Direct Approach to the Stability of a Cubic Functional Equation in Felbin's Type Fuzzy Normed Spaces

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Abstract

Using direct method, we prove the generalized Hyers-Ulam stability of the cubic functional equation

$$f(x+3y) - 3f(x+2y) + 4f(x+y) - 4f(x) + 3f(x-y) - f(x-2y) = 12f(y)$$

in Felbin's type fuzzy normed spaces.

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1 Introduction

In the field of fuzzy functional analysis, the concept of a fuzzy norm on a linear space is of comparatively recent origin. It was Katsaras [22], who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. His pioneering work provided some influence to several mathematicians who have defined fuzzy metrics and norms on a linear space from various points of view [1, 13, 14, 31]. In 1992, Felbin [9] introduced an idea of a fuzzy norm on a linear space (with the induced fuzzy metric of Kaleva and Seikkala type [21]) by assigning a non-negative fuzzy real number to each element of a linear space. In 1994, from a different approach, Cheng and Mordeson [7] defined another type of fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Michalek type [23]. Gähler and Gähler [11] defined fuzzy norm of a fuzzy real number as a difference of its positive and negative parts.

In [3], Bag and Samanta considered a fuzzy norm slightly different from the one defined by Cheng and Mordeson [7], and for which a suitable decomposition theorem was proved. This concept has been used in developing fuzzy functional analysis and its application and a large number of papers by different authors have been published [4, 5, 15, 18]. In this context it is worth mentioning the work done by Xiao and Zhu [33, 34] who have considered Felbin-type fuzzy norm in its general form and studied various properties of fuzzy normed linear spaces. In [6], Felbin's definition of 'fuzzy normed linear space' is slightly modified by Bag and Samanta in the sense that;

- (i) the value of the fuzzy norm is taken to be a fuzzy real number in the sense of Xiao and Zhu [33];
- (ii) the conditions (A) and (B) of Felbin in [10] (see Definition 2.5) is relaxed by the condition (A') (see Definition 2.6).

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [29] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [29] has provided a lot influence in the development of what we call generalized Hyers-Ulam stability or as Hyers-Ulam

Rassias stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias approach. We refer the curious readers for further information on such problems to the papers [8, 17, 19, 28, 30].

The functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$

is said to be the cubic functional equation, since cx^3 is its solution. In particular, every solution of a cubic functional equation is said to be a cubic mapping. The stability problem for a cubic type functional equation was proved by K.W. Jun and H.M. Kim [20] for mappings $f : X \rightarrow Y$, where X is a real normed space and Y is a Banach space. Recently considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic and cubic functional equations have been investigated [24, 25, 26, 27].

The main purpose of this paper is to prove the generalized Hyers-Ulam stability of the cubic functional equation

$$f(x + 3y) - 3f(x + 2y) + 4f(x + y) - 4f(x) + 3f(x - y) - f(x - 2y) = 12f(y) \quad (1)$$

in the modified Felbin's type fuzzy normed linear space. The organization of this paper is as follows: Section 2 comprises some useful definitions, notations and preliminary results. In section 3, the generalized Hyers-Ulam stability of the cubic functional equation (1) is discussed.

2 Preliminaries

In this paper, we consider the concept of fuzzy real numbers (fuzzy interval) in the sense of Xiao and Zhu [33] which is defined below.

Definition 2.1. [33] *A mapping $\eta : \mathbb{R} \rightarrow [0, 1]$ is called a fuzzy real number (fuzzy interval), whose α -level set is denoted by $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms;*

(N1) *There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*

(N2) *For each $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, where $-\infty < \eta_\alpha^1 \leq \eta_\alpha^2 < +\infty$.*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}$.

Definition 2.2. [33] Let $\eta \in \mathcal{F}$. Then η is called a positive fuzzy real number (fuzzy interval) if $\eta(t) = 0 \forall t < 0$.

The set of all positive fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}^+$.

Proposition 2.3. [7] Let $\eta, \delta \in \mathcal{F}$ and $[\eta]_\alpha = [a_\alpha^1, b_\alpha^1]$, $[\delta]_\alpha = [a_\alpha^2, b_\alpha^2]$, $\alpha \in (0, 1]$. Then

$$[\eta \oplus \delta]_\alpha = [a_\alpha^1 + a_\alpha^2, b_\alpha^1 + b_\alpha^2].$$

Definition 2.4. A partial ordering " $\preceq$ " in $\mathcal{F}$ is defined as follows, $\eta \preceq \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta_\alpha^1 \leq \delta_\alpha^1$ and $\eta_\alpha^2 \leq \delta_\alpha^2$ where, $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$ and $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$. The strict inequality in $\mathcal{F}$ is defined by $\eta < \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta_\alpha^1 < \delta_\alpha^1$ and $\eta_\alpha^2 < \delta_\alpha^2$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below.

