main result obtained by Farias and Pombo [8] by means of the tensor product technique. Let us also mention that polynomially bornological topological vector spaces over a very general class of topological fields were studied by Bernardes and Pombo [3].

2 Bornological modules over bornological rings

In the present work we consider the slightly more general notion of a bornological module over a bornological ring. It has the advantage of containing simultaneously the class of bornological modules over topological rings and the class of vectorially bornologized modules that were studied by Pombo in [21] and [18], respectively. For the first class, it is enough to consider the topological ring $(A, \mathcal{T})$ in question endowed with the bornology of its $\mathcal{T}$-bounded subsets and for the second class it is enough to consider the ring A in question endowed with its discrete bornology.

Our main goal in this section is to fix some terminology and to present the basic constructions (products, inverse limits, direct sums, direct limits, etc.) concerning bornological modules over bornological rings. These constructions are derived from the existence of initial and final module bornologies. Although our setting is slightly more general, the proofs of the existence of these bornologies are essentially the same as the ones presented by Pombo [21] in the case of bornological modules over topological rings. Nevertheless, since these proofs are very short, we shall present them here for the sake of completeness.

The basic notions and constructions concerning modules [7, 12] and bornological sets [9, 10] are assumed to be known. Moreover, we usually denote the bornology (resp. the topology) of a bornological set (resp. a topological space) X by $\mathcal{B}_X$ (resp. $\mathcal{T}_X$).

Let A be a ring (all rings are assumed to have a non-zero identity element). A *ring bornology* on A is a bornology $\mathcal{B}$ on A satisfying the following conditions:

(RB1) $B_1 + B_2 \in \mathcal{B}$ whenever $B_1, B_2 \in \mathcal{B}$;

(RB2) $B_1 B_2 \in \mathcal{B}$ whenever $B_1, B_2 \in \mathcal{B}$.

A *bornological ring* is a ring endowed with a ring bornology.

Note that if $\mathcal{B}$ is a ring bornology on a ring A, then $\mathcal{B}$ is also a ring bornology on the opposite ring A^0. We usually consider the opposite of a bornological ring as a bornological ring in this way.

Example 1. (a) The trivial bornology on a ring A is a ring bornology.
(b) The discrete bornology on a ring A is a ring bornology.
(c) Let A be a topological ring. Recall that a subset L of A is said to be left (resp. right) $\mathcal{T}_A$-bounded if for every neighborhood V of 0 in A there is a neighborhood W of 0 in A such that $WL \subset V$ (resp. $LW \subset V$); L is $\mathcal{T}_A$-bounded if it is both left and right $\mathcal{T}_A$-bounded. The collection $\mathcal{B}_\ell(\mathcal{T}_A)$ (resp. $\mathcal{B}_r(\mathcal{T}_A)$, $\mathcal{B}(\mathcal{T}_A)$) of all left $\mathcal{T}_A$-bounded (resp. right $\mathcal{T}_A$-bounded, $\mathcal{T}_A$-bounded) subsets of A is a ring bornology on A. Of course, these three bornologies coincide if A is commutative.

Let A be a bornological ring and let E be a left A-module (all modules are assumed to be unitary). A *left A-module bornology* on E is a bornology $\mathcal{B}$ on E satisfying the following conditions:

(MB1) $B_1 + B_2 \in \mathcal{B}$ whenever $B_1, B_2 \in \mathcal{B}$;

(MB2) $LB \in \mathcal{B}$ whenever $L \in \mathcal{B}_A$ and $B \in \mathcal{B}$.

A *bornological left A-module* is a left A-module endowed with a left A-module bornology.

Analogously we define the concepts of a *right A-module bornology* and a *bornological right A-module*.

If E is a bornological right A-module, then E is a bornological left A^0-module. For this reason, we shall restrict ourselves to left modules and so we shall omit the word "left".

Example 2. (a) For any bornological ring A, the trivial bornology on an A-module E is an A-module bornology.

(b) If A is a ring endowed with the discrete bornology, then the discrete bornology on an A-module E is an A-module bornology.
(c) Every bornological ring A may be regarded as a bornological A-module.
(d) Let X be a bornological set, A a bornological ring and F a bornological A-module. The set $B(X;F)$ of all bounded mappings from X into F is a submodule of the product A-module F^X of all mappings from X into F. Recall that a set $\mathcal{X}$ of mappings from X into F is said to be equibounded if

$$\mathcal{X}(B) := \bigcup_{f \in \mathcal{X}} f(B) \in \mathcal{B}_F \quad \text{whenever } B \in \mathcal{B}_X.$$

The collection of all equibounded subsets of $B(X;F)$ is an A-module bornology, called the *bornology of equiboundedness*.
(e) Let A be a topological ring endowed with the bornology $\mathcal{B}_\ell(\mathcal{T}_A)$. Let E be a topological A-module. Recall that a subset B of E is said to be $\mathcal{T}_E$-bounded if for every neighborhood V of 0 in E there is a neighborhood W of 0 in A such that $WB \subset V$. The collection $\mathcal{B}(\mathcal{T}_E)$ of all $\mathcal{T}_E$-bounded subsets of E is an A-module bornology on E.

Remark 3. Let G be a commutative group denoted additively. A *group bornology* on G is a bornology $\mathcal{B}$ on G such that

$$-B_1 \in \mathcal{B} \quad \text{and} \quad B_1 + B_2 \in \mathcal{B} \quad \text{whenever } B_1, B_2 \in \mathcal{B}.$$

A *bornological commutative group* is a commutative group endowed with a group bornology. By considering $\mathbb{Z}$ endowed with the discrete bornology and by regarding G as a $\mathbb{Z}$-module, a group bornology on G is the same as a $\mathbb{Z}$-module bornology on G. In this way, bornological commutative groups can be viewed as bornological $\mathbb{Z}$-modules.

For the remaining of this section, A denotes a bornological ring.

Theorem 4. *Let E be an A-module, let $(F_\alpha)_{\alpha \in I}$ be a family of bornological A-modules and, for each $\alpha \in I$, let $f_\alpha : E \to F_\alpha$ be an A-linear mapping. Then the initial bornology $\mathcal{B}$ on E for the family $(f_\alpha)_{\alpha \in I}$, which is given by*

$$\mathcal{B} = \{B \subset E; f_\alpha(B) \in \mathcal{B}_{F_\alpha} \text{ for every } \alpha \in I\},$$

is an A-module bornology. Hence, $\mathcal{B}$ is also the initial A-module bornology on E for the family $(f_\alpha)_{\alpha \in I}$ in the sense that if g is an A-linear mapping from a bornological A-module G into E, then g is bounded (E endowed with $\mathcal{B}$) if and only if each of the mappings $f_\alpha \circ g$ is bounded.

Proof. If $B_1, B_2 \in \mathcal{B}$ and $L \in \mathcal{B}_A$, then

$$f_\alpha(B_1 + B_2) = f_\alpha(B_1) + f_\alpha(B_2) \in \mathcal{B}_{F_\alpha} \quad \text{and} \quad f_\alpha(LB_1) = Lf_\alpha(B_1) \in \mathcal{B}_{F_\alpha}$$

for every $\alpha \in I$, which proves that $B_1 + B_2 \in \mathcal{B}$ and $LB_1 \in \mathcal{B}$. □

Example 5. (a) If E is a bornological A-module and M is a submodule of E, then the induced bornology on M is an A-module bornology. Recall that it consists of all bounded sets of E which are contained in M.
(b) If $(E_\alpha)_{\alpha\in I}$ is a family of bornological A-modules and $E = \prod_{\alpha\in I} E_\alpha$ is the product A-module, then the product bornology on E is an A-module bornology. Recall that the sets of the form $\prod_{\alpha\in I} B_\alpha$, with each B_α bounded in E_α, form a base for this bornology.
(c) Let $(E_\alpha, u_{\alpha\beta})_{\alpha\in I}$ be an inverse system of bornological A-modules. This means that I is a non-empty partially ordered set, E_α is a bornological A-module for every $\alpha \in I$, $u_{\alpha\beta} : E_\beta \to E_\alpha$ is a bounded A-linear mapping for every $\alpha, \beta \in I$ with $\alpha \leq \beta$, $u_{\alpha\alpha} = Id_{E_\alpha}$ for every $\alpha \in I$, and $u_{\alpha\beta} \circ u_{\beta\gamma} = u_{\alpha\gamma}$ for every $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta \leq \gamma$. Let $E = \varprojlim E_\alpha$ be the inverse limit A-module and let $u_\alpha : E \to E_\alpha$ be the canonical mapping ($\alpha \in I$). The *inverse limit bornology* on E is the initial (A-module) bornology on E for the family $(u_\alpha)_{\alpha\in I}$. By definition, it consists of all sets $B \subset E$ such that $u_\alpha(B)$ is bounded in E_α for every $\alpha \in I$. By considering E endowed with this bornology, the following universal property holds:

For every bornological A-module F and for every family $(v_\alpha)_{\alpha\in I}$ of bounded A-linear mappings $v_\alpha : F \to E_\alpha$ satisfying $v_\alpha = u_{\alpha\beta} \circ v_\beta$ whenever $\alpha \leq \beta$, there is a unique bounded A-linear mapping $v : F \to \varprojlim E_\alpha$ such that $v_\alpha = u_\alpha \circ v$ for every $\alpha \in I$.

