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Some Fixed Point Results for w-distances

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Abstract

In this paper, by using w-distances we improve and extend [Harjani, K. Sadrangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets" in Ninlinear Analysis 71 (2009) 3403-3410.

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1. Introduction

In 1996, Kada, Suzuki and Takahashi [1],[2] introduced the concept of w-distance on a metric space. They elaborated, with the help of examples, that the concept of w-distance is general than that of metric on a nonempty set. They also proved a generalization of Caristi fixed point theorem employing the definition of w-distance on a complete metric space.

In this paper, we prove some results for weakly contractive mappings in partially ordered sets by considering the concept of w-distance our results improve and extend [2].

Definition 1.1. *Let (X, d) be a metric space. Then a function $p : X \times X \longrightarrow [0, \infty)$ is called a w-distance on X if the followings are satisfied:*

(1) *$p(x,z) \leq p(x,y) + p(y,z)$ for any $x, y, x \in X$.*
(2) *For any $x \in X$, $p(x,.) : X \longrightarrow [0,\infty)$ is lower semicontinuous.*
(3) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.*

Example 1.2. *If $X = \{\frac{1}{n} \,|n \in \mathbb{N}\} \cup \{0\}$ for each $x, y \in X$, $d(x,y) = x + y$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$ is a metric on X and (X,d) is a complete metric space. Moreover, by defining $p(x,y) = y$, p is a w-distance on (X,d).*

Example 1.3. *Let (X,d) be a metric space. Then a function $p : X \times X \longrightarrow [0,\infty)$ defined by $p(x,y) = k$ for every $x, y \in X$ is a w-distance on X, where k is a positive real number. But p is not a metric since $p(x,x) = k \neq 0$ for any $x \in X$.*

Example 1.4. [10] *Let (X,d) be a metric space and let g be a continuous mapping from X into itself. Then a function $p : X \times X \longrightarrow [0,\infty)$ defined by*

$$p(x,y) = \max\{d(gx,y), d(gx,gy)\}$$

for every $x, y \in X$ is a w-distance on X.

Definition 1.5. *Let p be a w-distance on a metric space (X,d). Suppose that Φ is the set of functions*

$$\phi : [0,\infty) \longrightarrow [0,\infty)$$

where φ is non-decreasing, continuous and $\varphi(\varepsilon) > 0$ for each $\varepsilon > 0$. Moreover, let Ψ be the set of functions

$$\psi : [0,\infty) \longrightarrow [0,\infty)$$

where ψ is non-decreasing, right continuous and $\psi(t) < t$ for all $t > 0$.

Example 1.6. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are two non-negative sequences such that $\{a_n\}$ strictly decreasing, convergence to zero, and for each $n \in \mathbb{N}$,*

$$c_{n-1}a_n > a_{n+1} \qquad \textit{where} \qquad 0 < c_{n-1} < 1$$

define $\psi : [0,\infty) \longrightarrow [0,\infty)$ by $\psi(0) = 0$, $\psi(t) = c_n t$, if $a_{n+1} \leq t < a_n$, $\psi(t) = c_0 t$ if $t \geq a_1$, then ψ is in Ψ.

Lemma 1.7. [9] *If $\psi \in \Psi$, then $\lim_n \psi^n(t) = 0$ for each $t > 0$; and if $\varphi \in \Phi$, $\{a_n\} \subseteq [0, \infty)$ and $\lim_n \varphi(a_n) = 0$, then $\lim_n a_n = 0$.*

Lemma 1.8. [7] *Let (X, d) be a metric space and p be a w-distance on X. If $\{x_n\}$ is a sequence in X such that*

$$\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0,$$

then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.

If $p(a, b) = p(b, a) = 0$. Then by lemma 1.8, $a = b$.

Lemma 1.9. [7] *Let p be a w-distance on a metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ or*

$$\lim_{m,n} p(x_n, x_m) = 0,$$

then $\{x_n\}$ is a Cauchy sequence.