Definition 2.5. [9] Let X be a vector space over R . Let $\|\cdot\| : X \rightarrow R^*(I)$ and let the mappings, $L, U : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, non decreasing in both arguments and satisfying $L(0, 0) = 0$ and $U(0, 0) = 1$.

Write $[\|x\|]_\alpha = [\|x\|_\alpha^1, \|x\|_\alpha^2]$ for $x \in X$, $0 < \alpha \leq 1$ and suppose for all $x \in X$, $x \neq \underline{0}$, there exists $\alpha_0 \in (0, 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

$$(A) \quad \|x\|_\alpha^2 < \infty$$

$$(B) \quad \inf \|x\|_\alpha^1 > 0.$$

The quadruple $(X, \|\cdot\|, L, U)$ is called a fuzzy normed linear space and $\|\cdot\|$ is a fuzzy norm if

$$(i) \quad \|x\| = \bar{0} \text{ if and only if } x = \underline{0} \text{ (the null vector)}$$

$$(ii) \quad \|rx\| = |r|\|x\|, \quad x \in X, \quad r \in R,$$

$$(iii) \quad \text{for all } x, y \in X,$$

$$(a) \quad \text{whenever}$$

$$s \leq \|x\|_1^1, t \leq \|y\|_1^1 \text{ and } s+t \leq \|x+y\|_1^1, \\ \|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t)),$$

$$(b) \quad \text{whenever}$$

$$s \geq \|x\|_1^1, t \geq \|y\|_1^1 \text{ and } (s+t) \geq \|x+y\|_1^1, \\ \|x+y\|(s+t) \leq U(\|x\|(s), \|y\|(t)).$$

Next, we define the modified Felbin's type fuzzy norm on a linear space as given in [6]:

Definition 2.6. [6] Let X be a linear space over R . Let $\|\cdot\| : X \rightarrow \mathcal{F}^+$ be a mapping satisfying:

- (i) $\|x\| = \bar{0}$ iff $x = \underline{0}$.
- (ii) $\|rx\| = |r|\|x\|, \forall x \in X, r \in R$,
- (iii) for all $x, y \in X$,
 $\|x + y\| \preceq \|x\| \oplus \|y\|$
 and
 $(A') : x \neq \underline{0} \Rightarrow \|x\|(t) = 0 \forall t \leq 0$.

Then $(X, \|\cdot\|)$ is called a fuzzy normed linear space and $\|\cdot\|$ is called a fuzzy norm on X .

Example 2.7. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$\|x\|(t) = \begin{cases} \frac{\|x\|}{t} & \text{when } \|x\| \leq t, x \neq \underline{0} \\ 1 & \text{when } t = \|x\| = 0 \\ 0 & \text{otherwise.} \end{cases}$$

is a fuzzy norm on X , whose α -level set is given by, $[\|x\|]_\alpha = \left[\|x\|, \frac{\|x\|}{\alpha} \right]$, $x \in X$.

Definition 2.8. [6] Let $(X, \|\cdot\|)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to converge to $x \in X$ denoted by $\lim_{n \rightarrow \infty} x_n = x$ if and only if $\lim_{n \rightarrow \infty} \|x_n - x\| = \bar{0}$. i.e., $\lim_{n \rightarrow \infty} \|x_n - x\|_\alpha^1 = \lim_{n \rightarrow \infty} \|x_n - x\|_\alpha^2 = 0 \forall \alpha \in (0, 1]$.

Definition 2.9. [6] A sequence $\{x_n\}$ in a fuzzy normed linear space $(X, \|\cdot\|)$ is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_n - x_m\| = \bar{0}$, i.e., if $\lim_{m, n \rightarrow \infty} \|x_n - x_m\|_\alpha^1 = \lim_{m, n \rightarrow \infty} \|x_n - x_m\|_\alpha^2 = 0, \forall \alpha \in (0, 1]$.

Definition 2.10. [6] A fuzzy normed linear space $(X, \|\cdot\|)$ is said to be complete if every Cauchy sequence in X converges in X and the fuzzy normed space is called a fuzzy Banach space.

3 Stability of the Cubic Functional Equation (1)

In this section, using direct method, we prove the generalized Ulam-Hyers stability of the cubic functional equation (1) in modified Felbin's type fuzzy

normed spaces. For notational convenience, given a function $f : X \rightarrow Y$, we define the difference operator.

$$D_q f(x, y) := f(x + 3y) - 3f(x + 2y) + 4f(x + y) - 4f(x) + 3f(x - y) - f(x - 2y) - 12f(y) \quad (2)$$

for all $x, y \in X$.