Let us remark that if I is endowed with the equality relation, then $\varprojlim E_\alpha = \prod_{\alpha\in I} E_\alpha$ and the inverse limit bornology coincides with the product bornology.

Unless otherwise specified, whenever we consider a submodule of a bornological module (resp. a product of bornological modules, an inverse limit of an inverse system of bornological modules) as a bornological module, the induced bornology (resp. the product bornology, the inverse limit bornology) is implied.

Theorem 6. *Let E be an A-module, let $(F_\alpha)_{\alpha\in I}$ be a family of bornological A-modules and, for each $\alpha \in I$, let $f_\alpha : F_\alpha \to E$ be an A-linear mapping. Let $\mathcal{S}$ be the collection of all subsets of E of the form*

$$L_1x_1 + \cdots + L_mx_m + f_{\alpha_1}(B_{\alpha_1}) + \cdots + f_{\alpha_n}(B_{\alpha_n}),$$

where $m, n \in \mathbb{N}^*$, $L_1, \ldots, L_m \in \mathcal{B}_A$, $x_1, \ldots, x_m \in E$, $\alpha_1, \ldots, \alpha_n \in I$ and $B_{\alpha_j} \in \mathcal{B}_{F_{\alpha_j}}$ for $1 \leq j \leq n$. Then $\mathcal{S}$ is a base for an A-module bornology $\mathcal{B}$ on E, which is the final A-module bornology on E for the family $(f_\alpha)_{\alpha\in I}$ in the sense that if g is an A-linear mapping from E into a bornological A-module G, then g is bounded (E endowed with $\mathcal{B}$) if and only if each of the mappings $g \circ f_\alpha$ is bounded.

Proof. Obviously, $\mathcal{S}$ is a cover of E. Given $C, D \in \mathcal{S}$, there are $C', D' \in \mathcal{S}$ such that $C \cup \{0\} \subset C'$ and $D \cup \{0\} \subset D'$, and so

$$C \cup D \subset C' + D' \in \mathcal{S}.$$

Thus, $\mathcal{S}$ is a base for a bornology $\mathcal{B}$ on E. Clearly, $\mathcal{B}$ satisfies (MB1) and (MB2), that is, $\mathcal{B}$ is an A-module bornology. Moreover, each f_α is bounded, which implies that each $g \circ f_\alpha$ is bounded whenever g is bounded. Conversely, if each $g \circ f_\alpha$ is bounded, then g is bounded on each element of $\mathcal{S}$ and so g is bounded. $\square$

Remark 7. Under the conditions of Theorem 6, if E is generated by $\bigcup_{\alpha \in I} f_\alpha(F_\alpha)$, then the sets of the form $f_{\alpha_1}(B_{\alpha_1}) + \cdots + f_{\alpha_n}(B_{\alpha_n})$, where $n \in \mathbb{N}^*$, $\alpha_1, \ldots, \alpha_n \in I$ and $B_{\alpha_j} \in \mathcal{B}_{F_{\alpha_j}}$ for $1 \leq j \leq n$, form a base for the final A-module bornology.

Remark 8. Contrary to the case of initial bornologies, the final bornology on E and the final A-module bornology on E may be different. For example, consider $A = \mathbb{R}$ endowed with its usual bornology (which is given by its usual metric) and consider $F_1 = F_2 = \mathbb{R}$ regarded as bornological $\mathbb{R}$-modules. Let $E = \mathbb{R}^2$ regarded as an $\mathbb{R}$-module and let

$$f_1 : x \in F_1 \mapsto (x, 0) \in E \quad \text{and} \quad f_2 : y \in F_2 \mapsto (0, y) \in E.$$

We know that the final bornology $\mathcal{B}$ on E for the family $(f_\alpha)_{\alpha \in \{1,2\}}$ is formed by the sets of the form $X \cup f_1(B_1) \cup f_2(B_2)$, where $X \subset E$ is finite and $B_1, B_2 \subset F_1 = F_2$ are bounded. Since $f_1([0,1]) \in \mathcal{B}$, $f_2([0,1]) \in \mathcal{B}$ and $f_1([0,1]) + f_2([0,1]) = [0,1]^2 \notin \mathcal{B}$, we see that $\mathcal{B}$ is not an $\mathbb{R}$-module bornology.

Example 9. (a) If E is a bornological A-module and M is a submodule of E, then the quotient bornology on the quotient A-module E/M is an A-module bornology. Recall that it consists of the canonical images in E/M of the bounded sets of E.
(b) Let $(E_\alpha)_{\alpha \in I}$ be a family of bornological A-modules and $E = \bigoplus_{\alpha \in I} E_\alpha$ the direct sum A-module. The *direct sum bornology* on E is the final A-module bornology for the family $(\lambda_\alpha)_{\alpha \in I}$, where $\lambda_\alpha : E_\alpha \to E$ is the canonical injection ($\alpha \in I$). By Remark 7, the sets of the form $\lambda_{\alpha_1}(B_{\alpha_1}) + \cdots + \lambda_{\alpha_n}(B_{\alpha_n})$, where $n \in \mathbb{N}^*$, $\alpha_1, \ldots, \alpha_n \in I$ and $B_{\alpha_j} \in \mathcal{B}_{E_{\alpha_j}}$ for $1 \leq j \leq n$, form a base for this bornology.
(c) Let $(E_\alpha, u_{\beta\alpha})_{\alpha \in I}$ be a direct system of bornological A-modules. This means that I is a non-empty partially ordered set, E_α is a bornological A-module for every $\alpha \in I$, $u_{\beta\alpha} : E_\alpha \to E_\beta$ is a bounded A-linear mapping for every $\alpha, \beta \in I$ with $\alpha \leq \beta$, $u_{\alpha\alpha} = Id_{E_\alpha}$ for every $\alpha \in I$, and $u_{\gamma\beta} \circ u_{\beta\alpha} = u_{\gamma\alpha}$ for every $\alpha, \beta, \gamma \in I$ with $\alpha \leq \beta \leq \gamma$. Let $E = \varinjlim E_\alpha$ be the direct limit A-module and let $u_\alpha : E_\alpha \to E$ be the canonical mapping ($\alpha \in I$). The *direct*

limit bornology on E is the final A-module bornology on E for the family $(u_\alpha)_{\alpha\in I}$. By Remark 7, the sets of the form $u_{\alpha_1}(B_{\alpha_1})+\cdots+u_{\alpha_n}(B_{\alpha_n})$, where $n\in\mathbb{N}^*$, $\alpha_1,\ldots,\alpha_n\in I$ and $B_{\alpha_j}\in\mathcal{B}_{E_{\alpha_j}}$ for $1\le j\le n$, form a base for this bornology. By considering E endowed with this bornology, the following universal property holds:

For every bornological A-module F and for every family $(v_\alpha)_{\alpha\in I}$ of bounded A-linear mappings $v_\alpha : E_\alpha\to F$ satisfying $v_\alpha = v_\beta\circ u_{\beta\alpha}$ whenever $\alpha\le\beta$, there is a unique bounded A-linear mapping $v:\varinjlim E_\alpha\to F$ such that $v_\alpha = v\circ u_\alpha$ for every $\alpha\in I$.

Let us remark that if I is endowed with the equality relation, then $\varinjlim E_\alpha = \bigoplus_{\alpha\in I} E_\alpha$ and the direct limit bornology coincides with the direct sum bornology.

Unless otherwise specified, whenever we consider a quotient of a bornological module (resp. a direct sum of bornological modules, a direct limit of a direct system of bornological modules) as a bornological module, the quotient bornology (resp. the direct sum bornology, the direct limit bornology) is implied.

3 Bornological tensor products

Throughout this section A denotes a bornological commutative ring.

If $E_1,\ldots,E_n,F$ are A-modules, we denote by $\mathcal{L}_a(E_1,\ldots,E_n;F)$ the A-module of all A-multilinear mappings from $E_1\times\cdots\times E_n$ into F, by $E_1\otimes\cdots\otimes E_n$ the tensor product A-module of $E_1,\ldots,E_n$ and by

$$\phi:(x_1,\ldots,x_n)\in E_1\times\cdots\times E_n\mapsto x_1\otimes\cdots\otimes x_n\in E_1\otimes\cdots\otimes E_n$$

the canonical mapping. Moreover, given subsets $B_1\subset E_1,\ldots,B_n\subset E_n$, we define

$$B_1*\cdots*B_n=\{x_1\otimes\cdots\otimes x_n; x_1\in B_1,\ldots,x_n\in B_n\}.$$

We know that for each A-multilinear mapping f from $E_1\times\cdots\times E_n$ into F there exists a unique A-linear mapping from $E_1\otimes\cdots\otimes E_n$ into F, which we denote by u_f, such that

$$f(x_1,\ldots,x_n)=u_f(x_1\otimes\cdots\otimes x_n)$$

for all $(x_1,\ldots,x_n)\in E_1\times\cdots\times E_n$. Moreover, the mapping

$$f\in\mathcal{L}_a(E_1,\ldots,E_n;F)\mapsto u_f\in\mathcal{L}_a(E_1\otimes\cdots\otimes E_n;F)$$

is an A-module isomorphism.

