Some Fixed Point Results for $w$-distances

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Abstract

In this paper, by using $w$-distances we improve and extend [Harjani, K. Sadrangani, ”Fixed point theorems for weakly contractive mappings in partially ordered sets” in Ninlinear Analysis 71 (2009) 3403-3410].

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1. Introduction

In 1996, Kada, Suzuki and Takahashi [1],[2] introduced the concept of $w$-distance on a metric space. They elaborated, with the help of examples, that the concept of $w$-distance is general than that of metric on a nonempty set. They also proved a generalization of Caristi fixed point theorem employing the definition of $w$-distance on a complete metric space.

In this paper, we prove some results for weakly contractive mappings in partially ordered sets by considering the concept of $w$-distance our results improve and extend [2].

Definition 1.1. Let $(X,d)$ be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a $w$-distance on $X$ if the followings are satisfied:
(1) \( p(x, z) \leq p(x, y) + p(y, z) \) for any \( x, y, z \in X \).

(2) For any \( x \in X \), \( p(x, \cdot) : X \to [0, \infty) \) is lower semicontinuous.

(3) For each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) implies \( d(x, y) \leq \varepsilon \).

Example 1.2. If \( X = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \cup \{0\} \) for each \( x, y \in X \), \( d(x, y) = x + y \) if \( x \neq y \) and \( d(x, y) = 0 \) if \( x = y \) is a metric on \( X \) and \( (X, d) \) is a complete metric space. Moreover, by defining \( p(x, y) = y \), \( p \) is a \( w \)-distance on \( (X, d) \).

Example 1.3. Let \( (X, d) \) be a metric space. Then a function \( p : X \times X \to [0, \infty) \) defined by \( p(x, y) = k \) for every \( x, y \in X \) is a \( w \)-distance on \( X \), where \( k \) is a positive real number. But \( p \) is not a metric since \( p(x, x) = k \neq 0 \) for any \( x \in X \).

Example 1.4. [10] Let \( (X, d) \) be a metric space and let \( g \) be a continuous mapping from \( X \) into itself. Then a function \( p : X \times X \to [0, \infty) \) defined by

\[
p(x, y) = \max \{ d(gx, y), d(gx, gy) \}
\]

for every \( x, y \in X \) is a \( w \)-distance on \( X \).

Definition 1.5. Let \( p \) be a \( w \)-distance on a metric space \( (X, d) \). Suppose that \( \Phi \) is the set of functions

\[
\phi : [0, \infty) \to [0, \infty)
\]

where \( \varphi \) is non-decreasing, continuous and \( \varphi(\varepsilon) > 0 \) for each \( \varepsilon > 0 \). Moreover, let \( \Psi \) be the set of functions

\[
\psi : [0, \infty) \to [0, \infty)
\]

where \( \psi \) is non-decreasing, right continuous and \( \psi(t) < t \) for all \( t > 0 \).

Example 1.6. Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{c_n\}_{n=1}^{\infty} \) are two non-negative sequences such that \( \{a_n\} \) strictly decreasing, convergence to zero, and for each \( n \in \mathbb{N} \),

\[
c_{n-1}a_n > a_{n+1} \quad \text{where} \quad 0 < c_{n-1} < 1
\]

define \( \psi : [0, \infty) \to [0, \infty) \) by \( \psi(0) = 0 \), \( \psi(t) = c_n t \), if \( a_{n+1} \leq t < a_n \), \( \psi(t) = c_0 t \) if \( t \geq a_1 \), then \( \psi \) is in \( \Psi \).
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Lemma 1.7. [9] If $\psi \in \Psi$, then $\lim_n \psi^n(t) = 0$ for each $t > 0$; and if $\varphi \in \Phi$, \{a_n\} $\subseteq [0, \infty)$ and $\lim_n \varphi(a_n) = 0$, then $\lim_n a_n = 0$.

Lemma 1.8. [7] Let $(X, d)$ be a metric space and $p$ be a $w$-distance on $X$. If \{x_n\} is a sequence in $X$ such that $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$, then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.

If $p(a, b) = p(b, a) = 0$. Then by lemma 1.8, $a = b$.

Lemma 1.9. [7] Let $p$ be a $w$-distance on a metric space $(X, d)$ and \{x_n\} be a sequence in $X$ such that for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ or

$$\lim_{m,n} p(x_n, x_m) = 0,$$

then \{x_n\} is a Cauchy sequence.