Theorem 3.1. *Let X be a linear space and let $(Z, \| \cdot \|^\sim)$ be a fuzzy normed space. Let $\phi : X^2 \rightarrow Z$ be a function such that*

$$\lim_{n \rightarrow \infty} \frac{\|\phi(2^n x, 2^n y)\|_\alpha^{\sim 1}}{|8^n|} = \lim_{n \rightarrow \infty} \frac{\|\phi(2^n x, 2^n y)\|_\alpha^{\sim 2}}{|8^n|} = 0 \quad \forall x \in X, \alpha \in (0, 1] \quad (3)$$

Let $(Y, \| \cdot \|)$ be a fuzzy Banach space and $f : X \rightarrow Y$ be a mapping such that

$$\|D_q f(x, y)\| \preceq \|\phi(x, y)\|^\sim \quad (4)$$

for all $x \in X$, then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|C(x) - f(x)\| \preceq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M(2^k x)}{|8^k|} \quad (5)$$

for all $x \in X$, where

$$\begin{aligned} M(2^k x) = & \frac{1}{|12|} \|\phi(0, 2^k x)\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, 2^k x)\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(2^k x, -2^k x)\|^\sim \oplus \frac{1}{|12|} \|\phi(0, 2^{k+1} x)\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(0, 2^{k+1} x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(2^{k+1} x, -2^{k+1} x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2^{k+1} x, 2^k x)\|^\sim \\ & \oplus \frac{1}{|12|} \|\phi(3 \cdot 2^k x, 2^k x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2^k x, 2^k x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{9}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, 2^k x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(2^k x, -2^k x)\|^\sim. \end{aligned} \quad (6)$$

Proof. Replacing $x = y = 0$ in (4), we get

$$\|f(0)\| \preceq \frac{\|\phi(0, 0)\|^\sim}{|12|}. \quad (7)$$

Replacing x and y by 0 and x in (4), respectively, we get

$$\|f(3x) - 3f(2x) + 4f(x) - 4f(0) + 3f(-x) - f(-2x) - 12f(x)\| \preceq \|\phi(0, x)\|^\sim \quad (8)$$

for all $x \in X$. Again replacing x and y by x and $-x$ in (4), respectively, we have

$$\|f(-2x) - 3f(-x) + 4f(0) - 4f(x) + 3f(2x) - f(3x) - 12f(-x)\| \preceq \|\phi(x, -x)\|^\sim \quad (9)$$

for all $x \in X$. By (8) and (9), we obtain

$$\|f(x) + f(-x)\| \preceq \frac{1}{|12|} \|\phi(0, x)\|^\sim \oplus \frac{1}{|12|} \|\phi(x, -x)\|^\sim \quad (10)$$

for all $x \in X$. Replacing x and y by $3x$ and x in (4), respectively, we get

$$\|f(6x) - 3f(5x) + 4f(4x) - 4f(3x) + 3f(2x) - f(x) - 12f(x)\| \preceq \|\phi(3x, x)\|^\sim \quad (11)$$

for all $x \in X$. Replacing x and y by 0 and $2x$ in (4), respectively, we have

$$\|f(6x) - 3f(4x) + 4f(2x) - 4f(0) + 3f(-2x) - f(-4x) - 12f(2x)\| \preceq \|\phi(0, 2x)\|^\sim \quad (12)$$

for all $x \in X$. By (11) and (12), we obtain

$$\begin{aligned} & \|3f(5x) - 6f(4x) - 11f(2x) - 4f(0) + 3f(-2x) + 4f(3x) + 13f(x)\| \\ & \preceq \|\phi(0, 2x)\|^\sim \oplus \|\phi(3x, x)\|^\sim \end{aligned} \quad (13)$$

for all $x \in X$. By (7), (10) and (13), we have

$$\begin{aligned} & \|3f(5x) - 6f(4x) + 4f(3x) + 13f(x) - 14f(2x)\| \preceq \|\phi(0, 2x)\|^\sim \\ & \oplus \left| \frac{4}{12} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(0, 2x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2x, -2x)\|^\sim \oplus \|\phi(3x, x)\|^\sim \end{aligned} \quad (14)$$

for all $x \in X$. Replacing x and y by $2x$ and x in (4), respectively, we get

$$\|f(5x) - 3f(4x) + 4f(3x) - 4f(2x) + 3f(x) - f(0) - 12f(x)\| \preceq \|\phi(2x, x)\|^\sim \quad (15)$$

for all $x \in X$. By (7) and (15), we obtain

$$\|f(5x) - 3f(4x) + 4f(3x) - 4f(2x) - 9f(x)\| \preceq \|\phi(2x, x)\|^\sim \oplus \frac{\|\phi(0, 0)\|^\sim}{|12|} \quad (16)$$

