On the Curves of Constant Breadth in Four Dimensional Galilean Space

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Abstract

In this work, we obtained characterization curves of constant breadth in four dimensional Galilean space in terms of Frenet-Serret vector fields. First we investigate an explicit characterization of curves of constant breadth of four dimensional Galilean space. In addition, it has been observed that differential equation of 4th order of a constant breadth curve in four dimensional Galilean space. Finally, some differential and integral characterizations of the mentioned curves are expressed by the classical differential geometry methods.

1 Introduction

Curves of constant breadth were introduced by L. Euler in 1870. Ö. Köse (1984) wrote some geometric properties of plane curves of constant breadth. And, in another work Ö. Köse (1986) extended these properties to the Euclidean 3-space E^3 . F. Reuleaux gave a method obtaining some curves of constant breadth and found use in the kinematics of machinery.

Furthermore, W. Blaschke defined the curve of constant breadth on the sphere and M. Fujivara had obtained a problem to determine whether there exist “space curve of constant breadth” or not, and he defined “breadth” for space curves and obtained these curves on a surface of constant breadth. A. Mağden and Ö. Köse (1997) studied this kind curves in four dimensional Euclidean space E^4 . S. Yılmaz and M. Turgut expressed some characterizations of time-like curves of constant breadth in Minkowski 3-space and partially null curves of constant breadth in semi-Riemannian space in [14-15]. And recently this topic were studied in higher dimensions and further characterizations related to different geometries were obtained, see [1], [4], [6-8] and [11-13], S. Yılmaz (2010) defined Frenet-Serret frame in four dimensional Galilean Space, see [16].

Galilean geometry makes its extensive development an easy matter, and extensive development of a new geometric system is a precondition for an effective comparison of it with Euclidean geometry. Also, extensive development is likely to give the student the psychological assurance of the consistency of the investigated structure. Another distinction of Galilean geometry is the fact that it exemplifies the fruitful geometric idea of duality, see [17]. The popular name “Galilean geometry” is historically inaccurate: Galilean, whose works date from the beginning of the 17th century, did not in fact know this geometry, whose discovery was necessarily preceded by one of the greatest intellectual triumphs of the 19th century - the emergence of the idea that many legitimate geometric systems exist. A more accurate name would be “the geometry associated with the Galilean principle of the relativity”.

2 Preliminaries

The study of mechanics of plane-parallel motions reduces to the study of a geometry of three dimensional space with coordinates $\{x, y, t\}$ is given by the motion formula Yaglom (1979).

$$\begin{aligned}x^1 &= (\cos \alpha)x + (\sin \alpha)y + (v \cos \beta)t + a, \\y^1 &= -(\sin \alpha)x + (\cos \alpha)y + (v \sin \beta)t + b \\t^1 &= t + d.\end{aligned}$$

This geometry can be called three-dimensional Galilean Geometry. Yaglom (1979) stressed that four-dimensional Galilean Geometry, which studies all properties invariant under motions of objects in space, is even more complex. Yaglom (1979) also stated this geometry can be described more precisely as the study of those properties of four-dimensional space with coordinates that are invariant under the general Galilean transformations.

$$\begin{aligned}x^1 &= (\cos \beta \cos \alpha - \cos \gamma \sin \beta \sin \alpha)x + (\sin \beta \cos \alpha - \cos \gamma \cos \beta \sin \alpha)y + (\sin \gamma \sin \alpha)z + (v \cos \delta_1)t + a, \\y^1 &= -(\cos \beta \sin \alpha + \cos \gamma \sin \beta \cos \alpha)x + (-\sin \beta \sin \alpha + \cos \gamma \cos \beta \cos \alpha)y + (\sin \gamma \cos \alpha)z + (v \cos \delta_2)t + b, \\z^1 &= (\sin \gamma \sin \beta)x - (\sin \gamma \cos \beta)y + (\cos \gamma)z + (v \cos \delta_3)t + c, \\t^1 &= t + d.\end{aligned}$$

$$\text{with } \cos^2 \delta_1 + \cos^2 \delta_2 + \cos^2 \delta_3 = 1$$

In the light of the motion equations above, we observe that the geometry of four-dimensional Galilean geometry roughly restricted. In this work, we intend to study theory of constant breadth of curves expressed in Röschel(1986) and Öğrenmiş et al.(2009) to the Galilean 4-space.