If $E_1, \ldots, E_n, F$ are bornological A-modules, the set

$$\mathcal{L}_b(E_1, \ldots, E_n; F) = \mathcal{L}_a(E_1, \ldots, E_n; F) \cap B(E_1 \times \cdots \times E_n; F)$$

of all bounded A-multilinear mappings from $E_1 \times \cdots \times E_n$ into F is a submodule of both $\mathcal{L}_a(E_1, \ldots, E_n; F)$ and $B(E_1 \times \cdots \times E_n; F)$. The bornology of equiboundedness on $B(E_1 \times \cdots \times E_n; F)$ induces an A-module bornology on $\mathcal{L}_b(E_1, \ldots, E_n; F)$. Unless otherwise specified, we consider $\mathcal{L}_b(E_1, \ldots, E_n; F)$ endowed with this bornology.

Theorem 10. *Let $E_1, \ldots, E_n$ be bornological A-modules. Then there exists a unique A-module bornology $\mathcal{B}$ on $E_1 \otimes \cdots \otimes E_n$ such that the following property holds: for every bornological A-module F and for every A-multilinear mapping $f : E_1 \times \cdots \times E_n \to F$, we have that f is bounded if and only if u_f is bounded (where $E_1 \otimes \cdots \otimes E_n$ is endowed with $\mathcal{B}$).*

Proof. Let $\mathcal{S}$ be the collection of all sets of the form

$$(B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r}),$$

where $r \in \mathbb{N}^*$ and $B_{i,1}, \ldots, B_{i,r} \in \mathcal{B}_{E_i}$ for each $1 \leq i \leq n$. Clearly, $\mathcal{S}$ is a cover of $E_1 \otimes \cdots \otimes E_n$. If

$$B = (B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r}),$$

$$C = (C_{1,1} * \cdots * C_{n,1}) + \cdots + (C_{1,s} * \cdots * C_{n,s})$$

are two elements of $\mathcal{S}$, then by completing with sets of the form $\{0\}$, if necessary, we may assume $r = s$, and so

$$B \cup C \subset (D_{1,1} * \cdots * D_{n,1}) + \cdots + (D_{1,r} * \cdots * D_{n,r}),$$

where $D_{i,j} = B_{i,j} \cup C_{i,j} \in \mathcal{B}_{E_i}$ for each $1 \leq i \leq n$ and $1 \leq j \leq r$. Hence, $\mathcal{S}$ is a base for a bornology $\mathcal{B}$ on $E_1 \otimes \cdots \otimes E_n$. Since the sum of two elements of $\mathcal{S}$ is obviously an element of $\mathcal{S}$ and since

$$LB \subset (LB_{1,1} * B_{2,1} * \cdots * B_{n,1}) + \cdots + (LB_{1,r} * B_{2,r} * \cdots * B_{n,r}) \in \mathcal{S}$$

whenever $L \in \mathcal{B}_A$ and B is as above, it follows that $\mathcal{B}$ is an A-module bornology. Moreover, the canonical mapping $\phi : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is bounded if $E_1 \otimes \cdots \otimes E_n$ is endowed with $\mathcal{B}$. Hence, $f = u_f \circ \phi$ is bounded whenever u_f is bounded. Conversely, if f is bounded and B is as above, then

$$u_f(B) = f(B_{1,1} \times \cdots \times B_{n,1}) + \cdots + f(B_{1,r} \times \cdots \times B_{n,r}) \in \mathcal{B}_F,$$

proving that u_f is bounded.

It remains to prove the uniqueness of $\mathcal{B}$. Suppose $\mathcal{B}'$ is an A-module bornology on $E_1 \otimes \cdots \otimes E_n$ which has also the property stated in the theorem. Let $G = E_1 \otimes \cdots \otimes E_n$ endowed with $\mathcal{B}$ and $G' = E_1 \otimes \cdots \otimes E_n$ endowed with $\mathcal{B}'$. In the diagram

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[rr, "\phi"] \arrow[dr, "\phi"'] & & G' \\ & G' \arrow[ur, "Id"'] & \end{tikzcd}$$

the identity mapping Id is obviously bounded. Hence, by our hypothesis on $\mathcal{B}'$, $\phi = Id \circ \phi : E_1 \times \cdots \times E_n \to G'$ is bounded. This implies that $\mathcal{S} \subset \mathcal{B}'$, and so $\mathcal{B} \subset \mathcal{B}'$. Moreover, we see from the diagram

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[rr, "\phi"] \arrow[dr, "\phi"'] & & G \\ & G' \arrow[ur, "Id"'] & \end{tikzcd}$$

and from our hypothesis on $\mathcal{B}'$ that $Id : G' \to G$ is bounded, and so $\mathcal{B}' \subset \mathcal{B}$. $\square$

The bornology $\mathcal{B}$ obtained in Theorem 10 is called the *tensor product bornology* on $E_1 \otimes \cdots \otimes E_n$. It follows from the proof of Theorem 10 that if $\mathcal{S}_i$ is a base for the bornology $\mathcal{B}_{E_i}$ $(1 \le i \le n)$, then the sets of the form

$$(B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r}),$$

where $r \in \mathbb{N}^*$ and $B_{i,1}, \ldots, B_{i,r} \in \mathcal{S}_i$ for each $1 \le i \le n$, form a base for $\mathcal{B}$.

Unless otherwise specified, whenever we consider a tensor product of bornological modules as a bornological module the tensor product bornology is implied.

Proposition 11 (Commutativity)**.** *Let $E_1, \ldots, E_n$ be bornological A-modules and let π be a permutation of $\{1, \ldots, n\}$. Consider the unique A-linear mapping*

$$\alpha : E_1 \otimes \cdots \otimes E_n \to E_{\pi(1)} \otimes \cdots \otimes E_{\pi(n)}$$

which maps $x_1 \otimes \cdots \otimes x_n$ into $x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)}$. Then α is a bornological A-module isomorphism.

Proposition 12 (Associativity)**.** *Let $E_1, \ldots, E_n$ be bornological A-modules and let G be the bornological A-module obtained by putting some choice of parentheses in the expression $E_1 \otimes \cdots \otimes E_n$. Consider the unique A-linear mapping*

$$\beta : E_1 \otimes \cdots \otimes E_n \to G$$

which maps $x_1 \otimes \cdots \otimes x_n$ in the expression obtained from $x_1 \otimes \cdots \otimes x_n$ by putting the same choice of parentheses as before. Then β is a bornological A-module isomorphism.

In fact, we know that the mappings α and β are A-module isomorphisms. That these mappings are bounded and have bounded inverses follow immediately from the form of the basic bounded sets in the tensor product bornology.

Theorem 13. *Let $E_1, \dots, E_n, F$ be bornological A-modules. Then*

$$\psi : f \in \mathcal{L}_b(E_1, \dots, E_n; F) \mapsto u_f \in \mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F)$$

is a bornological A-module isomorphism.

Proof. We know that

$$f \in \mathcal{L}_a(E_1, \dots, E_n; F) \mapsto u_f \in \mathcal{L}_a(E_1 \otimes \cdots \otimes E_n; F)$$

is an A-module isomorphism and, by Theorem 10, f is bounded if and only if u_f is bounded. Thus, ψ is an A-module isomorphism. It remains to prove that ψ and ψ^{-1} are bounded.

Let $\mathcal{X}$ be an equibounded subset of $\mathcal{L}_b(E_1, \dots, E_n; F)$ and let

$$B = (B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r})$$

be a basic bounded set for the tensor product bornology on $E_1 \otimes \cdots \otimes E_n$. Then

$$\begin{aligned}\big(\psi(\mathcal{X})\big)(B) &= \bigcup_{f \in \mathcal{X}} \big(f(B_{1,1} \times \cdots \times B_{n,1}) + \cdots + f(B_{1,r} \times \cdots \times B_{n,r})\big) \\ &\subset \mathcal{X}(B_{1,1} \times \cdots \times B_{n,1}) + \cdots + \mathcal{X}(B_{1,r} \times \cdots \times B_{n,r}) \in \mathcal{B}_F.\end{aligned}$$

This proves that $\psi(\mathcal{X})$ is an equibounded subset of $\mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F)$.

Conversely, let $\mathcal{Y}$ be an equibounded subset of $\mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F)$ and let $C = C_1 \times \cdots \times C_s$ be a basic bounded set for the product bornology on $E_1 \times \cdots \times E_n$. Then

$$\begin{aligned}\big(\psi^{-1}(\mathcal{Y})\big)(C) &= \bigcup_{f \in \psi^{-1}(\mathcal{Y})} f(C_1 \times \cdots \times C_s) = \bigcup_{f \in \psi^{-1}(\mathcal{Y})} u_f(C_1 * \cdots * C_s) \\ &= \mathcal{Y}(C_1 * \cdots * C_s) \in \mathcal{B}_F,\end{aligned}$$

proving that $\psi^{-1}(\mathcal{Y})$ is an equibounded subset of $\mathcal{L}_b(E_1, \dots, E_n; F)$. $\square$

Proposition 14. *If $E_1, \dots, E_n, F_1, \dots, F_n$ are bornological A-modules and $u_1 : E_1 \to F_1, \dots, u_n : E_n \to F_n$ are bounded A-linear mappings, then the unique A-linear mapping*

$$u_1 \otimes \cdots \otimes u_n : E_1 \otimes \cdots \otimes E_n \to F_1 \otimes \cdots \otimes F_n$$

which satisfies

$$(u_1 \otimes \cdots \otimes u_n)(x_1 \otimes \cdots \otimes x_n) = u_1(x_1) \otimes \cdots \otimes u_n(x_n)$$

is bounded.