for all $x \in X$. Hence from (16), we have

$$\|3f(5x) - 9f(4x) + 12f(3x) - 12f(2x) - 27f(x)\| \preceq |3| \|\phi(2x, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(0, 0)\|^\sim \quad (17)$$

for all $x \in X$. By (14) and (17), we get

$$\begin{aligned} \|3f(4x) - 8f(3x) + 40f(x) - 2f(2x)\| &\preceq \|\phi(0, 2x)\|^\sim \oplus \left| \frac{4}{12} \right| \|\phi(0, 0)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(0, 2x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2x, -2x)\|^\sim \oplus |3| \|\phi(2x, x)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(0, 0)\|^\sim \oplus \|\phi(3x, x)\|^\sim \end{aligned} \quad (18)$$

for all $x \in X$. Replacing x and y by x and x in (4), respectively, we have

$$\|f(4x) - 3f(3x) + 4f(2x) - 4f(x) + 3f(0) - f(-x) - 12f(x)\| \preceq \|\phi(x, x)\|^\sim \quad (19)$$

for all $x \in X$. By (7), (10) and (19), we have

$$\begin{aligned} \|f(4x) - 3f(3x) + 4f(2x) - 15f(x)\| &\preceq \|\phi(x, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(0, 0)\|^\sim \\ &\oplus \frac{1}{|12|} \|\phi(0, x)\|^\sim \oplus \frac{1}{|12|} \|\phi(x, -x)\|^\sim \end{aligned} \quad (20)$$

for all $x \in X$. Hence from (20), we have

$$\begin{aligned} \|3f(4x) - 9f(3x) + 12f(2x) - 45f(x)\| &\preceq |3| \|\phi(x, x)\|^\sim \oplus \left| \frac{9}{12} \right| \|\phi(0, 0)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(0, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(x, -x)\|^\sim \end{aligned} \quad (21)$$

for all $x \in X$. By (18) and (21) we have

$$\begin{aligned} \|f(3x) + 85f(x) - 14f(2x)\| &\preceq \|\phi(0, 2x)\|^\sim \oplus \left| \frac{4}{12} \right| \|\phi(0, 0)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(0, 2x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2x, -2x)\|^\sim \oplus |3| \|\phi(2x, x)\|^\sim \\ &\oplus \|\phi(3x, x)\|^\sim \oplus |3| \|\phi(x, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{9}{12} \right| \|\phi(0, 0)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(0, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(x, -x)\|^\sim \end{aligned} \quad (22)$$

for all $x \in X$. By (7), (8) and (10) we have

$$\begin{aligned} \|f(3x) - 2f(2x) - 11f(x)\| &\preceq \|\phi(0, x)\|^\sim \oplus \left| \frac{4}{12} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(0, x)\|^\sim \\ &\oplus \left| \frac{3}{12} \right| \|\phi(x, -x)\|^\sim \end{aligned} \quad (23)$$

for all $x \in X$. Hence from (22) and (23), we obtain

$$\|f(2x) - 8f(x)\| \preceq M(x) \quad (24)$$

where

$$\begin{aligned} M(x) = & \frac{1}{|12|} \|\phi(0, x)\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|^\sim \oplus \frac{1}{|12|} \|\phi(0, 2x)\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(0, 2x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(2x, -2x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(2x, x)\|^\sim \\ & \oplus \frac{1}{|12|} \|\phi(3x, x)\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(x, x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{9}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|^\sim \end{aligned}$$

for all $x \in X$. Hence from (24), we have

$$\|f(2x) - 8f(x)\|_\alpha^1 \leq M_1(x) \quad (25)$$

and

$$\|f(2x) - 8f(x)\|_\alpha^2 \leq M_2(x) \quad (26)$$

where

$$\begin{aligned} M_1(x) = & \frac{1}{|12|} \|\phi(0, x)\|_\alpha^{1\sim} \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|_\alpha^{1\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|_\alpha^{1\sim} \\ & \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|_\alpha^{1\sim} \oplus \frac{1}{|12|} \|\phi(0, 2x)\|_\alpha^{1\sim} \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|_\alpha^{1\sim} \\ & \oplus \left| \frac{3}{144} \right| \|\phi(0, 2x)\|_\alpha^{1\sim} \oplus \left| \frac{3}{144} \right| \|\phi(2x, -2x)\|_\alpha^{1\sim} \oplus \left| \frac{3}{12} \right| \|\phi(2x, x)\|_\alpha^{1\sim} \\ & \oplus \frac{1}{|12|} \|\phi(3x, x)\|_\alpha^{1\sim} \oplus \left| \frac{3}{12} \right| \|\phi(x, x)\|_\alpha^{1\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, 0)\|_\alpha^{1\sim} \\ & \oplus \left| \frac{9}{144} \right| \|\phi(0, 0)\|_\alpha^{1\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|_\alpha^{1\sim} \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|_\alpha^{1\sim} \end{aligned} \quad (27)$$