2. Main Results

Theorem 2.1. Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric $d$ in $X$ such that $(X, d)$ is a complete metric space and let $p$ be a $w$-distance on $X$. Let $f : X \longrightarrow X$ be a continuous and nondecreasing mapping such that

$$p(f(x), f(y)) \leq p(x, y) - \psi(p(x, y))$$  \hspace{1cm} (2.1)

for $x \geq y$ where $\psi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) > 0$ for $t > 0$, $\psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then $f$ has a fixed point.

Proof. If $f(x_0) = x_0$, then the proof is finished. Suppose that $x_0 < f(x_0)$. Since $x_0 < f(x_0)$ and $f$ is a nondecreasing function, we obtain by induction that

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots$$

Put $x_{n+1} = f(x_n)$. Then for each integer $n \geq 1$, from 2.1 and, as the elements $x_n$ and $x_{n+1}$ are comparable, we get

$$p(x_{n+1}, x_n) = p(f(x_n), f(x_{n-1})) \leq p(x_n, x_{n-1}) - \psi(p(x_n, x_{n-1})).$$
If there exists \( n_0 \in \mathbb{N} \) such that \( p(x_{n_0}, x_{n_0-1}) = 0 \), then
\[
x_{n_0} = f(x_{n_0-1}) = x_{n_0-1},
\]
and \( x_{n_0-1} \) is a fixed point and the proof is finished.

Now, suppose that \( p(x_{n+1}, x_n) \neq 0 \) for all \( n \in \mathbb{N} \). Then by 2.1 and our assumptions about \( \psi \)
\[
p(x_{n+1}, x_n) = p(x_n, x_{n-1}) - \psi(p(x_n, x_{n-1})) < p(x_n, x_{n-1}).
\]
Put \( \rho_n = p(x_{n+1}, x_n) \). Then we have
\[
\rho_n \leq \rho_{n-1} - \psi(\rho_{n-1}) \leq \rho_{n-1}.
\]
Therefore \( \{\rho_n\} \) is a nonnegative nonincreasing sequence and hence possesses a limit \( \rho^* \) by 2.2, when \( n \to \infty \) we have
\[
\rho^* \leq \rho^* - \psi(\rho^*) \leq \rho^*
\]
and, consequently, \( \psi(\rho^*) = 0 \). So, \( \rho^* = 0 \).

In what follows we will show that \( \{x_n\} \) is a Couchy sequence. Fix \( \varepsilon > 0 \). As \( p_n = p(x_{n+1}, x_n) \to 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[
p(x_{n_0+1}, x_{n_0}) \leq \min \left\{ \frac{\varepsilon}{2}, \psi(\frac{\varepsilon}{2}) \right\}.
\]
Note that \( \psi(\frac{\varepsilon}{2}) > 0 \). We claim that
\[
f\left( B(x_{n_0}, \varepsilon) \cap \{ y \in X : y \geq x_{n_0} \} \right) \subset B(x_{n_0}, \varepsilon).
\]
Let \( z \in B(x_{n_0}, \varepsilon) \cap \{ y \in X : y \geq x_{n_0} \} \). Then are two cases:

**case 1.** \( p(z, x_{n_0}) \leq \frac{\varepsilon}{2} \).

In this case, as \( z \) and \( x_{n_0} \) are comparable, we have
\[
p(f(z), x_{n_0}) \leq p(f(z), f(x_{n_0})) + p(f(x_{n_0}), x_{n_0})
\]
\[
= p(f(z), f(x_{n_0})) + p(x_{n_0+1}, x_{n_0})
\]
\[
\leq p(z, x_{n_0}) - \psi(p(z, x_{n_0})) + p(x_{n_0+1} x_{n_0})
\]
\[
\leq p(z, x_{n_0}) + p(x_{n_0+1} x_{n_0}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

**case 2.** \( \frac{\varepsilon}{2} < p(z, x_{n_0}) \leq \varepsilon \).

In this case, as \( \psi \) is nondecreasing,
\[
\psi(p(z, x_{n_0})) \geq \psi(\frac{\varepsilon}{2}).
\]
Therefore from 2.3 we have
\[
p(f(z), x_{n_0}) \leq p(f(z), f(x_{n_0})) + p(f(x_{n_0}), x_{n_0}) \\
= p(f(z), f(x_{n_0})) + p(x_{n_0+1}, x_{n_0}) \\
\leq p(z, x_{n_0}) - \psi(p(z, x_{n_0})) + p(x_{n_0+1}x_{n_0}) \\
\leq p(z, x_{n_0}) - \psi\left(\frac{\varepsilon}{2}\right) + \psi\left(\frac{\varepsilon}{2}\right) \leq p(z, x_{n_0}) \leq \varepsilon.
\]

This proves the claim.