Proof. Indeed, if $B = (B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r})$ is a basic bounded set in $E_1 \otimes \cdots \otimes E_n$, then $(u_1 \otimes \cdots \otimes u_n)(B) = (u_1(B_{1,1}) * \cdots * u_n(B_{n,1})) + \cdots + (u_1(B_{1,r}) * \cdots * u_n(B_{n,r}))$ is bounded in $F_1 \otimes \cdots \otimes F_n$. □

Proposition 15. *If $E_1, \ldots, E_n, F_1, \ldots, F_n$ are bornological A-modules, then*

$$\Phi : \mathcal{L}_b(E_1; F_1) \times \cdots \times \mathcal{L}_b(E_n; F_n) \to \mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F_1 \otimes \cdots \otimes F_n)$$

given by

$$\Phi(u_1, \ldots, u_n) = u_1 \otimes \cdots \otimes u_n$$

is a bounded A-multilinear mapping.

Proof. Clearly, Φ is A-multilinear. If $\mathcal{X}_j$ is an equibounded subset of $\mathcal{L}_b(E_j; F_j)$ $(1 \le j \le n)$ and $B = (B_{1,1} * \cdots * B_{n,1}) + \cdots + (B_{1,r} * \cdots * B_{n,r})$ is a basic bounded set in $E_1 \otimes \cdots \otimes E_n$, then $\Phi(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n)(B) \subset (\mathcal{X}_1(B_{1,1}) * \cdots * \mathcal{X}_n(B_{n,1})) + \cdots + (\mathcal{X}_1(B_{1,r}) * \cdots * \mathcal{X}_n(B_{n,r}))$, which is a bounded set in $F_1 \otimes \cdots \otimes F_n$. This proves that $\Phi(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n)$ is an equibounded subset of $\mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F_1 \otimes \cdots \otimes F_n)$. □

In view of Theorem 10, there corresponds to the mapping Φ of Proposition 15 a unique bounded A-linear mapping (called *canonical*)

$$\mathcal{L}_b(E_1; F_1) \otimes \cdots \otimes \mathcal{L}_b(E_n; F_n) \to \mathcal{L}_b(E_1 \otimes \cdots \otimes E_n; F_1 \otimes \cdots \otimes F_n),$$

which associates to each element $u_1 \otimes \cdots \otimes u_n$ of the tensor product the linear mapping $u_1 \otimes \cdots \otimes u_n : E_1 \otimes \cdots \otimes E_n \to F_1 \otimes \cdots \otimes F_n$. This canonical mapping is not necessarily injective nor surjective, so that the notation $u_1 \otimes \cdots \otimes u_n$ can lead to confusion. So, unless otherwise specified, $u_1 \otimes \cdots \otimes u_n$ will denote the bounded A-linear mapping given by Proposition 14.

Proposition 16. *Let $(E^{(1)}_{\alpha_1})_{\alpha_1 \in I_1}, \ldots, (E^{(n)}_{\alpha_n})_{\alpha_n \in I_n}$ be families of bornological A-modules and let $I = I_1 \times \cdots \times I_n$. Then the canonical A-linear mapping*

$$\Psi : \Big(\prod_{\alpha_1 \in I_1} E^{(1)}_{\alpha_1} \Big) \otimes \cdots \otimes \Big(\prod_{\alpha_n \in I_n} E^{(n)}_{\alpha_n} \Big) \to \prod_{(\alpha_1, \ldots, \alpha_n) \in I} \big(E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n} \big),$$

which satisfies

$$\Psi\big((x^{(1)}_{\alpha_1})_{\alpha_1 \in I_1} \otimes \cdots \otimes (x^{(n)}_{\alpha_n})_{\alpha_n \in I_n} \big) = \big(x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(n)}_{\alpha_n} \big)_{(\alpha_1, \ldots, \alpha_n) \in I},$$

is bounded.

Proof. Consider the A-multilinear mapping

$$f : \Big(\prod_{\alpha_1 \in I_1} E^{(1)}_{\alpha_1} \Big) \times \cdots \times \Big(\prod_{\alpha_n \in I_n} E^{(n)}_{\alpha_n} \Big) \to \prod_{(\alpha_1, \ldots, \alpha_n) \in I} \big(E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n} \big)$$

given by $f\big((x^{(1)}_{\alpha_1})_{\alpha_1\in I_1},\dots,(x^{(n)}_{\alpha_n})_{\alpha_n\in I_n}\big) = \big(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n}\big)_{(\alpha_1,\dots,\alpha_n)\in I}$. The inclusion

$$f\Big(\Big(\prod_{\alpha_1\in I_1} B^{(1)}_{\alpha_1}\Big)\times\cdots\times\Big(\prod_{\alpha_n\in I_n} B^{(n)}_{\alpha_n}\Big)\Big)\subset\prod_{(\alpha_1,\dots,\alpha_n)\in I}\big(B^{(1)}_{\alpha_1}*\cdots*B^{(n)}_{\alpha_n}\big)$$

shows that f is bounded. Hence $\Psi = u_f$ is bounded by Theorem 10. □

We know that the mapping Ψ in the above proposition is not necessarily injective nor surjective. There is a well-known particular case in which Ψ is an A-module isomorphism, namely:

$$\Psi : \Big(\prod_{\alpha\in I} E_\alpha\Big)\otimes F\to\prod_{\alpha\in I}(E_\alpha\otimes F),$$

where $(E_\alpha)_{\alpha\in I}$ is any family of A-modules and F is a finitely generated free A-module ([7], Chapter II, §3, Corollary 3 to Proposition 7). Nevertheless, even in this particular case, if E_α $(\alpha\in I)$ and F are endowed with A-module bornologies, it is still not necessarily true that Ψ is a bornological A-module isomorphism, as the next example shows. However, we shall see later (Proposition 20) that under some additional conditions we get a positive result.

Example 17. Let K be a non-trivially valued commutative field with absolute value $|\cdot|$ and identity element 1. Consider K endowed with the bornology formed by the sets which are bounded with respect to the absolute value $|\cdot|$. Let $E = K$ regarded as a bornological K-vector space and let $F = K$ regarded as a K-vector space but endowed with the trivial bornology. Then the canonical mapping $\Psi : E^{\mathbb{N}}\otimes F\to(E\otimes F)^{\mathbb{N}}$ is a K-vector space isomorphism and is bounded by Proposition 16. We shall prove that Ψ^{-1} is not bounded. Since $\Psi^{-1}\big((x_n\otimes y_n)_{n\in\mathbb{N}}\big) = (x_ny_n)_{n\in\mathbb{N}}\otimes 1$,

$$\Psi^{-1}\big((\{1\}*F)^{\mathbb{N}}\big) = E^{\mathbb{N}}*\{1\}.$$

Clearly $(\{1\}*F)^{\mathbb{N}}$ is a bounded subset of $(E\otimes F)^{\mathbb{N}}$, but we claim that $E^{\mathbb{N}}*\{1\}$ is not a bounded subset of $E^{\mathbb{N}}\otimes F$. In fact, suppose that our claim is false. Then $E^{\mathbb{N}}*\{1\}$ must be contained in a basic bounded set of the form

$$\Big[\Big(\prod_{n\in\mathbb{N}} B_{n,1}\Big)*F\Big]+\cdots+\Big[\Big(\prod_{n\in\mathbb{N}} B_{n,r}\Big)*F\Big],$$

where $B_{n,j}$ is bounded in E $(1\le j\le r,\ n\in\mathbb{N})$. For each $n\in\mathbb{N}$, let

$$c_n = \sup\{|\lambda|;\lambda\in B_{n,1}\cup\ldots\cup B_{n,r}\}<\infty$$

and choose $x_n\in E$ such that $|x_n|>nc_n$. We may write

$$(x_n)_{n\in\mathbb{N}}\otimes 1 = \sum_{j=1}^{r}\big((b_{n,j})_{n\in\mathbb{N}}\otimes y_j\big) = \Big(\sum_{j=1}^{r} y_jb_{n,j}\Big)_{n\in\mathbb{N}}\otimes 1,$$

where $b_{n,j} \in B_{n,j}$ and $y_j \in F$ $(1 \le j \le r,\, n \in \mathbb{N})$. Hence,

$$nc_n < |x_n| = \Big|\sum_{j=1}^{r} y_j b_{n,j}\Big| \le c_n \sum_{j=1}^{r} |y_j|$$

for every $n \in \mathbb{N}$, which is impossible.