and

$$\begin{aligned}
 M_2(x) = & \frac{1}{|12|} \|\phi(0, x)\|_\alpha^{2\sim} \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|_\alpha^{2\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|_\alpha^{2\sim} \\
 & \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|_\alpha^{2\sim} \oplus \frac{1}{|12|} \|\phi(0, 2x)\|_\alpha^{2\sim} \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|_\alpha^{2\sim} \\
 & \oplus \left| \frac{3}{144} \right| \|\phi(0, 2x)\|_\alpha^{2\sim} \oplus \left| \frac{3}{144} \right| \|\phi(2x, -2x)\|_\alpha^{2\sim} \oplus \left| \frac{3}{12} \right| \|\phi(2x, x)\|_\alpha^{2\sim} \\
 & \oplus \frac{1}{|12|} \|\phi(3x, x)\|_\alpha^{2\sim} \oplus \left| \frac{3}{12} \right| \|\phi(x, x)\|_\alpha^{2\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, 0)\|_\alpha^{2\sim} \\
 & \oplus \left| \frac{9}{144} \right| \|\phi(0, 0)\|_\alpha^{2\sim} \oplus \left| \frac{3}{144} \right| \|\phi(0, x)\|_\alpha^{2\sim} \oplus \left| \frac{3}{144} \right| \|\phi(x, -x)\|_\alpha^{2\sim}
 \end{aligned} \tag{28}$$

for all $x \in X$ and $\alpha \in (0, 1]$. From (25), we have

$$\left\| \frac{f(2x)}{8} - f(x) \right\|_\alpha^1 \leq \frac{M_1(x)}{|8|} \tag{29}$$

for all $x \in X$ and $\alpha \in (0, 1]$. Replacing x by $2^n x$ in (29) and dividing both sides of (29) by $|8^n|$, we get

$$\left\| \frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n} \right\|_\alpha^1 \leq \frac{M_1(2^n x)}{|8^{n+1}|} \tag{30}$$

for all $x \in X$ and all non-negative integers n . Hence

$$\begin{aligned}
 \left\| \frac{f(2^n x)}{8^n} - f(x) \right\|_\alpha^1 &= \left\| \sum_{k=0}^{n-1} \left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \right) \right\|_\alpha^1 \\
 &\leq \sum_{k=0}^{n-1} \left\| \frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \right\|_\alpha^1 \\
 &\leq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M_1(2^k x)}{|8^k|}
 \end{aligned} \tag{31}$$

for all $x \in X$ and $\alpha \in (0, 1]$. Replacing x by $2^m x$ and dividing by $|8^m|$ in (31),

for any $m, n > 0$, we have

$$\begin{aligned} \left\| \frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^n} \right\|_\alpha^1 &= \frac{1}{|8^m|} \left\| \frac{f(2^n 2^m x)}{8^m} - f(2^m x) \right\|_\alpha^1 \\ &\leq \frac{1}{|8|} \sum_{k=0}^{n-1} \frac{M_1(2^{k+m}x)}{|8^{k+m}|} \\ &\leq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M_1(2^{k+m}x)}{|8^{k+m}|} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned} \quad (32)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Similarly we can show from (26)

$$\left\| \frac{f(2^n x)}{8^n} - f(x) \right\|_\alpha^2 \leq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M_2(2^k x)}{|8^k|} \quad (33)$$

and

$$\begin{aligned} \left\| \frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m} \right\|_\alpha^2 &\leq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M_2(2^{k+m}x)}{|8^{k+m}|} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \end{aligned} \quad (34)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Therefore from (32) and (34), we conclude $\left\{ \frac{f(2^n x)}{8^n} \right\}$ is a Cauchy sequence in Y for all $x \in X$, i.e.,

$$\begin{aligned} \left\| \frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m} \right\| &\preceq \frac{1}{|8|} \sum_{k=0}^{\infty} \frac{M(2^{k+m}x)}{|8^{k+m}|} \\ &\rightarrow \bar{0} \quad \text{as } m \rightarrow \infty \end{aligned} \quad (35)$$

