The Optimal Convex Combination Bounds of Harmonic Arithmetic and Contraharmonic Means for the Neuman means

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Abstract

In the paper, we find the greatest values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and the least values $\beta_1, \beta_2, \beta_3, \beta_4$ such that the double inequalities

$\alpha_1 A(a,b) + (1 - \alpha_1) H(a,b) < N(A(a,b), G(a,b)) < \beta_1 A(a,b) + (1 - \beta_1) H(a,b),$

$\alpha_2 A(a,b) + (1 - \alpha_2) H(a,b) < N(G(a,b), A(a,b)) < \beta_2 A(a,b) + (1 - \beta_2) H(a,b),$

$\alpha_3 C(a,b) + (1 - \alpha_3) A(a,b) < N(Q(a,b), A(a,b)) < \beta_3 C(a,b) + (1 - \beta_3) A(a,b),$

$\alpha_4 C(a,b) + (1 - \alpha_4) A(a,b) < N(A(a,b), Q(a,b)) < \beta_4 C(a,b) + (1 - \beta_4) A(a,b)$

hold for all $a, b > 0$ with $a \neq b$. Here $H(a,b)$, $G(a,b)$, $A(a,b)$, $Q(a,b)$, $C(a,b)$ and $N(a,b)$ denote the classical harmonic, geometric, arithmetic, quadratic, contraharmonic and Neumant means of $a$ and $b$, respectively.

Mathematics Subject Classification: 26E60
Keywords: Schwab-Borchardt mean, Neuman mean, harmonic mean, geometric mean, arithmetic mean, quadratic mean, contraharmonic mean

1 Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a, b)[1, 2]$ is defined by

$$SB(a, b) = \begin{cases} \sqrt{b^2 - a^2} \cos^{-1}(a/b), & \text{if } a < b, \\ \sqrt{a^2 - b^2} \cosh^{-1}(a/b), & \text{if } a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that $SB(a, b)$ is strictly increasing in both $a$ and $b$, nonsymmetric and homogeneous of degree 1 with respect to $a$ and $b$. Several bivariate means are special of the Schwab-Borchardt mean. We recall definitions of the first and the second Seiffert means, denoted by $P(a, b)$ and $T(a, b)$, respectively, the Neuman-Sándor mean $M(a, b)$, and the logarithmic mean $L(a, b)$ (see [3], [4]).

$$P(a, b) = \frac{a - b}{2 \sin^{-1}[(a - b)/(a + b)]} = SB(G, A)$$
$$T(a, b) = \frac{a - b}{2 \tan^{-1}[(a - b)/(a + b)]} = SB(A, Q)$$
$$M(a, b) = \frac{a - b}{2 \sinh^{-1}[(a - b)/(a + b)]} = SB(Q, A)$$
$$T(a, b) = \frac{a - b}{2 \tanh^{-1}[(a - b)/(a + b)]} = SB(A, G)$$

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$, $C(a, b) = (a^2 + b^2)/(a + b)$ are the classical harmonic, geometric, arithmetic, quadratic and contraharmonic means of $a$ and $b$, respectively. Then we have

$$H(a, b) < G(a, b) < L(a, b) < P(a, b) < A(a, b)$$
$$< M(a, b) < T(a, b) < Q(a, b) < C(a, b)$$

for $a, b > 0$ with $a \neq b$. 

Recently, the Schwab-Borchardt mean has been the subject of intensive research. The bivariate means $S_{AH}(a,b), S_{HA}(a,b), S_{CA}(a,b)$ and $S_{AC}(a,b)$ derived from the Schwab-Borchardt mean are defined by Neuman [7] as follows:

$$S_{AH}(a,b) = A(a,b) \frac{\tanh(p)}{p}, \quad S_{HA}(a,b) = A(a,b) \frac{\sin(q)}{q},$$

$$S_{CA}(a,b) = A(a,b) \frac{\sinh(r)}{r}, \quad S_{AC}(a,b) = A(a,b) \frac{\tan(s)}{s},$$

where $v = (a-b)/(a+b) \in (-1,1)$ and $p, q, r$ and $s$ are defined implicitly as $\text{sech}(p) = 1 - v^2$, $\cos(q) = 1 - v^2$, $\cosh(r) = 1 + v^2$ and $\sec(s) = 1 + v^2$, respectively. Clearly $p \in (0, +\infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$ and $s \in (0, \pi/3)$. Many remarkable inequalities for Schwab-Borchardt mean and its generated means can be found in the literature [5-9].