As \(x_{n_0+1} \in B(x_{n_0}, \varepsilon) \cap \{y \in X : y \geq x_{n_0}\}\), we have
\[
f(x_{n_0+1}) = x_{n_0+2} \in B(x_{n_0}, \varepsilon) \cap \{y \in X : y \geq x_{n_0}\}
\]
Repeating this process, yields that \(x_n \in B(x_{n_0}, \varepsilon)\) for \(n \geq n_0\). Since \(E\) is arbitrary, \(\{x_n\}\) is a Cauchy sequence. Then there exists \(z \in X\) such that \(\lim_{n \to \infty} x_n = z\).
The continuity of \(f\) implies that \(z\) is a fixed point. This, the proof is complete. □

**Theorem 2.2.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space and let \(p\) be a \(w\)-distance on \(X\). Assume that \(X\) Satisfies:

If \(\{x_n\}\) is a nondecreasing sequence in \(X\) such that \(x_n \to x\) then
\[
x_n \leq x \quad \text{for all } n \in \mathbb{N} \quad (2.4)
\]
Let \(f : X \to X\) be a continuous and nondecreasing mapping such that
\[
p(f(x), f(y)) \leq p(x, y) - \psi(p(x, y))
\]
for \(x \geq y\) where \(\psi : [0, \infty) \to [0, \infty)\) is a continuous and nondecreasing function such that \(\psi(t) > 0\) for \(t > 0\), \(\psi(0) = 0\) and \(\lim_{t \to \infty} \psi(t) = \infty\). If there exists \(x_0 \in X\) with \(x_0 \leq f(x_0)\), then \(f\) has a fixed point.

**proof.** Following the proof of theorem 2.1, we only check that \(f(z) = z\). In fact, by using 2.4,
\[
p(f(z), z) \leq p(f(z), f(x_n)) + p(f(x_n), z)
\]
\[
= p(z, x_n) - \psi(p(z, x_n)) + p(x_{n+1}, z)
\]
and taking limit as \(n \to \infty\), \(p(f(z), z) \leq 0\) and this proves that \(p(f(z), z) = 0\). Consequently, \(f(z) = z\). □
Theorem 2.3. Adding condition

For \( x, y \in X \) there exists \( z \in X \) which is comparable to \( x \) and \( y \)

\[
(2.5)
\]

to the hypotheses of theorem 2.1 (resp. theorem 2.2) we obtain uniqueness of the fixed point of \( f \).

proof. Suppose that there exist \( z, y \in X \) which are fixed points. We distinguish two cases:

Case 1. If \( y \) is comparable to \( z \), then \( f^n(y) = y \) is comparable to \( f^n(z) = z \) for \( n = 0, 1, 2, \ldots \), and

\[
p(z, y) = p(f^n(z), f^n(y)) \leq p(f^{n-1}(z), f^{n-1}(y)) - \psi(p(f^{n-1}(z), f^{n-1}(y))) \leq p(z, y) - \psi(p(z, y)) \leq p(z, y).
\]

Consequently, \( \psi(p(z, y)) = 0 \) and this gives us that \( p(z, y) = 0 \).

Case 2. If \( y \) is not comparable to \( z \), then there exists \( x \in X \) comparable to \( y \) and \( z \). Monotonicity implies that \( f^n(x) \) is comparable to \( f^n(y) = y \) and \( f^n(z) = z \) for \( n = 0, 1, 2, \ldots \). Moreover,

\[
p(z, f^n(x)) = p(f^n(z), f^n(x)) \leq p(f^{n-1}(z), f^{n-1}(x)) - \psi(p(f^{n-1}(z), f^{n-1}(x))) \leq p(f^{n-1}(z), f^{n-1}(x)).
\]

Consequently, \( p(z, f^n(x)) = p(f^n(z), f^n(x)) \) is a nonnegative nonincreasing sequence and hence possesses limit \( \gamma \).

From the last inequality we can obtain \( \gamma \leq \gamma - \psi(\gamma) \leq \gamma \) and hence \( \psi(\gamma) = 0 \), so \( \gamma = 0 \). Analogously, it can be proved that

\[
\lim_{n \to \infty} p(y, f^n(x)) = 0.
\]

Finally,

\[
p(z, y) \leq p(z, f^n(x)) + p(f^n(x), y)
\]

and taking limit we have \( p(z, y) = 0 \). \(\square\)

Now, we deal with nonincreasing functions.
Theorem 2.4. Let \((X, \leq)\) be a partially ordered set verifying 2.5 and suppose that there exists a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space and \(p\) is a \(w\)-distance on \(X\).