Let $(E^{(1)}_{\alpha_1}, u^{(1)}_{\beta_1\alpha_1})_{\alpha_1 \in I_1}, \ldots, (E^{(n)}_{\alpha_n}, u^{(n)}_{\beta_n\alpha_n})_{\alpha_n \in I_n}$ be direct systems of bornological A-modules. For each $1 \le j \le n$, consider the bornological A-module

$$E^{(j)} = \varinjlim E^{(j)}_{\alpha_j}$$

and let $u^{(j)}_{\alpha_j} : E^{(j)}_{\alpha_j} \to E^{(j)}$ be the canonical bounded A-linear mapping $(\alpha_j \in I_j)$. Consider the product set $I = I_1 \times \cdots \times I_n$ endowed with the following partial order relation:

$$(\alpha_1, \ldots, \alpha_n) \le (\beta_1, \ldots, \beta_n) \Longleftrightarrow \alpha_1 \le \beta_1, \ldots, \alpha_n \le \beta_n.$$

For each $(\alpha_1, \ldots, \alpha_n) \le (\beta_1, \ldots, \beta_n)$ in I, we define

$$w_{(\beta_1,\ldots,\beta_n)(\alpha_1,\ldots,\alpha_n)} = u^{(1)}_{\beta_1\alpha_1} \otimes \cdots \otimes u^{(n)}_{\beta_n\alpha_n}$$

which is a bounded A-linear mapping. It is easy to show that

$$\big(E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n}, w_{(\beta_1,\ldots,\beta_n)(\alpha_1,\ldots,\alpha_n)}\big)_{(\alpha_1,\ldots,\alpha_n) \in I}$$

is a direct system of bornological A-modules. Consider the bornological A-module

$$E = \varinjlim (E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n})$$

and, for each $(\alpha_1, \ldots, \alpha_n) \in I$, let

$$w_{(\alpha_1,\ldots,\alpha_n)} : E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n} \to E$$

be the canonical bounded A-linear mapping. Now, for each $(\alpha_1, \ldots, \alpha_n) \in I$, we define

$$h_{(\alpha_1,\ldots,\alpha_n)} = u^{(1)}_{\alpha_1} \otimes \cdots \otimes u^{(n)}_{\alpha_n}$$

which is a bounded A-linear mapping. Since

$$h_{(\alpha_1,\ldots,\alpha_n)} = h_{(\beta_1,\ldots,\beta_n)} \circ w_{(\beta_1,\ldots,\beta_n)(\alpha_1,\ldots,\alpha_n)}$$

whenever $(\alpha_1, \ldots, \alpha_n) \le (\beta_1, \ldots, \beta_n)$ in I, there exists a unique bounded A-linear mapping

$$h : E \to E^{(1)} \otimes \cdots \otimes E^{(n)}$$

such that $h_{(\alpha_1,\ldots,\alpha_n)} = h \circ w_{(\alpha_1,\ldots,\alpha_n)}$ for every $(\alpha_1, \ldots, \alpha_n) \in I$.

Theorem 18. *The mapping*

$$h : \varinjlim (E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n}) \to (\varinjlim E^{(1)}_{\alpha_1}) \otimes \cdots \otimes (\varinjlim E^{(n)}_{\alpha_n})$$

is a bornological A-module isomorphism.

Proof. Let

$$\lambda^{(j)}_{\alpha_j} : E^{(j)}_{\alpha_j} \to \bigoplus_{\beta_j \in I_j} E^{(j)}_{\beta_j}$$

$(\alpha_j \in I_j, 1 \leq j \leq n)$ and

$$\lambda_{(\alpha_1,\ldots,\alpha_n)} : E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n} \to \bigoplus_{(\beta_1,\ldots,\beta_n)\in I} (E^{(1)}_{\beta_1} \otimes \cdots \otimes E^{(n)}_{\beta_n})$$

$((\alpha_1, \ldots, \alpha_n) \in I)$ be the canonical injections. We define a bounded A-multilinear mapping $f : E^{(1)} \times \cdots \times E^{(n)} \to E$ in the following way: given $x^{(1)} \in E^{(1)}, \ldots, x^{(n)} \in E^{(n)}$, we choose a representative $(x^{(j)}_{\alpha_j})_{\alpha_j \in I_j} \in \bigoplus_{\alpha_j \in I_j} E^{(j)}_{\alpha_j}$ of the class $x^{(j)}$ in $E^{(j)} = \varinjlim E^{(j)}_{\alpha_j}$ for each $1 \leq j \leq n$, and then define

$$f(x^{(1)}, \ldots, x^{(n)}) = \text{ the class of } (x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(n)}_{\alpha_n})_{(\alpha_1,\ldots,\alpha_n)\in I} \text{ in } E.$$

In order to prove that f is well-defined, it is enough to show that if

$$(y^{(1)}_{\alpha_1})_{\alpha_1 \in I_1} = (x^{(1)}_{\alpha_1})_{\alpha_1 \in I_1} + \lambda^{(1)}_{\alpha}(z^{(1)}_{\alpha}) - \lambda^{(1)}_{\beta}(u^{(1)}_{\beta\alpha}(z^{(1)}_{\alpha}))$$

(where $\alpha \leq \beta$ in I_1 and $z^{(1)}_{\alpha} \in E^{(1)}_{\alpha}$ are fixed) and

$$(y^{(2)}_{\alpha_2})_{\alpha_2 \in I_2} = (x^{(2)}_{\alpha_2})_{\alpha_2 \in I_2}, \ldots, (y^{(n)}_{\alpha_n})_{\alpha_n \in I_n} = (x^{(n)}_{\alpha_n})_{\alpha_n \in I_n},$$

then the class of $(y^{(1)}_{\alpha_1} \otimes \cdots \otimes y^{(n)}_{\alpha_n})_{(\alpha_1,\ldots,\alpha_n)\in I}$ is equal to the class of $(x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(n)}_{\alpha_n})_{(\alpha_1,\ldots,\alpha_n)\in I}$. But this follows from the fact that $(y^{(1)}_{\alpha_1} \otimes \cdots \otimes y^{(n)}_{\alpha_n})_{(\alpha_1,\ldots,\alpha_n)\in I} = (x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(n)}_{\alpha_n})_{(\alpha_1,\ldots,\alpha_n)\in I} + (t_{(\alpha_1,\ldots,\alpha_n)})_{(\alpha_1,\ldots,\alpha_n)\in I}$, where

$$\begin{aligned}(t_{(\alpha_1,\ldots,\alpha_n)})_{(\alpha_1,\ldots,\alpha_n)\in I} = \sum_{(\alpha_2,\ldots,\alpha_n)\in I_2\times\cdots\times I_n} &\big[\lambda_{(\alpha,\alpha_2,\ldots,\alpha_n)}(z^{(1)}_{\alpha} \otimes x^{(2)}_{\alpha_2} \otimes \cdots \otimes x^{(n)}_{\alpha_n}) \\ &- \lambda_{(\beta,\alpha_2,\ldots,\alpha_n)}\big(w_{(\beta,\alpha_2,\ldots,\alpha_n)(\alpha,\alpha_2,\ldots,\alpha_n)}(z^{(1)}_{\alpha} \otimes x^{(2)}_{\alpha_2} \otimes \cdots \otimes x^{(n)}_{\alpha_n})\big)\big].\end{aligned}$$

Now, it is easy to see that f is A-multilinear. Since

$$f\big(u^{(1)}_{\alpha_1}(B^{(1)}_{\alpha_1}) \times \cdots \times u^{(n)}_{\alpha_n}(B^{(n)}_{\alpha_n})\big) = w_{(\alpha_1,\ldots,\alpha_n)}(B^{(1)}_{\alpha_1} * \cdots * B^{(n)}_{\alpha_n})$$

is bounded in E whenever $B^{(j)}_{\alpha_j}$ is bounded in $E^{(j)}_{\alpha_j}$ for each $1 \leq j \leq n$, it follows that f is bounded. By Theorem 10, there is a unique bounded A-linear mapping $g : E^{(1)}\otimes\cdots\otimes E^{(n)} \to E$ such that $f(x^{(1)},\dots,x^{(n)}) = g(x^{(1)}\otimes\cdots\otimes x^{(n)})$ for every $(x^{(1)},\dots,x^{(n)}) \in E^{(1)}\times\cdots\times E^{(n)}$. It remains to prove that $g = h^{-1}$. Since

$$\begin{aligned} g\big(h\big(w_{(\alpha_1,\dots,\alpha_n)}(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n})\big)\big) &= g\big(h_{(\alpha_1,\dots,\alpha_n)}(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n})\big) \\ &= g\big(u^{(1)}_{\alpha_1}(x^{(1)}_{\alpha_1})\otimes\cdots\otimes u^{(n)}_{\alpha_n}(x^{(n)}_{\alpha_n})\big) \\ &= w_{(\alpha_1,\dots,\alpha_n)}(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n}), \end{aligned}$$

we see that $g\circ h$ is the identity mapping on E. On the other hand, since

$$\begin{aligned} h\big(g\big(u^{(1)}_{\alpha_1}(x^{(1)}_{\alpha_1})\otimes\cdots\otimes u^{(n)}_{\alpha_n}(x^{(n)}_{\alpha_n})\big)\big) &= h\big(w_{(\alpha_1,\dots,\alpha_n)}(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n})\big) \\ &= h_{(\alpha_1,\dots,\alpha_n)}(x^{(1)}_{\alpha_1}\otimes\cdots\otimes x^{(n)}_{\alpha_n}) \\ &= u^{(1)}_{\alpha_1}(x^{(1)}_{\alpha_1})\otimes\cdots\otimes u^{(n)}_{\alpha_n}(x^{(n)}_{\alpha_n}), \end{aligned}$$

$h\circ g$ is the identity mapping on $E^{(1)}\otimes\cdots\otimes E^{(n)}$. □

The isomorphism given in the above theorem is said to be canonical.