Since Y is complete the sequence $\left\{ \frac{f(2^n x)}{8^n} \right\}$ converges in Y for all $x \in X$. So one can define the mapping $C : X \rightarrow Y$ by

$$C(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{8^n} \quad (36)$$

for all $x \in X$. That is

$$\lim_{n \rightarrow \infty} \left\| \frac{f(2^n x)}{8^n} - C(x) \right\| = \bar{0} \quad (37)$$

for all $x \in X$. Letting $n \rightarrow \infty$ in (31) and (33), we see that (5) holds for all $x \in X$. Now we show that $C : X \rightarrow Y$ is a cubic mapping. Replace (x, y) by

$(2^n x, 2^n y)$ and divide by $|8^n|$ in (4), we obtain

$$\begin{aligned} \frac{1}{|8^n|} \|f(2^n(x+3y)) - 3f(2^n(x+2y)) + 4f(2^n(x+y)) - 4f(2^n x) \\ + 3f(2^n(x-y)) - f(2^n(x-2y)) - 12f(2^n y)\| \preceq \left\| \frac{\phi(2^n x, 2^n y)}{8^n} \right\|^\sim \end{aligned}$$

for all $x, y \in X$. Letting $n \rightarrow \infty$ in the above inequality, we see that

$$\|C(x+3y) - 3C(x+2y) + 4C(x+y) - 4C(x) + 3C(x-y) - C(x-2y) - 12C(y)\| \preceq \bar{0}$$

which gives

$$C(x+3y) - 3C(x+2y) + 4C(x+y) - 4C(x) + 3C(x-y) - C(x-2y) - 12C(y) = \underline{0}$$

for all $x, y \in X$. So the mapping $C : X \rightarrow Y$ is cubic.

To prove the uniqueness of C . Let $T : X \rightarrow Y$ be another cubic mapping satisfying (1) and (5). For fix $x \in X$, clearly $C(2^n x) = 8^n C(x)$ and $T(2^n x) = 8^n T(x)$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} \|C(x) - T(x)\| &= \frac{1}{|8^n|} \|C(2^n x) - T(2^n x)\| \\ &= \frac{1}{|8^n|} \|C(2^n x) - f(2^n x) + f(2^n x) - T(2^n x)\| \\ &\preceq \frac{1}{|8^n|} \|C(2^n x) - f(2^n x)\| \oplus \frac{1}{|8^n|} \|f(2^n x) - T(2^n x)\| \\ &\preceq \frac{1}{|8^{n+1}|} \left\{ \sum_{k=0}^{\infty} \frac{M(2^{k+n}x)}{|8^k|} \oplus \sum_{k=0}^{\infty} \frac{M(2^{k+n}x)}{|8^k|} \right\} \\ &\rightarrow \bar{0} \text{ as } n \rightarrow \infty \end{aligned}$$

for all $x \in X$. Hence $C(x) - T(x) = \underline{0}$, for all $x \in X$. So $C(x) = T(x)$. Hence C is unique. $\square$

Corollary 3.2. Let p be a non negative real number with $p < 3$, X be a normed space with norm $\|\cdot\|$, $(Z, \|\cdot\|^\sim)$ be a fuzzy normed space, $(Y, \|\cdot\|)$ be a complete fuzzy normed space, and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that

$$\|D_q f(x, y)\| \preceq \|(\|x\|^p + \|y\|^p)z_0\|^\sim \quad (38)$$

for all $x, y \in X$, then there exist a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|C(x) - f(x)\| \preceq \frac{\|x\|^p z_0\|^\sim}{|8| - |2|^p} \left[\left| \frac{102}{144} \right| \oplus \left| \frac{21(|2|^p)}{144} \right| \oplus \left| \frac{3(|2|^p + 1)}{12} \right| \oplus \left| \frac{|3|^p + 1}{12} \right| \right] \quad (39)$$

for all $x \in X$.

Proof. By choosing $\phi : X^2 \rightarrow Z$ in Theorem 3.1 as $(\|x\|^p + \|y\|^p)z_0$, we get the required result. $\square$

Corollary 3.3. *Let r, s be non negative real numbers with $\lambda := r + s < 3$, X be a normed space with norm $\|\cdot\|$. $(Z, \|\cdot\|^\sim)$ be a fuzzy normed space, $(Y, \|\cdot\|)$ be a complete fuzzy normed space, and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that*

$$\|D_q f(x, y)\| \preceq \|[\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0\|^\sim \quad (40)$$

for all $x, y \in X$, then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\begin{aligned} \|C(x) - f(x)\| \preceq & \frac{\| \|x\|^\lambda z_0 \|^\sim}{|8| - |2|^\lambda} \left[|1| \oplus \left| \frac{|2|^\lambda}{6} \right| \oplus \left| \frac{3}{12}(|2|^r + |2|^\lambda + 1) \right| \right. \\ & \left. \oplus \left| \frac{1}{12}(|3|^r + |3|^\lambda + 1) \right| \right] \end{aligned} \quad (41)$$