2. Main Results

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d in X such that (X, d) is a complete metric space and let p be a w-distance on X. Let $f : X \longrightarrow X$ be a continuous and nondecreasing mapping such that*

$$p\big(f(x), f(y)\big) \leq p(x, y) - \psi\big(p(x, y)\big) \tag{2.1}$$

for $x \geq y$ where $\psi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) > 0$ for $t > 0$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

proof. *If $f(x_0) = x_0$, then the proof is finished. Suppose that $x_0 < f(x_0)$. Since $x_0 < f(x_0)$ and f is a nondecreasing function, we obtain by induction that*

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq ... \leq f^n(x_0) \leq f^{n+1}(x_0) \leq ....$$

Put $x_{n+1} = f(x_n)$. Then for each integer $n \geq 1$, from 2.1 and, as the elements x_n and x_{n+1} are comparable, we get

$$p(x_{n+1}, x_n) = p\big(f(x_n), f(x_{n-1})\big) \leq p(x_n, x_{n-1}) - \psi\big(p(x_n, x_{n-1})\big).$$

If there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, x_{n_0-1}) = 0$, then

$$x_{n_0} = f(x_{n_0-1}) = x_{n_0-1},$$

and x_{n_0-1} is a fixed point and the proof is finished.

Now, suppose that $p(x_{n+1}, x_n) \neq 0$ for all $n \in \mathbb{N}$. Then by 2.1 and our assumptions about ψ

$$p(x_{n+1}, x_n) = p(x_n, x_{n-1}) - \psi\big(p(x_n, x_{n-1})\big) < p(x_n, x_{n-1}).$$

Put $\rho_n = p(x_{n+1}, x_n)$. Then we have

$$\rho_n \leq \rho_{n-1} - \psi\big(\rho_{n-1}\big) \leq \rho_{n-1}. \tag{2.2}$$

Therefore $\{\rho_n\}$ is a nonnegative nonincreasing sequence and hence possesses a limit ρ^* by 2.2, when $n \to \infty$ we have

$$\rho^* \leq \rho^* - \psi(\rho^*) \leq \rho^*$$

and, consequently , $\psi(\rho^*) = 0$. So, $\rho^* = 0$.

In what follows we will show that $\{x_n\}$ is a Couchy sequence. Fix $\varepsilon > 0$. As $p_n = p(x_{n+1}, x_n) \to 0$, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_{n_0+1}, x_{n_0}) \leq \min\Big\{\frac{\varepsilon}{2}, \psi(\frac{\varepsilon}{2})\Big\}. \tag{2.3}$$

Note that $\psi(\frac{\varepsilon}{2}) > 0$. We claim that

$$f\Big(\overline{B(x_{n_0}, \varepsilon)} \cap \big\{y \in X : y \geq x_{n_0}\big\}\Big) \subset \overline{B(x_{n_0}, \varepsilon)}.$$

Let $z \in \overline{B(x_{n_0}, \varepsilon)} \cap \big\{y \in X : y \geq x_{n_0}\big\}$. Then are two cases:

case 1. $p(z, x_{n_0}) \leq \frac{\varepsilon}{2}$.

In this case, as z and x_{n_0} are comparable, we have

$$\begin{aligned} p(f(z), x_{n_0}) &\leq p\big(f(z), f(x_{n_0})\big) + p\big(f(x_{n_0}), x_{n_0}\big) \\ &= p\big(f(z), f(x_{n_0})\big) + p\big(x_{n_0+1}, x_{n_0}\big) \\ &\leq p(z, x_{n_0}) - \psi\big(p(z, x_{n_o})\big) + p(x_{n_0+1} x_{n_0}) \\ &\leq p(z, x_{n_0}) + p(x_{n_0+1} x_{n_0}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

case 2. $\frac{\varepsilon}{2} < p(z, x_{n_0}) \leq \varepsilon$.

In this case, as ψ is nondecreasing,

$$\psi\big(p(z, x_{n_o})\big) \geq \psi(\frac{\varepsilon}{2}).$$

Therefore from 2.3 we have

$$\begin{aligned} p(f(z), x_{n_0}) &\le p\big(f(z), f(x_{n_0})\big) + p\big(f(x_{n_0}), x_{n_0}\big) \\ &= p\big(f(z), f(x_{n_0})\big) + p\big(x_{n_0+1}, x_{n_0}\big) \\ &\le p(z, x_{n_0}) - \psi\big(p(z, x_{n_o})\big) + p(x_{n_0+1}x_{n_0}) \\ &\le p(z, x_{n_0}) - \psi\big(\frac{\varepsilon}{2}\big) + \psi\big(\frac{\varepsilon}{2}\big) \le p(z, x_{n_0}) \le \varepsilon. \end{aligned}$$

This proves the claim.