Very recently, Neuman [10] found a new bivariate mean $N(a,b)$ derived from the Schwab-Borchardt mean, defined as follows

$$N(a,b) = \frac{1}{2} \left( a + \frac{b^2}{SB(a,b)} \right),$$

and proved that the inequalities

$$G(a,b) < N(A(a,b), G(a,b)) < N(G(a,b), A(a,b)) < A(a,b)$$

$$< N(Q(a,b), A(a,b)) < N(A(a,b), Q(a,b)) < Q(a,b)$$

and

$$L(a,b) < N(A(a,b), G(a,b)) < P(a,b) < N(G(a,b), A(a,b))$$

$$< M(a,b) < N(Q(a,b), A(a,b)) < T(a,b) < N(A(a,b), Q(a,b))$$

for $a, b > 0$ with $a \neq b$.

Let $a > b$ and $v = (a-b)/(a+b) \in (0,1)$, then the following explicit formulas for $N(A(a,b), G(a,b))$, $N(G(a,b), A(a,b))$, $N(Q(a,b), A(a,b))$ and $N(A(a,b), Q(a,b))$ are presented in [10]

$$N(A(a,b), G(a,b)) = \frac{1}{2} A(a,b) \left[ 1 + (1 - v^2) \frac{\tanh^{-1}(v)}{v} \right], \quad (1)$$

$$N(G(a,b), A(a,b)) = \frac{1}{2} A(a,b) \left[ \sqrt{1 - v^2} + \frac{\sin^{-1}(v)}{v} \right], \quad (2)$$

$$N(Q(a,b), A(a,b)) = \frac{1}{2} A(a,b) \left[ \sqrt{1 + v^2} + \frac{\sinh^{-1}(v)}{v} \right], \quad (3)$$

$$N(A(a,b), Q(a,b)) = \frac{1}{2} A(a,b) \left[ 1 + (1 + v^2) \frac{\tan^{-1}(v)}{v} \right], \quad (4)$$
where \( \tanh^{-1}(x) \), \( \sin^{-1}(x) \), \( \sinh^{-1}(x) \) and \( \tan^{-1}(x) \) are the inverse hyperbolic tangent, inverse sine, inverse hyperbolic sine and inverse tangent functions, respectively.

In [10], Neuman also proved that the double inequalities

\[
\frac{2}{3} A(a, b) + \frac{1}{3} G(a, b) < N\left(G(a, b), A(a, b)\right) < \frac{\pi}{4} A(a, b) + (1 - \frac{\pi}{4}) G(a, b)
\]

\[
\frac{2}{3} Q(a, b) + \frac{1}{3} A(a, b) < N\left(A(a, b), Q(a, b)\right) < \frac{\pi - 2}{4(\sqrt{2} - 1)} A(a, b)
\]

\[
\frac{1}{3} A(a, b) + \frac{2}{3} G(a, b) < N\left(A(a, b), G(a, b)\right) < \frac{1}{2} A(a, b) + \frac{1}{2} G(a, b),
\]

\[
\frac{1}{3} Q(a, b) + \frac{2}{3} A(a, b) < N\left(Q(a, b), A(a, b)\right)
\]

\[
< \frac{\log(1 + \sqrt{2}) + \sqrt{2} - 2}{2(\sqrt{2} - 1)} Q(a, b) + \left(1 - \frac{\log(1 + \sqrt{2}) + \sqrt{2} - 2}{2(\sqrt{2} - 1)}\right) A(a, b)
\]

for \( a, b > 0 \) with \( a \neq b \).

