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On the Proof of the Analytic Form of the Hahn Banach Theorem in Real Linear Spaces

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Abstract

Let X be a real linear space and M be a proper subspace of X. Let $p : X \to \Re$ be a sublinear functional on X and $f : M \to \Re$ be a linear functional on M satisfying $f(x) \leq p(x)$ for all $x \in M$. We observe that the proof of the well-definedness of the upper bound functionals for chains in the set of linear functionals extending f, appears not to be plausible in many expositions. We put this aright.

Introduction

We give some definitions which we shall use in the sequel.

Definition 1.1 Let X and Y be linear spaces, over a scalar field f. Let $T : X \to Y$ be any map. Then

(i) T is said to be linear if:

(a) $T(\alpha x) = \alpha T(x), \forall x \in X, \alpha \in f$

(b) $T(x + y) = T(x) + T(y) \ \ \forall x, y \in X$

(ii) $T(0) = 0$ if T is a linear map

(iii) T is said to be well-defined if:
(a) for each $x, y \in X$ such that $x = y$, we have $Tx = Ty$
(b) T is one - to - one
Condition $(iii)(a)$ is weaker than condition $(iii)(b)$. Condition $(iii)(b)$ is more of interest when the invertibility of T is in question. The question of well-definedness of functions, in Analysis, is a crucial one, as it prevents us from working with ambiguous relations.

Preliminaries

The Hahn Banach theorem (analytic form) is a theorem which deals with the extensions of linear functionals from subspaces of linear spaces to the whole space, satisfying certain properties. It is a very important theorem as it is extensively applied in Functional Analysis for the existence of linear functionals, separation of convex sets, proofs of weak topological concepts (for example in weakly closed sets) etc. The analytic form in real linear spaces is the basic result from which the various other/geometric forms are obtained. We give a statement of the theorem.

Theorem (Analytic Form of Hahn Banach): Let X be a real linear space and M be a proper subspace of X. Let $p : X \to \Re$ be a sublinear functional and $f : M \to \Re$ be a linear functional from M into $\Re$, satisfying $f(x) \leq p(x), \forall x \in M$. Then (i) there exist a linear functional $F : X \to \Re$ which is an extension of f. (ii) $F(x) \leq p(x), \forall x \in X$

Sketch of Proof

The proof of this theorem is usually divided into two parts. The first part consist of picking a fixed $x_0 \in X - M$ and generating the subspace $N = [\{x_0\} \bigcup M] := \{m + \lambda x_0 : m \in M, \lambda \in \Re\}$. The proof is given for $X = N$ (i.e we extend f from M to N, satisfying the desired properties). Finally, the second part consists of applying Zorn's lemma, to give a proof for the whole space X. To do this, we define the set S of all linear functionals,G, extending f and satisfying $G(x) \leq p(x), \forall x \in dom(G)$, where dom(G) denotes domain of G. We now introduce a partial ordering ,$<$, on S by domain-set inclusion i.e $G_1 < G_2$ if and only if $domG_1 \subset domG_2$ and $G_2|_{domG_1} = G_1$. Let I be an index set and let $C = \{G_\alpha, \alpha \in I\}$ be an arbitrary chain in S. Consider $D = \bigcup_{\alpha \in I} domG_\alpha$ and define a functional $G^* : D \to \Re$ by $G^*(x) = G_{\alpha_0}(x), \forall x \in domG_{\alpha_0}$, for some $\alpha_0 \in I$. Among the conditions that G^* must satisfy, is the well-definedness of

G^*.

Many authors (see e.g [1], [2], [3], [4], [5], [6]) have taken on the proof of this theorem. However, in their expositions, many (see e.g [5], [6]) choose to remain silent about the proof of the well-definedness of G^* while others (see e.g [1], [2], [3], [4]) give a proof of the well-definedness of G^* which does not appear to be plausible. They consider $x \in G_\alpha$ and $x \in G_\beta$, and using the definition of G^*, end up with $G_\alpha(x) = G_\beta(x)$ and then conclude that this yields the well-definedness of G^* since C is a chain and either G_α extends G_β or vice versa. This does not appear to be plausible.

A plausible Proof of the Well-definedness of G^*

Here, we give a more plausible proof of the well-definedness of G^*. We let $x, y \in D$ be such that $x = y$. Then $x \in G_{\alpha_0}$ and $y \in G_{\alpha_0}$, for some $\alpha_0 \in I$. From the definition of G^*, we have $G^*(x) = G_{\alpha_0}(x)$ and $G^*(y) = G_{\alpha_0}(y)$.
Using the linearity if G_{α_0}, for each $\alpha_0 \in I$, this implies

$$\begin{aligned} G^*(x) - G^*(y) &= G_{\alpha_0}(x) - G_{\alpha_0}(y) \\ &= G_{\alpha_0}(x - y) \\ &= G_{\alpha_0}(0), \text{ since x = y} \\ &= 0 \end{aligned}$$

This implies $G^*(x) = G^*(y)$, i.e G^* is well-defined.

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