Let $\alpha : I \subset \mathbb{R} \longrightarrow G^4$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t), w(t)),$$

where $x(t), y(t), z(t), w(t) \in C^4$ (the set of three times continuously differentiable functions) and t run through in a real interval. Let α be a curve in G^4 , parameterized by arc length $t=s$, given coordinate form

$$\alpha(s) = (s, y(s), z(s), w(s))$$

In affine coordinates between two points the scalar product is defined by

$$g(P_1, P_2) = \begin{cases} |x_{21} - x_{11}| & , \text{ if } x_{11} \neq x_{21} \\ \sqrt{(x_{22} - x_{12})^2 + (x_{23} - x_{13})^2 + (x_{24} - x_{14})^2} & , \text{ if } x_{11} = x_{21} \end{cases}.$$

where, $P_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$, $i = 1, 2$

We define the Galilean cross product in G^4 for the vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4)$ as follows:

$$x \wedge y \wedge z = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where e_i are standard basis vectors. ($i=1, 2, 3, 4$)

Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc length s in G^4 . Here, we denote differentiation with respect to s by a dash. The real valued function

$$\kappa(s) = \|t'\| = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2}$$

is called the first curvature of the curve α . we define the second and third curvature function as, respectively

$$\tau(s) = \|n'(s)\|$$

$$\sigma(s) = \langle b', e \rangle_G$$

In the rest of the paper, we shall suppose $\kappa(s) \neq 0, \tau(s) \neq 0, \sigma(s) \neq 0$ at everywhere.

The orthonormal quad of the curve α is defined by

$$t(s) = \alpha'(s) = (1, y'(s), z'(s), w'(s))$$

$$n(s) = \frac{1}{\kappa(s)} (0, y''(s), z''(s), w''(s))$$

$$b(s) = \frac{1}{\tau(s)} \left(0, \left(\frac{1}{\kappa(s)} y''(s) \right)', \left(\frac{1}{\kappa(s)} z''(s) \right)', \left(\frac{1}{\kappa(s)} w''(s) \right)' \right)$$

$$e(s) = \mu t \wedge n \wedge b.$$

Here the coefficient μ is taken ± 1 to make $+1$ the matrix $[t, n, b, e]$

The vectors $t(s)$, $n(s)$, $b(s)$ are called the tangent, principal normal, the binormal line and trinormal line respectively. For their derivatives, the following Frenet formulas hold

$$\begin{aligned}t'(s) &= \kappa(s)n(s) \\n'(s) &= -\tau(s)b(s) \\b'(s) &= \tau(s)n(s) + \sigma(s)e(s) \\e'(s) &= \sigma(s)b(s) \\&\text{see [16].}\end{aligned}$$

3 Curves of Constant Breadth in Galilean Space G^4

Let $\vec{\varphi} = \vec{\varphi}(s)$ and $\vec{\varphi}^* = \vec{\varphi}^*(s)$ be simple closed curve in Galilean space G^4 . These curves will be denoted by C . The normal plane at every point P on the curve meets the curve at a single point Q other than P . We call the point Q the opposite point of P . We consider a curve in the class Γ as in M. Fujivara (1914) having parallel tangents $\vec{T}$ and $\vec{T}^*$ in opposite directions at the opposite points φ and φ^* of the curve. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\vec{\varphi}^* = \vec{\varphi} + m_1 \vec{T} + m_2 \vec{N} + m_3 \vec{B} + m_4 \vec{E} \quad (3.1)$$

where $m_i(s)$, $1 \leq i \leq 4$ arbitrary functions of s and $\vec{\varphi}$ and $\vec{\varphi}^*$ are opposite points. Vector $d = \vec{\varphi}^* - \vec{\varphi}$ is called “the distance vector” of C . Differentiating both sides of (3.1) and considering Frenet equations, we have

$$\begin{aligned}\frac{d\varphi^*}{ds} = \vec{T}^* \frac{ds^*}{ds} &= \left(1 + \frac{dm_1}{ds}\right) \vec{T} + \left(m_1 \kappa + \frac{dm_2}{ds} - m_3 \tau\right) \vec{N} \\&+ \left(m_2 \tau + \frac{dm_3}{ds} - m_4 \sigma\right) \vec{B} + \left(m_3 \sigma + \frac{dm_4}{ds}\right) \vec{E}\end{aligned} \quad (3.2)$$