The purpose of the present paper is to find the greatest values \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and the least value \( \beta_1, \beta_2, \beta_3, \beta_4 \) such that the double inequalities

\[
\alpha_1 A(a, b) + (1 - \alpha_1) H(a, b) < N\left(A(a, b), G(a, b)\right) < \beta_1 A(a, b) + (1 - \beta_1) H(a, b),
\]

\[
\alpha_2 A(a, b) + (1 - \alpha_2) H(a, b) < N\left(G(a, b), A(a, b)\right) < \beta_2 A(a, b) + (1 - \beta_2) H(a, b),
\]

\[
\alpha_3 C(a, b) + (1 - \alpha_3) A(a, b) < N\left(Q(a, b), A(a, b)\right) < \beta_3 C(a, b) + (1 - \beta_3) A(a, b),
\]

\[
\alpha_4 C(a, b) + (1 - \alpha_4) A(a, b) < N\left(A(a, b), Q(a, b)\right) < \beta_4 C(a, b) + (1 - \beta_4) A(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \).

## 2. Lemma

In order to prove the desired theorems, we need following Lemmas.

**Lemma 2.1.** (see [11, Theorem 1.25]). For \(-\infty < a < b < +\infty\), let \( f, g : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\), and be differentiable on \((a, b)\), let \( g'(x) \neq 0 \) on \((a, b)\). If \( f'(x)/g'(x) \) is increasing (decreasing) on \((a, b)\), then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}
\]

If \( f'(x)/g'(x) \) is strictly monotone, then the monotonicity in the conclusion is also strict.
Lemma 2.2. (see [12, Theorem 1.1]). Suppose that the power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) have the radius of convergence \( r > 0 \) and \( a_n > 0, b_n > 0 \) for all \( n \in \{0, 1, 2, \ldots\} \).

Let \( h(x) = f(x)/g(x) \), if the sequence series \( \{a_n/b_n\}_{n=0}^{\infty} \) is (strictly) increasing (decreasing), then \( h(x) \) is also (strictly) increasing (decreasing) on \((0, r)\).

Lemma 2.3. The function

\[
f(x) = \frac{[2x + \sinh(2x)] [1 + \cosh(2x)] - 4 \sinh(2x)}{2 \sinh(2x) [\cosh(2x) - 1]}
\]

is strictly decreasing on \((0, +\infty)\), where \( \sinh(x) = (e^x - e^{-x})/2 \) and \( \cosh(x) = (e^x + e^{-x})/2 \) are the hyperbolic sine and cosine functions, respectively.

Proof. Let

\[
f_1(x) = [2x + \sinh(2x)] [1 + \cosh(2x)] - 4 \sinh(2x)
= 2x + 2x \cosh(2x) + \frac{1}{2} \sinh(4x) - 3 \sinh(2x)
\]

(5)

\[
f_2(x) = 2 \sinh(2x) [\cosh(2x) - 1]
= \sinh(4x) - 2 \sinh(2x)
\]

(6)

Then

\[
f_1(0^+) = f_2(0) = 0, \quad f(x) = f_1(x)/f_2(x)
\]

(7)

Using the power series \( \sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)! \) and \( \cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)! \), we can express \( f_1(x) \) and \( f_2(x) \) as follows

\[
f_1(x) = 2x + 2x \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+1)!} x^{2n+1} - 3 \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}
\]

\[
= 2x \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{4^{2n+1}}{(2n+1)!} x^{2n+1} - 3 \sum_{n=1}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}
\]

\[
= x^3 \sum_{n=0}^{\infty} \frac{2^{2n+1}(2^{2n+1} + n)}{(2n+3)!} x^{2n}
\]

(8)
It follows from (8) and (9) that

\[ f(x) = \sum_{n=0}^{\infty} a_n x^{2n} \tag{10} \]

with \( a_n = \frac{2^{2n+1}(2^{2n+1} + n)}{(2n + 3)!} \) and \( b_n = \frac{2^{2n+4}(2^{2n+2} - 1)}{(2n + 3)!} \).

Simple computations lead to

\[ \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{(6n + 1)2^{2n+1} + 1}{(2^{n+2} - 1)(2^{2n+4} - 1)} < 0 \tag{11} \]

for all \( n \geq 0 \).