Now, let $(E^{(1)}_{\alpha_1})_{\alpha_1\in I_1},\dots,(E^{(n)}_{\alpha_n})_{\alpha_n\in I_n}$ be families of bornological A-modules. By partially ordering I_j through the equality relation and by defining $u^{(j)}_{\alpha_j\alpha_j}$ as the identity mapping on $E^{(j)}_{\alpha_j}$ ($\alpha_j\in I_j$), we obtain a direct system $(E^{(j)}_{\alpha_j}, u^{(j)}_{\beta_j\alpha_j})_{\alpha_j\in I_j}$ of bornological A-modules so that

$$\varinjlim E^{(j)}_{\alpha_j} = \bigoplus_{\alpha_j\in I_j} E^{(j)}_{\alpha_j}$$

as bornological A-modules ($1\le j\le n$). In the present case, the partial order relation on the product $I = I_1\times\cdots\times I_n$ is also the equality relation and so

$$\varinjlim(E^{(1)}_{\alpha_1}\otimes\cdots\otimes E^{(n)}_{\alpha_n}) = \bigoplus_{(\alpha_1,\dots,\alpha_n)\in I}(E^{(1)}_{\alpha_1}\otimes\cdots\otimes E^{(n)}_{\alpha_n})$$

as bornological A-modules. Therefore, we obtain from the previous theorem the following

Corollary 19. *The bornological A-modules*

$$\bigoplus_{(\alpha_1,\dots,\alpha_n)\in I}(E^{(1)}_{\alpha_1}\otimes\cdots\otimes E^{(n)}_{\alpha_n}) \text{ and } \Big(\bigoplus_{\alpha_1\in I_1} E^{(1)}_{\alpha_1}\Big)\otimes\cdots\otimes\Big(\bigoplus_{\alpha_n\in I_n} E^{(n)}_{\alpha_n}\Big)$$

are canonically isomorphic.

As an application of this corollary, let us establish the following

Proposition 20. *Let K be a complete non-trivially valued commutative field and consider K endowed with the bornology defined by its absolute value. Let $(E_\alpha)_{\alpha\in I}$ be a family of bornological K-vector spaces and let F be a bornological K-vector space. If F is finite-dimensional and separated, then the canonical mapping*

$$\Psi : \Big(\prod_{\alpha\in I} E_\alpha\Big) \otimes F \to \prod_{\alpha\in I}(E_\alpha \otimes F)$$

is a bornological K-vector space isomorphism.

Recall that F separated means that $\{0\}$ is the only bounded vector subspace of F ([10], Chapter 1, §3, Definition 2).

Proof. If $n = \dim F$ then F is isomorphic as a bornological K-vector space to the product vector space K^n endowed with the product bornology ([10], Chapter 1, §3, Proposition 12). Thus, it is enough to prove the proposition in the case $F = K^n$. For this purpose, let us observe that for every bornological K-vector space E, the mapping

$$\theta : x \in E \mapsto x \otimes 1 \in E \otimes K$$

is a bornological K-vector space isomorphism. Indeed, this follows easily from the fact that $\theta^{-1}(x \otimes a) = ax$ for all $a \in K$ and $x \in E$.

Now, consider the following finite sequence of canonical bornological K-vector space isomorphisms:

$$\begin{aligned}(\prod_{\alpha\in I} E_\alpha) \otimes K^n &\to \big((\prod_{\alpha\in I} E_\alpha) \otimes K\big)^n \to \big(\prod_{\alpha\in I} E_\alpha\big)^n \to \prod_{\alpha\in I}(E_\alpha)^n \\ &\to \prod_{\alpha\in I}(E_\alpha \otimes K)^n \to \prod_{\alpha\in I}(E_\alpha \otimes K^n),\end{aligned}$$

where the first and the fifth isomorphisms come from Corollary 19, the second and the fourth isomorphisms come from the observation in the previous paragraph, and the third isomorphism is obvious. Since the composition of these isomorphisms is exactly the mapping Ψ, the proof is complete. □

For the remaining of this section we shall assume all the notations fixed in the paragraph before Theorem 18.

Let $(F_\lambda, v_{\lambda\mu})_{\lambda\in J}$ be an inverse system of bornological A-modules. Consider the product set $K = I_1 \times \cdots \times I_n \times J$ endowed with the following partial order relation:

$$(\alpha_1, \ldots, \alpha_n, \lambda) \le (\beta_1, \ldots, \beta_n, \mu) \Longleftrightarrow \alpha_1 \le \beta_1, \ldots, \alpha_n \le \beta_n, \lambda \le \mu.$$

For each $(\alpha_1, \dots, \alpha_n, \lambda) \leq (\beta_1, \dots, \beta_n, \mu)$ in K, consider the bounded A-linear mapping

$$\Phi_{(\alpha_1,\dots,\alpha_n,\lambda)(\beta_1,\dots,\beta_n,\mu)} : \mathcal{L}_b(E^{(1)}_{\beta_1}, \dots, E^{(n)}_{\beta_n}; F_\mu) \to \mathcal{L}_b(E^{(1)}_{\alpha_1}, \dots, E^{(n)}_{\alpha_n}; F_\lambda)$$

given by

$$\Phi_{(\alpha_1,\dots,\alpha_n,\lambda)(\beta_1,\dots,\beta_n,\mu)}(\varphi) = v_{\lambda\mu} \circ \varphi \circ (u^{(1)}_{\beta_1\alpha_1} \times \cdots \times u^{(n)}_{\beta_n\alpha_n}),$$

where

$$(u^{(1)}_{\beta_1\alpha_1} \times \cdots \times u^{(n)}_{\beta_n\alpha_n})(x^{(1)}_{\alpha_1}, \dots, x^{(n)}_{\alpha_n}) = (u^{(1)}_{\beta_1\alpha_1}(x^{(1)}_{\alpha_1}), \dots, u^{(n)}_{\beta_n\alpha_n}(x^{(n)}_{\alpha_n})).$$

$$\begin{tikzcd}
E^{(1)}_{\alpha_1} \times \cdots \times E^{(n)}_{\alpha_n} \arrow[r, "u^{(1)}_{\beta_1\alpha_1} \times \cdots \times u^{(n)}_{\beta_n\alpha_n}"] \arrow[d, "\Phi_{(\alpha_1,\dots,\alpha_n,\lambda)(\beta_1,\dots,\beta_n,\mu)}(\varphi)"'] & E^{(1)}_{\beta_1} \times \cdots \times E^{(n)}_{\beta_n} \arrow[d, "\varphi"] \\
F_\lambda & F_\mu \arrow[l, "v_{\lambda\mu}"]
\end{tikzcd}$$

It is easy to show that

$$\big(\mathcal{L}_b(E^{(1)}_{\alpha_1}, \dots, E^{(n)}_{\alpha_n}; F_\lambda), \Phi_{(\alpha_1,\dots,\alpha_n,\lambda)(\beta_1,\dots,\beta_n,\mu)}\big)_{(\alpha_1,\dots,\alpha_n,\lambda)\in K}$$

is an inverse system of bornological A-modules. Consider the bornological A-module

$$F = \varprojlim F_\lambda$$

and let $v_\lambda : F \to F_\lambda$ be the canonical bounded A-linear mapping ($\lambda \in J$). For each $(\alpha_1, \dots, \alpha_n, \lambda) \in K$, consider the bounded A-linear mapping

$$\Psi_{(\alpha_1,\dots,\alpha_n,\lambda)} : \mathcal{L}_b(E^{(1)}, \dots, E^{(n)}; F) \to \mathcal{L}_b(E^{(1)}_{\alpha_1}, \dots, E^{(n)}_{\alpha_n}; F_\lambda)$$

given by

$$\Psi_{(\alpha_1,\dots,\alpha_n,\lambda)}(\varphi) = v_\lambda \circ \varphi \circ (u^{(1)}_{\alpha_1} \times \cdots \times u^{(n)}_{\alpha_n}).$$

$$\begin{tikzcd}
E^{(1)}_{\alpha_1} \times \cdots \times E^{(n)}_{\alpha_n} \arrow[r, "u^{(1)}_{\alpha_1} \times \cdots \times u^{(n)}_{\alpha_n}"] \arrow[d, "\Psi_{(\alpha_1,\dots,\alpha_n,\lambda)}(\varphi)"'] & E^{(1)} \times \cdots \times E^{(n)} \arrow[d, "\varphi"] \\
F_\lambda & F \arrow[l, "v_\lambda"]
\end{tikzcd}$$

Clearly,

$$\Psi(\varphi) = \big(\Psi_{(\alpha_1,\dots,\alpha_n,\lambda)}(\varphi)\big)_{(\alpha_1,\dots,\alpha_n,\lambda)\in K} \in \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1}, \dots, E^{(n)}_{\alpha_n}; F_\lambda).$$

In this way we obtain an A-linear mapping

$$\Psi : \mathcal{L}_b(E^{(1)}, \dots, E^{(n)}; F) \to \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1}, \dots, E^{(n)}_{\alpha_n}; F_\lambda).$$

Theorem 21. *The mapping*

$$\Psi : \mathcal{L}_b(\varinjlim E^{(1)}_{\alpha_1}, \ldots, \varinjlim E^{(n)}_{\alpha_n}; \varprojlim F_\lambda) \to \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1}, \ldots, E^{(n)}_{\alpha_n}; F_\lambda)$$

is a bornological A-module isomorphism.