Proof. By choosing $\phi : X^2 \rightarrow Z$ in Theorem 3.1 as $[\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0$, we arrive at the above result. $\square$

Theorem 3.4. *Let X be a linear space and let $(Z, \|\cdot\|^\sim)$ be a fuzzy normed space. Let $\phi : X^2 \rightarrow Z$ be a function such that*

$$\lim_{n \rightarrow \infty} |8^n| \left\| \phi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\|_\alpha^\sim = \lim_{n \rightarrow \infty} |8^n| \left\| \phi \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\|_\alpha^{\sim^2} = 0 \quad (42)$$

$\forall x \in X, \alpha \in (0, 1]$. Let $(Y, \|\cdot\|)$ be a fuzzy Banach space and $f : X \rightarrow Y$ be a mapping such that

$$\|D_q f(x, y)\| \preceq \|\phi(x, y)\|^\sim \quad \forall x \in X,$$

then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|C(x) - f(x)\| \preceq \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{k+1}| M \left(\frac{x}{2^{k+1}} \right) \quad (43)$$

for all $x \in X$, where

$$\begin{aligned} M \left(\frac{x}{2^{k+1}} \right) = & \frac{1}{|12|} \|\phi(0, \frac{x}{2^{k+1}})\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, \frac{x}{2^{k+1}})\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(\frac{x}{2^{k+1}}, -\frac{x}{2^{k+1}})\|^\sim \oplus \frac{1}{|12|} \|\phi(0, \frac{x}{2^k})\|^\sim \oplus \left| \frac{4}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{3}{144} \right| \|\phi(0, \frac{x}{2^k})\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(\frac{x}{2^k}, -\frac{x}{2^k})\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(\frac{x}{2^k}, \frac{x}{2^{k+1}})\|^\sim \\ & \oplus \frac{1}{|12|} \|\phi(3\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}})\|^\sim \oplus \left| \frac{3}{12} \right| \|\phi(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}})\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, 0)\|^\sim \\ & \oplus \left| \frac{9}{144} \right| \|\phi(0, 0)\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(0, \frac{x}{2^{k+1}})\|^\sim \oplus \left| \frac{3}{144} \right| \|\phi(\frac{x}{2^{k+1}}, -\frac{x}{2^{k+1}})\|^\sim \end{aligned} \quad (44)$$

Proof. By (25), we have

$$\left\| f(x) - 8f\left(\frac{x}{2}\right) \right\|_{\alpha}^1 \leq M_1\left(\frac{x}{2}\right) \quad (45)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Replacing x by $\frac{x}{2^n}$ in (45) and multiplying $|8^n|$ on both sides of (45), we obtain

$$\left\| 8^n f\left(\frac{x}{2^n}\right) - 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\|_{\alpha}^1 \leq |8^n| M_1\left(\frac{x}{2^{n+1}}\right) \quad (46)$$

for all $x, y \in X$ and all non-negative integers n . Hence,

$$\begin{aligned} \left\| 8^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\alpha}^1 &= \left\| \sum_{k=0}^{n-1} 8^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 8^k f\left(\frac{x}{2^k}\right) \right\|_{\alpha}^1 \\ &\leq \sum_{k=0}^{n-1} \left\| 8^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 8^k f\left(\frac{x}{2^k}\right) \right\|_{\alpha}^1 \\ &\leq \sum_{k=0}^{\infty} |8^k| M_1\left(\frac{x}{2^{k+1}}\right) = \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{k+1}| M_1\left(\frac{x}{2^{k+1}}\right) \end{aligned} \quad (47)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Replacing x by $\frac{x}{2^m}$ and multiplying by $|8^m|$ in (47), for any $m, n > 0$, we have

$$\begin{aligned} \left\| 8^{n+m} f\left(\frac{x}{2^{n+m}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\|_{\alpha}^1 &= |8^m| \left\| 8^n f\left(\frac{x}{2^n 2^m}\right) - f\left(\frac{x}{2^m}\right) \right\|_{\alpha}^1 \\ &\leq |8^m| \sum_{k=0}^{\infty} |8^k| M_1\left(\frac{x}{2^{k+m+1}}\right) \\ &\leq \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{m+k+1}| M_1\left(\frac{x}{2^{k+m+1}}\right) \\ &\rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned} \quad (48)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Similarly, it can be shown from (26)

$$\left\| 8^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\alpha}^2 \leq \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{k+1}| M_2\left(\frac{x}{2^{k+1}}\right) \quad (49)$$

and

$$\begin{aligned} \left\| 8^{n+m} f\left(\frac{x}{2^{n+m}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\|_{\alpha}^2 &\leq \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{m+k+1}| M_2\left(\frac{x}{2^{k+m+1}}\right) \\ &\rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned} \quad (50)$$

for all $x \in X$ and $\alpha \in (0, 1]$. Hence from (48) and (50), we conclude $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x \in X$. That is

$$\begin{aligned} \left\| 8^{n+m} f\left(\frac{x}{2^{n+m}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\| &\preceq \frac{1}{|8|} \sum_{k=0}^{\infty} |8^{m+k+1}| M_1\left(\frac{x}{2^{k+m+1}}\right) \\ &\rightarrow \bar{0} \text{ as } m \rightarrow \infty \end{aligned} \quad (51)$$