Therefore, from (5) - (11) and Lemma 2.2 we know that \( f(x) \) is strictly decreasing on \((0, +\infty)\).

**Lemma 2.4.** The function

\[ g(x) = \frac{x + \sin(x)\cos(x) - 2\sin(x)\cos^2(x)}{2\sin^3(x)} \]

is strictly decreasing on \((0, \pi/2)\).

**Proof.** Let

\[ g_1(x) = x + \sin(x)\cos(x) - 2\sin(x)\cos^2(x) \]
\[ g_2(x) = 2\sin^3(x) \]

Then

\[ g(x) = g_1(x)/g_2(x), \quad g_1(0^+) = g_2(0) = 0 \tag{14} \]

and

\[ g'_1(x)/g'_2(x) = \frac{\cos^2(x) - \sin^2(x) + 1 - 2\cos^3(x) + 4\cos(x)\sin^2(x)}{6\sin^2(x)\cos(x)} \]
\[ = \frac{\cos(x) - \cos^2(x) + 2\sin^2(x)}{3\sin^2(x)} \]
\[ = 1 - \frac{1}{3[1 + \cos(x)]} \tag{15} \]
Let
\[ g_3(x) = 1 - \frac{1}{3[1 + \cos(x)]}, \quad (16) \]

Then
\[ g'_3(x) = -\frac{\sin(x)}{3[1 + \cos(x)]^2} < 0, \quad (17) \]

for \( x \in (0, \pi/2) \).

Then from (15)-(17) we clearly see that \( g'_1(x)/g'_2(x) \) is strictly decreasing on \( x \in (0, \pi/2) \).

Therefore, Lemma 2.4 follows easily from (12)-(14) and Lemma 2.1.

**Lemma 2.5.** The function

\[ h(x) = \frac{x + \sinh(x) \cosh(x) - 2 \sinh(x)}{2 \sinh^3(x)} \]

is strictly decreasing on \((0, \log(1 + \sqrt{2}))\).

Proof. Let

\[ h_1(x) = x + \sinh(x) \cosh(x) - 2 \sinh(x) \quad (18) \]
\[ h_2(x) = 2 \sinh^3(x) \quad (19) \]

Then

\[ h(x) = h_1(x)/h_2(x), \quad h_1(0^+) = h_2(0) = 0 \quad (20) \]

Using the power series \( \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \) and \( \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \),

we can express \( h_1(x) \) and \( h_2(x) \) as follows

\[ h_1(x) = x + \frac{1}{2} \sinh(2x) - 2 \sinh(x) \]
\[ = x + \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1} - 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \]
\[ = \sum_{n=1}^{\infty} \frac{(2^n - 2)}{(2n+1)!} x^{2n+1} = x^3 \sum_{n=0}^{\infty} \frac{(2^{2n+2} - 2)}{(2n+3)!} x^{2n}, \quad (21) \]

\[ h_2(x) = 2 \sinh^3(x) = \frac{1}{2} [ \sinh(3x) - 3 \sinh(x) ] \]
\[ = \frac{1}{2} \left[ \sum_{n=0}^{\infty} \frac{3^{2n+1}}{(2n+1)!} x^{2n+1} - 3 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \right] \]
\[ = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(3^{2n+1} - 3)}{(2n+1)!} x^{2n+1} = x^3 \sum_{n=0}^{\infty} \frac{([3^{2n+3} - 3]/2)}{(2n+3)!} x^{2n}, \quad (22) \]
It follows from (21) and (22) that

\[ h(x) = \frac{\sum_{n=0}^{\infty} c_n x^{2n}}{\sum_{n=0}^{\infty} d_n x^{2n}} \]  

(23)

with \( c_n = \frac{(2^{2n+2} - 2)/(2n + 3)!}{(3^{2n+3} - 3)/2}/(2n + 3)! \) and \( d_n = \frac{(3^{2n+3} - 3)/2}{(3^{2n+5} - 3)} \).

Simple computations lead to

\[ \frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = \frac{4(4 - 5 * 2^{2n})3^{2n+3} - 9 * 2^{2n+2}}{(3^{2n+3} - 3)(3^{2n+5} - 3)} < 0 \]  

(24)

for all \( n \geq 0 \).