As $x_{n_0+1} \in \overline{B(x_{n_0}, \varepsilon)} \cap \{y \in X : y \ge x_{n_0}\}$, *we have*

$$f(x_{n_0+1}) = x_{n_0+2} \in \overline{B(x_{n_0}, \varepsilon)} \cap \{y \in X : y \ge x_{n_0}\}$$

Repeating this process, yields that $x_n \in \overline{B(x_{n_0}, \varepsilon)}$ *for* $n \ge n_0$. *Since* E *is arbitrary,* $\{x_n\}$ *is a Cauchy sequence. Then there exists* $z \in X$ *such that* $\lim_{n\to\infty} x_n = z$.

The continuity of f *implies that* z *is a fixed point. This, the proof is complete.* □

Theorem 2.2. *Let* $(X, \le)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space and let* p *be a* w*-distance on* X. *Assume that* X *Satisfies:*

If $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to x$ *then*

$$x_n \le x \qquad \text{for all} \quad n \in \mathbb{N} \tag{2.4}$$

Let $f : X \longrightarrow X$ *be a continuous and nondecreasing mapping such that*

$$p\big(f(x), f(y)\big) \le p(x, y) - \psi\big(p(x, y)\big)$$

for $x \ge y$ *where* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *is a continuous and nondecreasing function such that* $\psi(t) > 0$ *for* $t > 0$, $\psi(0) = 0$ *and* $\lim_{t\to\infty} \psi(t) = \infty$. *If there exists* $x_0 \in X$ *with* $x_0 \le f(x_0)$, *then* f *has a fixed point.*

proof. *Following the proof of theorem 2.1, we only check that* $f(z) = z$. *In fact, by using 2.4,*

$$\begin{aligned} p(f(z), z) &\le p\big(f(z), f(x_n)\big) + p\big(f(x_n), z\big) \\ &= p\big(z, x_n\big) - \psi\Big(p\big(z, x_n\big)\Big) + p(x_{n+1}, z) \end{aligned}$$

and taking limit as $n \to \infty$, $p(f(z), z) \le 0$ *and this proves that* $p(f(z), z) = 0$. *Consequently,* $f(z) = z$. □

Theorem 2.3. Adding condition
For $x, y \in X$ there exists $z \in X$ which is comparable to

$$x \quad and \quad y \tag{2.5}$$

to the hypotheses of theorem 2.1 (resp. theorem 2.2) we obtain uniqueness of the fixed point of f.

proof. *Suppose that there exist $z, y \in X$ which are fixed points. We distinguish two cases:*

Case 1. *If y is comparable to z, then $f^n(y) = y$ is comparable to $f^n(z) = z$ for $n = 0, 1, 2, ...,$ and*

$$\begin{aligned} p(z,y) &= p\big(f^n(z), f^n(y)\big) \\ &\leq p\big(f^{n-1}(z), f^{n-1}(y)\big) - \psi\Big(p\big(f^{n-1}(z), f^{n-1}(y)\big)\Big) \\ &\leq p(z,y) - \psi(p(z,y) \leq p(z,y). \end{aligned}$$

Consequently, $\psi\big(p(z,y)\big) = 0$ and this gives us that $p(z,y) = 0$.

Case 2. *If y is not comparable to z, then there exists $x \in X$ comparable to y and z. Monotonicity implies that $f^n(x)$ is comparable to $f^n(y) = y$ and $f^n(z) = z$ for $n = 0, 1, 2, ...$ Moreover,*

$$\begin{aligned} p(z, f^n(x)) &= p\big(f^n(z), f^n(x)\big) \\ &\leq p\big(f^{n-1}(z), f^{n-1}(x)\big) - \psi\Big(p\big(f^{n-1}(z), f^{n-1}(x)\big)\Big) \\ &\leq p\big(f^{n-1}(z), f^{n-1}(x)\big). \end{aligned}$$

Consequently, $p(z, f^n(x)) = p\big(f^n(z), f^n(x)\big)$ is a nonnegative nonincreasing sequence and hence possesses limit γ.
From the last inequality we can obtain $\gamma \leq \gamma - \psi(\gamma) \leq \gamma$ and hence $\psi(\gamma) = 0$, so $\gamma = 0$. Analogously, it can be proved that

$$\lim_{n\to\infty} p\big(y, f^n(x)\big) = 0.$$

Finally,

$$p(z,y) \leq p\big(z, f^n(x)\big) + p\big(f^n(x), y\big)$$

and taking limit we have $p(z,y) = 0$. □

Now, we deal with nonincreasing functions.