The case $n = 1$ of the above theorem (linear case) was obtained in [10] in the context of bornological vector spaces over a non-discrete complete valued field. The linear and the multilinear cases were established in [8] in the context of bornological modules over a commutative topological ring. The proof presented in [8] of the linear case works as well in the present context and so we shall omit it. Our goal here is to give a different proof of the multilinear case by using tensor products.

Proof. By Theorems 13 and 18,

$$\Psi_1 : \varphi \in \mathcal{L}_b(E^{(1)}, \ldots, E^{(n)}; F) \mapsto u_\varphi \in \mathcal{L}_b(E^{(1)} \otimes \cdots \otimes E^{(n)}; F),$$

$$\Psi_2 : u \in \mathcal{L}_b(E^{(1)} \otimes \cdots \otimes E^{(n)}; F) \mapsto u \circ h \in \mathcal{L}_b(E; F)$$

are bornological A-module isomorphisms. By the case $n = 1$ of the theorem, the mapping

$$\Psi_3 : \mathcal{L}_b(E; F) \to \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n}; F_\lambda)$$

given by $\Psi_3(w) = (v_\lambda \circ w \circ w_{(\alpha_1,\ldots,\alpha_n)})_{(\alpha_1,\ldots,\alpha_n,\lambda)\in K}$ is a bornological A-module isomorphism. In view of Theorem 13, the mapping

$$\Psi_4 : \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1}, \ldots, E^{(n)}_{\alpha_n}; F_\lambda) \to \varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1} \otimes \cdots \otimes E^{(n)}_{\alpha_n}; F_\lambda)$$

given by $\Psi_4\big((f_{(\alpha_1,\ldots,\alpha_n,\lambda)})_{(\alpha_1,\ldots,\alpha_n,\lambda)\in K}\big) = \big(u_{f_{(\alpha_1,\ldots,\alpha_n,\lambda)}}\big)_{(\alpha_1,\ldots,\alpha_n,\lambda)\in K}$ is also a bornological A-module isomorphism. Now, fix $\varphi \in \mathcal{L}_b(E^{(1)}, \ldots, E^{(n)}; F)$. By definition,

$$\Psi_3(\Psi_2(\Psi_1(\varphi))) = (v_\lambda \circ u_\varphi \circ h \circ w_{(\alpha_1,\ldots,\alpha_n)})_{(\alpha_1,\ldots,\alpha_n,\lambda)\in K}.$$

Since

$$\begin{aligned}(v_\lambda \circ u_\varphi \circ h \circ w_{(\alpha_1,\ldots,\alpha_n)})&(x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(n)}_{\alpha_n}) \\ &= (v_\lambda \circ \varphi \circ (u^{(1)}_{\alpha_1} \times \cdots \times u^{(n)}_{\alpha_n}))(x^{(1)}_{\alpha_1}, \ldots, x^{(n)}_{\alpha_n}),\end{aligned}$$

we see that $\Psi_3(\Psi_2(\Psi_1(\varphi))) = \Psi_4(\Psi(\varphi))$, and so $\Psi = \Psi_4^{-1} \circ \Psi_3 \circ \Psi_2 \circ \Psi_1$. □

The isomorphism given in the above theorem is said to be canonical.

Now, let $(E^{(1)}_{\alpha_1})_{\alpha_1\in I_1},\dots,(E^{(n)}_{\alpha_n})_{\alpha_n\in I_n},(F_\lambda)_{\lambda\in J}$ be families of bornological A-modules. By partially ordering I_j through the equality relation and by defining $u^{(j)}_{\alpha_j\alpha_j}$ as the identity mapping on $E^{(j)}_{\alpha_j}$ ($\alpha_j\in I_j$), we obtain a direct system $(E^{(j)}_{\alpha_j},u^{(j)}_{\beta_j\alpha_j})_{\alpha_j\in I_j}$ of bornological A-modules so that

$$\varinjlim E^{(j)}_{\alpha_j}=\bigoplus_{\alpha_j\in I_j}E^{(j)}_{\alpha_j}$$

as bornological A-modules ($1\le j\le n$). Analogously, by partially ordering J through the equality relation and by defining $v_{\lambda\lambda}$ as the identity mapping on F_λ ($\lambda\in J$), we obtain an inverse system $(F_\lambda,v_{\lambda\mu})_{\lambda\in J}$ of bornological A-modules so that

$$\varprojlim F_\lambda=\prod_{\lambda\in J}F_\lambda$$

as bornological A-modules. In the present case, the partial order relation on the product $K=I_1\times\cdots\times I_n\times J$ is also the equality relation and so

$$\varprojlim \mathcal{L}_b(E^{(1)}_{\alpha_1},\dots,E^{(n)}_{\alpha_n};F_\lambda)=\prod_{(\alpha_1,\dots,\alpha_n,\lambda)\in K}\mathcal{L}_b(E^{(1)}_{\alpha_1},\dots,E^{(n)}_{\alpha_n};F_\lambda)$$

as bornological A-modules. Therefore, we obtain from the previous theorem the following

Corollary 22. *The bornological A-modules*

$$\mathcal{L}_b\Big(\bigoplus_{\alpha_1\in I_1}E^{(1)}_{\alpha_1},\dots,\bigoplus_{\alpha_n\in I_n}E^{(n)}_{\alpha_n};\prod_{\lambda\in J}F_\lambda\Big)\ \textit{and}\ \prod_{(\alpha_1,\dots,\alpha_n,\lambda)\in K}\mathcal{L}_b(E^{(1)}_{\alpha_1},\dots,E^{(n)}_{\alpha_n};F_\lambda)$$

are canonically isomorphic.

4 Bounded homogeneous polynomials on bornological modules

Throughout this section $m\in\mathbb{N}^*$ is fixed and A denotes a bornological commutative ring with identity element $e\neq 0$ such that $m!e$ is invertible in A.

Let E and F be A-modules. Recall that a mapping $p:E\to F$ is said to be an m-homogeneous polynomial if there is an A-multilinear mapping $f:E^m\to F$ such that

$$p(x)=f(x,\dots,x)\quad\text{for all } x\in E.$$

Note that in this case the mapping $g : E^m \to F$ given by

$$g(x_1, \ldots, x_m) = (m!e)^{-1} \sum_{\pi \in S_m} f(x_{\pi(1)}, \ldots, x_{\pi(m)})$$

(where S_m denotes the symmetric group of $\{1, \ldots, m\}$) is a *symmetric* A-multilinear mapping which also satisfies

$$p(x) = g(x, \ldots, x) \quad \text{for all } x \in E.$$

Thus, we can always get a symmetric multilinear mapping in the definition of a homogeneous polynomial. We denote by $\mathcal{P}_a({}^m E; F)$ the A-module of all m-homogeneous polynomials from E into F. Moreover, we define

$$E^{\otimes m} = \underbrace{E \otimes \cdots \otimes E}_{m \text{ times}} \quad \text{and} \quad x^{\otimes m} = \underbrace{x \otimes \cdots \otimes x}_{m \text{ times}} \ (x \in E).$$

Finally, $\gamma^m(E)$ denotes the submodule of $E^{\otimes m}$ generated by the set $\{x^{\otimes m}; x \in E\}$ and

$$\theta : x \in E \mapsto x^{\otimes m} \in \gamma^m(E)$$

is called the *canonical mapping*.

The following result will be very useful for our purposes:

Lemma 23. *Let G, G' be two commutative groups and f a symmetric $\mathbb{Z}$-multilinear mapping from G^m into G'. Then, for every $(x_1, \ldots, x_m) \in G^m$,*

$$m! f(x_1, \ldots, x_m) = \sum_{\epsilon_1, \ldots, \epsilon_m \in \{0,1\}} (-1)^{m-(\epsilon_1+\cdots+\epsilon_m)} \hat{f}(\epsilon_1 x_1 + \cdots + \epsilon_m x_m),$$

where $\hat{f}(x) = f(x, \ldots, x)$ $(x \in G)$.

Proof. Argue as in [15] or [6]. □

Given an A-module E, we define $\delta : E^m \to E^{\otimes m}$ by

$$\delta(x_1, \ldots, x_m) = (m!e)^{-1} \sum_{\pi \in S_m} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(m)}.$$

Note that δ is a symmetric A-multilinear mapping and $\delta(x, \ldots, x) = x^{\otimes m}$ for all $x \in E$. Hence, by Lemma 23,

$$\delta(x_1, \ldots, x_m) = (m!e)^{-1} \sum_{\epsilon_1, \ldots, \epsilon_m \in \{0,1\}} (-1)^{m-(\epsilon_1+\cdots+\epsilon_m)} (\epsilon_1 x_1 + \cdots + \epsilon_m x_m)^{\otimes m},$$

which proves that δ actually maps E^m into $\gamma^m(E)$. From now on, we consider δ as a mapping from E^m into $\gamma^m(E)$, which is said to be *canonical*. Note that $\theta(x) = \delta(x, \ldots, x)$ for all $x \in E$, where $\theta : E \to \gamma^m(E)$ is the canonical mapping. This shows that θ is an m-homogeneous polynomial.

Lemma 24. *Let E and F be A-modules. For each m-homogeneous polynomial p from E into F there exists a unique A-linear mapping from $\gamma^m(E)$ into F, which we denote by v_p, such that*

$$p(x) = v_p(x^{\otimes m}) \quad \textit{for all } x \in E.$$

Moreover, the mapping

$$p \in \mathcal{P}_a({}^m E; F) \mapsto v_p \in \mathcal{L}_a(\gamma^m(E); F)$$

is an A-module isomorphism.