Since $\vec{T}^* = -\vec{T}$. Rewriting (3.2) we obtain following system of equations,

$$\begin{aligned} 1 + \frac{dm_1}{ds} + \frac{ds^*}{ds} &= 0, \\ m_1\kappa + \frac{dm_2}{ds} - m_3\tau &= 0, \\ m_2\tau + \frac{dm_3}{ds} - m_4\sigma &= 0. \\ m_3\sigma + \frac{dm_4}{ds} &= 0 \end{aligned} \quad (3.3)$$

If we call θ as the angle between the tangent of the curve C at point $\vec{\varphi}$ with a given fixed direction and s arc length parameter of $\vec{\varphi}(s)$, consider $\frac{d\theta}{ds} = \kappa$, we have (3.3) as following;

$$\begin{aligned} \frac{dm_1}{d\theta} &= f(\theta), \\ \frac{dm_2}{d\theta} &= -m_1 + m_3\rho\tau \\ \frac{dm_3}{d\theta} &= -m_2\rho\tau + m_4\rho\sigma \\ \frac{dm_4}{d\theta} &= -m_3\rho\sigma \end{aligned} \quad (3.4)$$

where $f(\theta) = \rho + \rho^*$, $\rho = \frac{1}{\kappa}$ and $\rho^* = \frac{1}{\kappa^*}$ denote the radius of curvature at $\vec{\varphi}$ and $\vec{\varphi}^*$, respectively. If m_1, m_3 and their derivatives are eliminated in equations (3.4), we obtain the following equation with respect to m_2 :

$$\begin{aligned} \frac{d^4 m_2}{d\theta^4} - [\ln(\rho\tau)]' \cdot \frac{df}{d\theta} \cdot H(\theta) \Big] \frac{d^3 m_2}{d\theta^3} + \left\{ (\rho\tau)^{2'} + (\rho\tau)^2 + \rho^2 \sigma^2 + [2(\ln(\rho\tau))^{''}] \right\} \frac{d^2 m_2}{d\theta^2} \\ + \left[2(\rho\tau)^{2'} + (\rho^2 \sigma^2)' + (\rho^2 \tau \sigma)' \right] \frac{dm_2}{d\theta} + \left[(\rho\tau)^{2''} + [\ln(\rho\tau)]^{'''} \right] m_2 + \frac{d^2 f}{d\theta^2} - [\ln(\rho\tau)]' \frac{df}{d\theta} \\ + \left\{ \rho^2 \sigma^2 + [\ln(\rho\tau)]^{''} \right\} f + (\rho^2 \sigma^2)' + \frac{\sigma}{\tau} (\rho^2 \tau \sigma) + [\ln(\rho\tau)]^{'''} F(\theta) + H(\theta) = 0 \end{aligned} \quad (3.5)$$

where $H(\theta) = \int \left[\frac{dm_2}{d\theta} + \int f d\theta \right] \frac{\sigma}{\tau} d\theta$, $F(\theta) = \int f d\theta$

These equations are characterizations for the curve $\vec{\varphi}^*$. If the distance between opposite points of C and C^* is constant, then, due to null frame vectors, we can write that

$$\|\vec{\varphi}^* - \vec{\varphi}\| = \left\{ \begin{array}{ll} m_2^2 + m_3^2 + m_4^2 & , \quad m_1 = m_2 \\ m_1^2 & , \quad m_1 \neq m_2 \end{array} \right\}, \quad (3.6)$$

and if $m_1 = m_2$, we can write that

$$m_2^2 + m_3^2 + m_4^2 = k^2 = \text{constant}. \quad (3.7)$$

Hence, by the differentiation we have

$$m_2 \frac{dm_2}{d\theta} + m_3 \frac{dm_3}{d\theta} + m_4 \frac{dm_4}{d\theta} = 0. \quad (3.8)$$

Considering system (3.4), we obtain

$$m_1 \cdot m_2 = 0. \quad (3.9)$$

Therefore, we arrive at $m_1 = 0$ or $m_2 = 0$. As a next step, we shall study in the following cases.