Therefore, from (18) - (24) and Lemma 2.2 we know that \( h(x) \) is strictly decreasing on \((0, \log(1 + \sqrt{2}))\).

**Lemma 2.6.** The function

\[ k(x) = \frac{x \cos(x) - \sin(x) \cos^2(x)}{2 \sin^3(x)} \]

is strictly decreasing on \((0, \pi/4)\).

Proof. Differentiating \( k(x) \) gives

\[ k'(x) = \frac{3 \sin(x) \cos(x) + 2x \sin^2(x) - 3x}{2 \sin^4(x)} \]  

(25)

Let

\[ k_1(x) = 3 \sin(x) \cos(x) + 2x \sin^2(x) - 3x. \]  

(26)

Then simple computations lead to

\[ k_1(0) = 0, \]  

(27)

\[ k'_1(x) = 4 \sin(x) \left[ x \cos(x) - \sin(x) \right]. \]  

(28)

Let

\[ k_2(x) = x \cos(x) - \sin(x). \]  

(29)

and

\[ k'_2(x) = -x \sin(x) < 0. \]  

(30)

for \( x \in (0, \pi/4) \).

Therefore, Lemma 2.6 follows easily from (25)-(30).
3 Proof of Theorem

Theorem 3.1. The double inequality
\[
\alpha_1 A(a, b) + (1 - \alpha_1) H(a, b) < N(A(a, b), G(a, b)) < \beta_1 A(a, b) + (1 - \beta_1) H(a, b)
\]
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_1 \leq 1/2\) and \(\beta_1 \geq 2/3\).

Proof. Since \(H(a, b), G(a, b)\) and \(A(a, b)\) are symmetric and homogeneous of degree 1. Without loss of generality, we assume that \(a > b\). Let \(v = (a - b)/(a + b)\) and \(x = \tanh^{-1}(v)\). Then \(v \in (0, 1), x \in (0, +\infty)\),
\[
H(a, b) = A(a, b) \text{sech}^2(x)
\] (31)
and (1) leads to
\[
N(A(a, b), G(a, b)) = \frac{1}{2} A(a, b) \left[ 1 + \frac{2x}{\sinh(2x)} \right]
\] (32)

It follows from (31),(32) that
\[
\frac{N(A(a, b), G(a, b)) - H(a, b)}{A(a, b) - H(a, b)} = \frac{[2x + \sinh(2x)] [1 + \cosh(2x)] - 4 \sinh(2x)}{2 \sinh(2x) [ \cosh(2x) - 1 ]}
\] (33)
\[= f(x) \]
where \(f(x)\) is defined as in Lemma 2.3.

Note that
\[
\lim_{x \to 0^+} f(x) = \frac{2}{3}, \quad (34)
\]
\[
\lim_{x \to \infty} f(x) = \frac{1}{2}, \quad (35)
\]

Therefore, Theorem 3.1 follows easily from (34) and (35) together with Lemma 2.3. \(\square\)

Theorem 3.2. The double inequality
\[
\alpha_2 A(a, b) + (1 - \alpha_2) H(a, b) < N(G(a, b), A(a, b)) < \beta_2 A(a, b) + (1 - \beta_2) H(a, b)
\]
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_2 \leq \pi/4\) and \(\beta_2 \geq 5/6\).
Proof. We follow the same idea in the proof of theorem 3.1. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b)$ and $x = \sin^{-1}(v)$. Then $v \in (0, 1)$, $x \in (0, \pi/2)$,

$$H(a, b) = A(a, b) \cos^2(x) \quad (36)$$

and (2) leads to

$$N\left(G(a, b), A(a, b)\right) = \frac{1}{2} A(a, b) \left[ \cos(x) + \frac{x}{\sin(x)} \right] \quad (37)$$

It follows from (36), (37) that

$$\frac{N\left(G(a, b), A(a, b)\right) - H(a, b)}{A(a, b) - H(a, b)} = \frac{x + \sin(x) \cos(x) - 2 \sin(x) \cos^2(x)}{2 \sin^3(x)}$$

$$=: g(x) \quad (38)$$

where $g(x)$ is defined as in Lemma 2.4.