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "\theta"'] & & F \\ & \gamma^m(E) \arrow[ur, "v_p"'] & \end{tikzcd}$$

Proof. Given $p \in \mathcal{P}_a({}^m E; F)$ there exists by definition an A-multilinear mapping $f_p : E^m \to F$ such that $p(x) = f_p(x, \dots, x)$ for all $x \in E$. If v_p is the restriction of u_{f_p} to $\gamma^m(E)$, then

$$p(x) = f_p(x, \dots, x) = u_{f_p}(x \otimes \cdots \otimes x) = v_p(x^{\otimes m}) \quad \text{for all } x \in E.$$

The uniqueness of v_p follows immediately from the fact that $\gamma^m(E)$ is generated by the elements of the form $x^{\otimes m}$. Clearly the mapping $p \to v_p$ is A-linear. So, it remains to show that it is onto. For this purpose, fix $v \in \mathcal{L}_a(\gamma^m(E); F)$ and let $\delta : E^m \to \gamma^m(E)$ be the canonical mapping. Then $f = v \circ \delta : E^m \to F$ is an A-multilinear mapping and so

$$p(x) = f(x, \dots, x) \quad (x \in E)$$

defines an m-homogeneous polynomial from E into F. Since

$$p(x) = f(x, \dots, x) = v(\delta(x, \dots, x)) = v(x^{\otimes m}) \quad \text{for all } x \in E,$$

$v = v_p$ and the proof is complete. □

If E and F are bornological A-modules, the set

$$\mathcal{P}_b({}^m E; F) = \mathcal{P}_a({}^m E; F) \cap B(E; F)$$

of all bounded m-homogeneous polynomials from E into F is a submodule of both $\mathcal{P}_a({}^m E; F)$ and $B(E; F)$. The bornology of equiboundedness on $B(E; F)$ induces an A-module bornology on $\mathcal{P}_b({}^m E; F)$. Unless otherwise specified, we consider $\mathcal{P}_b({}^m E; F)$ endowed with this bornology. Moreover, we consider $\gamma^m(E)$ endowed with the bornology induced by the tensor product bornology on $E^{\otimes m}$, and so the canonical mappings

$$\theta : E \to \gamma^m(E) \quad \text{and} \quad \delta : E^m \to \gamma^m(E)$$

are bounded.

Theorem 25. *Let E and F be bornological A-modules. If $p : E \to F$ is an m-homogeneous polynomial, then p is bounded if and only if v_p is bounded. Moreover,*

$$\psi : p \in \mathcal{P}_b(^m E; F) \mapsto v_p \in \mathcal{L}_b(\gamma^m(E); F)$$

is a bornological A-module isomorphism.

Proof. Let $\mathcal{X}$ be an equibounded subset of $\mathcal{P}_b(^m E; F)$. For each $p \in \mathcal{X}$, let $f_p : E^m \to F$ be a symmetric A-multilinear mapping such that

$$p(x) = f_p(x, \dots, x) \quad \text{for all } x \in E.$$

By Lemma 23,

$$f_p(x_1, \dots, x_m) = (m!e)^{-1} \sum_{\epsilon_1,\dots,\epsilon_m \in \{0,1\}} (-1)^{m-(\epsilon_1+\cdots+\epsilon_m)} p(\epsilon_1 x_1 + \cdots + \epsilon_m x_m)$$

for every $(x_1, \dots, x_m) \in E^m$. This formula shows that $\{f_p; p \in \mathcal{X}\}$ is an equibounded subset of $\mathcal{L}_b(E, \cdots, E; F)$, and therefore $\{u_{f_p}; p \in \mathcal{X}\}$ is an equibounded subset of $\mathcal{L}_b(E^{\otimes m}; F)$ by Theorem 13. Since v_p is the restriction of u_{f_p} to $\gamma^m(E)$, we conclude that

$$\psi(\mathcal{X}) = \{v_p; p \in \mathcal{X}\}$$

is an equibounded subset of $\mathcal{L}_b(\gamma^m(E); F)$. This proves that ψ really maps $\mathcal{P}_b(^m E; F)$ into $\mathcal{L}_b(\gamma^m(E); F)$ and is a bounded mapping. On the other hand, if $\mathcal{Y}$ is an equibounded subset of $\mathcal{L}_b(\gamma^m(E); F)$, then

$$\psi^{-1}(\mathcal{Y}) = \mathcal{Y} \circ \theta$$

is an equibounded subset of $\mathcal{P}_b(^m E; F)$ because θ is bounded. Thus, ψ is onto and ψ^{-1} is also a bounded mapping. □

If E, F are two bornological A-modules and $u : E \to F$ is a bounded A-linear mapping, we define

$$u^{\otimes m} = \underbrace{u \otimes \cdots \otimes u}_{m \text{ times}},$$

which is a bounded A-linear mapping from $E^{\otimes m}$ into $F^{\otimes m}$. Since

$$u^{\otimes m}(x^{\otimes m}) = \big(u(x)\big)^{\otimes m} \quad \text{for all } x \in E,$$

it follows that $u^{\otimes m}(\gamma^m(E)) \subset \gamma^m(F)$. We denote by

$$\gamma^m(u) : \gamma^m(E) \to \gamma^m(F)$$

the bounded A-linear mapping obtained by restricting $u^{\otimes m}$ to $\gamma^m(E)$.

Let $(E_\alpha, u_{\beta\alpha})_{\alpha\in I}$ be a direct system of bornological A-modules. Consider the bornological A-module

$$E = \varinjlim E_\alpha$$

and let $u_\alpha : E_\alpha \to E$ be the canonical bounded A-linear mapping ($\alpha \in I$). For each $\alpha \leq \beta$ in I, we define

$$w_{\beta\alpha} = \gamma^m(u_{\beta\alpha}) : \gamma^m(E_\alpha) \to \gamma^m(E_\beta),$$

which is a bounded A-linear mapping. It is easy to show that

$$\big(\gamma^m(E_\alpha), w_{\beta\alpha}\big)_{\alpha\in I}$$

is a direct system of bornological A-modules. For each $\alpha \in I$, let

$$w_\alpha : \gamma^m(E_\alpha) \to \varinjlim \gamma^m(E_\alpha)$$

be the canonical bounded A-linear mapping. Now, for each $\alpha \in I$, we define

$$h_\alpha = \gamma^m(u_\alpha) : \gamma^m(E_\alpha) \to \gamma^m(E),$$

which is a bounded A-linear mapping. Since $h_\alpha = h_\beta \circ w_{\beta\alpha}$ whenever $\alpha \leq \beta$ in I, there exists a unique bounded A-linear mapping

$$h : \varinjlim \gamma^m(E_\alpha) \to \gamma^m(E)$$

such that $h_\alpha = h \circ w_\alpha$ for every $\alpha \in I$. The mapping h is said to be *canonical*. We summarize this discussion in the following

Proposition 26. *The canonical mapping*

$$h : \varinjlim \gamma^m(E_\alpha) \to \gamma^m(\varinjlim E_\alpha)$$

is a bounded A-linear mapping.

Since a direct sum is a special case of direct limit, we obtain the following

Corollary 27. *For every family $(E_\alpha)_{\alpha\in I}$ of bornological A-modules, the canonical mapping*

$$h : \bigoplus_{\alpha\in I} \gamma^m(E_\alpha) \to \gamma^m\Big(\bigoplus_{\alpha\in I} E_\alpha\Big),$$

which satisfies

$$h\big((x_\alpha^{\otimes m})_{\alpha\in I}\big) = \sum_{\alpha\in I} \big(\lambda_\alpha(x_\alpha)\big)^{\otimes m}$$

(where $\lambda_\alpha : E_\alpha \to \bigoplus_{\beta\in I} E_\beta$ is the canonical injection), is a bounded A-linear mapping.

The mapping h in the above corollary (and, in particular, the mapping h in Proposition 26) is not necessarily a bornological A-module isomorphism for algebraic reasons. For example, let K be a commutative field of characteristic zero and consider $E_1 = E_2 = F = K$ regarded as K-vector spaces. Since $\dim \gamma^2(E_1) = \dim \gamma^2(E_2) = 1$ and $\dim \gamma^2(E_1 \oplus E_2) = \dim \mathcal{L}_a(\gamma^2(E_1 \oplus E_2); F) = \dim \mathcal{P}_a(^2(E_1 \oplus E_2); F) = 3$, the K-vector spaces

$$\gamma^2(E_1 \oplus E_2) \quad \text{and} \quad \gamma^2(E_1) \oplus \gamma^2(E_2)$$

are not isomorphic.

The result corresponding to Corollary 22 (and, in particular, to Theorem 21) is also not necessarily true for modules of homogeneous polynomials. For example, let K, E_1, E_2 and F be as in the previous paragraphy. Since $\dim \mathcal{P}_a(^2E_1; F) = \dim \mathcal{P}_a(^2E_2; F) = 1$, the K-vector spaces

$$\mathcal{P}_a(^2(E_1 \oplus E_2); F) \quad \text{and} \quad \mathcal{P}_a(^2E_1; F) \times \mathcal{P}_a(^2E_2; F)$$

are not isomorphic.

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