Case 1: If $m_1 = 0$. then $f = 0$, in this case, equation (3.5) gives

$$\begin{aligned} \frac{d^4 m_2}{d\theta^4} + \left\{ (\rho\tau)^{2'} + (\rho\tau)^2 + \rho^2 \sigma^2 + [2(\ln(\rho\tau))^{''}] \right\} \frac{d^2 m_2}{d\theta^2} + \\ + \left[2(\rho\tau)^{2'} + (\rho^2 \sigma^2)' + (\rho^2 \tau \sigma)' \right] \frac{dm_2}{d\theta} + \left[(\rho\tau)^{2''} + [\ln(\rho\tau)]^{'''} \right] m_2 = 0 \end{aligned} \quad (3.10)$$

If m_1 is constant, besides, let us suppose that $m_2 = 0$. Then we obtain

$$f(\theta) = 0, \quad m_3 = \frac{m_1}{\rho\tau}, \quad m_4 = -\int \rho\sigma m_3 d\theta$$

If $m_1 = k_1$, ($k_1 \in \mathbb{R}$), in this case equation (3.1) transforms to

$$\vec{\varphi}^* = \vec{\varphi} + k_1 \vec{T} + \frac{m_1}{\rho\tau} \vec{B} - \left[\int \rho\sigma m_3 d\theta \right] \vec{E} \quad (3.11)$$

The change of variable $t(\Psi) = \int_0^\Psi \rho\sigma d\Psi$ gives

$$\frac{d^2 m_3}{dt^2} + m_3 = 0$$

General solution for $m_3(t)$ is

$$m_3 = A \cos\left(\int_0^\Psi \rho\sigma dt\right) + B \sin\left(\int_0^\Psi \rho\sigma dt\right)$$

If we consider m_3 , then

$$m_4 = -A \sin\left(\int_0^\Psi \rho\sigma dt\right) + B \cos\left(\int_0^\Psi \rho\sigma dt\right),$$

where A and B are arbitrary constants. Therefore, the general solution set of the system

$$\left\{ \begin{array}{l} m_1 = k_1, (k_1 \in \mathbb{R}), \\ m_2 = 0, \\ m_3 = A \cos\left(\int_0^\Psi \rho\sigma dt\right) + B \sin\left(\int_0^\Psi \rho\sigma dt\right), \\ m_4 = -A \sin\left(\int_0^\Psi \rho\sigma dt\right) + B \cos\left(\int_0^\Psi \rho\sigma dt\right). \end{array} \right\}$$

Thus equation (3.1) becomes

$$\vec{\varphi}^* = \vec{\varphi} + k_1 \vec{T} + \left\{ A \cos(\int_0^\Psi \rho \sigma dt) + B \sin(\int_0^\Psi \rho \sigma dt) \right\} \vec{B} + \left\{ -A \sin(\int_0^\Psi \rho \sigma dt) + B \cos(\int_0^\Psi \rho \sigma dt) \right\} \vec{E}$$

Case 2: i) If $m_2 = 0$, then the equation (3.5) yields

$$\frac{d^2 f}{d\theta^2} - [\ln(\rho\tau)] \frac{df}{d\theta} + \{\rho^2 \sigma^2 + [(\ln(\rho\tau))']\} f + (\rho^2 \sigma^2)' + \frac{\sigma}{\tau} (\rho^2 \tau \sigma) + [\ln(\rho\tau)]''' F(\theta) + H(\theta) = 0 \quad (3.12)$$

ii) if $m_1 \neq m_2$, we get

$$m_1^2 = k_1^2, \quad (3.13)$$

where $k_1^2 \in \mathbb{R}$. Hence, by the differentiation we have

$$m_1 \frac{dm_1}{d\theta} = 0. \quad (3.14)$$

If we carry (3.4) into (3.14) we obtain same cases and equation (3.10).

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