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set varifying 2.5 and suppose that there exists a metric d in X such that (X, d) is a complete metric space and p is a w-distance on X.*

Let $f : X \longrightarrow X$ be a nonincreasing function such that

$$p\big(f(x), f(y)\big) \leq p(x, y) - \psi\big(p(x, y)\big)$$

for $x \geq y$ where $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfies the conditions appearing in theorem 2.2. Suppose also that either f is a continuous,
or
X is such that if $(x_n) \to x$ is a sequence in X whose consecutive terms are comparable,
then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x.

$$(2.6)$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$ then f has a unique fixed point.

proof. *If $f(x_0) = x_0$, then the existence of a fixed point is proved. Suppose that $f(x_0) \neq x_0$. Following the lines of the proof of theorem 2.1, we obtain that $\{f^n(x_0)\}$ is a convergent sequence in X.*
Indeed, by our assumption $f^{n+1}(x_0)$ and $f^n(x_0)$ are comparable, for every $n = 0, 1, 2, ...$ and therefore, by induction

$$\begin{aligned} p(f^{n+1}(x_0), f^n(x_0)) &\leq p\big(f^n(x_0), f^{n-1}(x_0)\big) - \psi\Big(p\big(f^n(x_0), f^{n-1}(x_0)\big)\Big) \\ &\leq p\big(f^n(x_0), f^{n-1}(x_0)\big). \end{aligned}$$

This proves that the sequence $\{p(f^{n+1}(x), f^n(x_0))\}$ is a nonnegative nonincreasing sequence with limit ρ^*. Using the same argument that in theorem 2.1, we prove that $\rho^* = 0$.
The same reasoning that in theorem 2.1 gives us that $\{f^n(x_0)\}$ is a cauchy sequence and consequently $\{f^n(x_0)\}$ is convergent to some $z \in X$.
In the case that f is continuous it is easily seen that z is a fixed point.

Suppose that condition 2.6 holds. Since f is nonincreasing $\{f^n(x_0)\}$ is not necessarily monotone, but it is a convergent sequence with comparable consecutive terms. Then, by condition 2.5, there exists a subsequence $\{f^{n_k}(x_0)\}$ consisting of terms which are comparable to the limit z. Hence, for $k \in \mathbb{N}$

$$\begin{aligned} p(f(z), z) &\leq p\big(f(z), f^{n_k+1}(x_0)\big) + p\big(f^{n_k+1}(x_0), z\big) \\ &\leq p\big(z, f^{n_k}(x_0)\big) - \psi\Big(p\big(z, f^{n_k}(x_0)\big)\Big) + p(f^{n_k+1}(x_0), z) \end{aligned}$$

$$\leq p\big(z, f^{n_k}(x_0)\big) + p(f^{n_k+1}(x_0), z).$$

Taking limit as $k \to \infty$, we obtain that $p(f(z), z) = 0$. The uniqueness of the fixed point is proved as in theorem 2.3. □

Finally, we show that the monotonicity of f is not essential for the existence of a fixed point, we replace this condition by the preservation of comparable elements which is trivially verified if X is totally ordered.

Theorem 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose that 2.5 holds and that there exists a metric d in X such that (X, d) is a complete metric space and let p be a w-distance on X. Let $f : X \longrightarrow X$ be such that f maps comparable elements into comparable elements, that is, for $x, y \in X$,*

$$x \leq y \quad \Longrightarrow \quad f(x) \leq f(y) \qquad \textit{or} \qquad f(x) \geq f(y)$$

and such that for $x, y \in X$ with $x \geq y$

$$p\big(f(x), f(y)\big) \leq p(x, y) - \psi\big(p(x, y)\big)$$

where $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfies the conditions appearing in theorem 2.1. Suppose that either f is continuous or X is such that condition 2.6 holds. If there exists $x_0 \in X$ with x_0 comparable to $f(x_0)$, then f has a unique fixed point $\overline{x}$. Moreover, for $x \in X$, $\lim_{n\to\infty} f^n(x) = \overline{x}$.

proof. *Since $x_0 \in X$ is comparable to $f(x_0)$, then $f^{n+1}(x_0)$ and $f^n(x_0)$ are comparable for $n = 0, 1, 2, ...$ the argument exposed in the proof of theorem 2.4 is valid.* □

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