It is not difficult to verify that

$$\lim_{x \to 0^+} g(x) = \frac{5}{6}, \quad (39)$$

$$\lim_{x \to \pi/2} g(x) = \frac{\pi}{4}, \quad (40)$$

Therefore, Theorem 3.2 follows easily from (39) - (40) and Lemma 2.4. \qed

Theorem 3.3. The double inequality

$$\alpha_3 C(a, b) + (1 - \alpha_3) A(a, b) < N\left(Q(a, b), A(a, b)\right) < \beta_3 C(a, b) + (1 - \beta_3) A(a, b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq \left[ \log(1 + \sqrt{2}) + \sqrt{2} - 2 \right]/2 = 0.1477 \cdots$ and $\beta_3 \geq 1/6$.

Proof. Since $A(a, b)$, $Q(a, b)$ and $C(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b)$ and $x = \sinh^{-1}(v)$. Then $v \in (0, 1)$, $x \in (0, \log(1 + \sqrt{2}))$,

$$C(a, b) = A(a, b) \cosh^2(x) \quad (41)$$

and (3) leads to

$$N\left(Q(a, b), A(a, b)\right) = \frac{1}{2} \left[ \cosh(x) + \frac{x}{\sinh(x)} \right] \quad (42)$$
Then from (41) and (42) we have

\[
\frac{N(Q(a, b), A(a, b)) - A(a, b)}{C(a, b) - A(a, b)} = \frac{x + \sinh(x) \cosh(x) - 2 \sinh(x)}{2 \sinh^3(x)} =: h(x)
\]

(43)

where \(h(x)\) is defined as in Lemma 2.5.

Note that

\[
\lim_{x \to 0^+} h(x) = \frac{1}{6},
\]

(44)

\[
\lim_{x \to \log(1 + \sqrt{2})} h(x) = \frac{\log(1 + \sqrt{2}) + \sqrt{2} - 2}{2} = 0.1477 \cdots,
\]

(45)

Therefore, Theorem 3.3 follows easily from (44) and (45) together with Lemma 2.5. 

**Theorem 3.4.** The double inequality

\[
\alpha_4 C(a, b) + (1 - \alpha_4) A(a, b) < N(A(a, b), Q(a, b)) < \beta_4 C(a, b) + (1 - \beta_4) A(a, b)
\]

holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_4 \leq (\pi - 2)/4 = 0.2853 \cdots\) and \(\beta_4 \geq 1/3\).

**Proof.** We follow the same idea in the proof of theorem 3.3. Without loss of generality, we assume that \(a > b\). Let \(v = (a - b)/(a + b)\) and \(x = \tan^{-1}(v)\). Then \(v \in (0, 1), \ x \in (0, \pi/4), \)

\[
C(a, b) = A(a, b) \sec^2(x)
\]

(46)

and (1.4) leads to

\[
N(A(a, b), Q(a, b)) = \frac{1}{2} A(a, b) \left[ 1 + \frac{2x}{\sin(2x)} \right]
\]

(47)

It follows from (46), (47) that

\[
\frac{N(A(a, b), Q(a, b)) - A(a, b)}{C(a, b) - A(a, b)} = \frac{x \cos(x) - \sin(x) \cos^2(x)}{2 \sin^3(x)} =: k(x)
\]

(48)

where \(k(x)\) is defined as in Lemma 2.6.
It is not difficult to verify that
\[
\lim_{x \to 0^+} k(x) = \frac{1}{3}, \quad (49)
\]
\[
\lim_{x \to \frac{\pi}{2}} k(x) = \frac{\pi - 2}{4} = 0.2853 \ldots, \quad (50)
\]
Therefore, Theorem 3.4 follows easily from (49), (50) and Lemma 2.6. □

Acknowledgements.
This research was supported by the Natural Science Foundation of Zhejiang Province under Grant LY13A010004, the Natural Science Foundation of the Department of Education of Zhejiang Province under Grant Y201431915 and the Natural Science Foundation of the Open University of China under Grant Q1601E-Y.

References


**Received: July 9, 2014**