Let \(f : X \rightarrow X\) be a nonincreasing function such that
\[
p(f(x), f(y)) \leq p(x, y) - \psi(p(x, y))
\]
for \(x \geq y\) where \(\psi : [0, \infty) \rightarrow [0, \infty)\) satisfies the conditions appearing in theorem 2.2. Suppose also that either \(f\) is a continuous, or \(X\) is such that if \((x_n) \rightarrow x\) is a sequence in \(X\) whose consecutive terms are comparable,
then there exists a subsequence \(\{x_{nk}\}\) of \(\{x_n\}\) such that every term is comparable to the limit \(x\).

(2.6)

If there exists \(x_0 \in X\) with \(x_0 \leq f(x_0)\) or \(x_0 \geq f(x_0)\) then \(f\) has a unique fixed point.

proof. If \(f(x_0) = x_0\), then the existence of a fixed point is proved. Suppose that \(f(x_0) \neq x_0\). Following the lines of the proof of theorem 2.1, we obtain that \(\{f^n(x_0)\}\) is a convergent sequence in \(X\). Indeed, by our assumption \(f^{n+1}(x_0)\) and \(f^n(x_0)\) are comparable, for every \(n = 0, 1, 2, \ldots\) and therefore, by induction
\[
p(f^{n+1}(x_0), f^n(x_0)) \leq p(f^n(x_0), f^{n-1}(x_0)) - \psi(p(f^n(x_0), f^{n-1}(x_0)))
\]
\[
\leq p(f^n(x_0), f^{n-1}(x_0)).
\]
This proves that the sequence \(\{p(f^{n+1}(x), f^n(x))\}\) is a nonnegative nonincreasing sequence with limit \(\rho^*\). Using the same argument that in theorem 2.1, we prove that \(\rho^* = 0\).

The same reasoning that in theorem 2.1 gives us that \(\{f^n(x_0)\}\) is a Cauchy sequence and consequently \(\{f^n(x_0)\}\) is convergent to some \(z \in X\).

In the case that \(f\) is continuous it is easily seen that \(z\) is a fixed point.

Suppose that condition 2.6 holds. Since \(f\) is nonincreasing \(\{f^n(x_0)\}\) is not necessarily monotone, but it is a convergent sequence with comparable consecutive terms. Then, by condition 2.5, there exists a subsequence \(\{f_{nk}(x_0)\}\) consisting of terms which are comparable to the limit \(z\). Hence, for \(k \in \mathbb{N}\)
\[
p(f(z), z) \leq p(f(z), f_{nk+1}(x_0)) + p(f_{nk+1}(x_0), z)
\]
\[
\leq p(z, f_{nk}(x_0)) - \psi(p(z, f_{nk}(x_0))) + p(f_{nk+1}(x_0), z)
\]
\[ \leq p(z, f^{nk}(x_0)) + p(f^{nk+1}(x_0), z). \]

Taking limit as \( k \to \infty \), we obtain that \( p(f(z), z) = 0 \). The uniqueness of the fixed point is proved as in theorem 2.3. □

Finally, we show that the monotonicity of \( f \) is not essential for the existence of a fixed point, we replace this condition by the preservation of comparable elements which is trivially verified if \( X \) is totally ordered.

**Theorem 2.5.** Let \((X, \leq)\) be a partially ordered set and suppose that 2.5 holds and that there exists a metric \( d \) in \( X \) such that \((X, d)\) is a complete metric space and let \( p \) be a \( w \)-distance on \( X \). Let \( f : X \to X \) be such that \( f \) maps comparable elements into comparable elements, that is, for \( x, y \in X \),
\[ x \leq y \implies f(x) \leq f(y) \quad \text{or} \quad f(x) \geq f(y) \]
and such that for \( x, y \in X \) with \( x \geq y \)
\[ p(f(x), f(y)) \leq p(x, y) - \psi(p(x, y)) \]
where \( \psi : [0, \infty) \to [0, \infty) \) satisfies the conditions appearing in theorem 2.1. Suppose that either \( f \) is continuous or \( X \) is such that condition 2.6 holds. If there exists \( x_0 \in X \) with \( x_0 \) comparable to \( f(x_0) \), then \( f \) has a unique fixed point \( \bar{x} \). Moreover, for \( x \in X \), \( \lim_{n \to \infty} f^n(x) = \bar{x} \).

**proof.** Since \( x_0 \in X \) is comparable to \( f(x_0) \), then \( f^{n+1}(x_0) \) and \( f^n(x_0) \) are comparable for \( n = 0, 1, 2, ... \) the argument exposed in the proof of theorem 2.4 is valid. □

**References**


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