Since Y is complete, the sequence $\{8^n f(\frac{x}{2^n})\}$ converges in Y for all $x \in X$. So, we can define the mapping $C : X \rightarrow Y$ by

$$C(x) = \lim_{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. That is

$$\lim_{n \rightarrow \infty} \left\| 8^n f\left(\frac{x}{2^n}\right) - C(x) \right\| = \bar{0}$$

for all $x \in X$. Letting $n \rightarrow \infty$ in (47) and (49), we see that (43) holds for all $x \in X$. Using the same method as in the proof of Theorem 3.1, we conclude that $C(x)$ is the unique cubic type mapping satisfying (43). $\square$

Corollary 3.5. *Let p be a non-negative real number with $p > 3$, X be a normed space with norm $\|\cdot\|$, $(Z, \|\cdot\|^\sim)$ be a fuzzy normed space, $(Y, \|\cdot\|)$ be a complete fuzzy normed space and let $z_0 \in Z$.*

If $f : X \rightarrow Y$ is a mapping such that

$$\|D_q f(x, y)\| \preceq \|(\|x\|^p + \|y\|^p)z_0\|^\sim$$

for all $x, y \in X$, then there exist a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|C(x) - f(x)\| \preceq \frac{\| \|x\|^p z_0 \|^\sim}{|2|^p - |8|} \left[\left| \frac{102}{144} \right| \oplus \left| \frac{21(|2|^p)}{144} \right| \oplus \left| \frac{3(|2|^p + 1)}{12} \right| \oplus \left| \frac{|3|^p + 1}{12} \right| \right] \quad (52)$$

for all $x \in X$.

Proof. The result follows immediately by choosing $\phi(x, y) = [\|x\|^p + \|y\|^p]z_0$ in Theorem 3.4. $\square$

Corollary 3.6. *Let r, s be non-negative real numbers with $\lambda := r + s > 3$, X be a normed space with norm $\|\cdot\|$, $(Z, \|\cdot\|^\sim)$ be a fuzzy normed space, $(Y, \|\cdot\|)$ be a complete fuzzy normed space and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that*

$$\|D_q f(x, y)\| \preceq \|[\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0\|^\sim$$

for all $x, y \in X$, then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|C(x) - f(x)\| \preceq \frac{\| \|x\|^\lambda z_0 \| \sim}{|2|^\lambda - |8|} \left[|1| \oplus \left| \frac{24(|2|^\lambda)}{144} \right| \oplus \left| \frac{3(|2|^r + |2|^\lambda + 1)}{12} \right| \oplus \left| \frac{|3|^r + |3|^\lambda + 1}{12} \right| \right] \quad (53)$$

Proof. By choosing $\phi : X^2 \rightarrow Z$ in Theorem 3.4 as $[\|x\|^r \|y\|^s + (\|x\|^{r+s} + \|y\|^{r+s})]z_0$, we get the required result. $\square$

Example 3.7. Let $(X, \|\cdot\|)$ be a real Banach algebra, $(X, \|\cdot\|^*)$ be a complete fuzzy norm space in which

$$\|x\|^*(t) = \begin{cases} \frac{\|x\|}{t}, & \text{when } \|x\| \leq t, x \neq \underline{0}, \\ 1, & \text{when } t = \|x\| = 0, \\ 0, & \text{otherwise.} \end{cases}$$

whose α -level set is given by $[\|x\|^*]_\alpha = \left[\|x\|, \frac{\|x\|}{\alpha} \right]$ for all $x \in X$. Define $f : X \rightarrow X$ by $f(x) = x^3 + \|x\|x_0$, where x_0 is a unit vector in X . A straight forward computation shows that

$$\begin{aligned} & \|f(x+3y) - 3f(x+2y) + 4f(x+y) - 4f(x) + 3f(x-y) - f(x-2y) - 12f(y)\| \\ & \leq (16\|x\| + 30\|y\|)x_0 \end{aligned}$$

for all $x, y \in X$, hence

$$\|D_q f(x, y)\|^* \preceq \|(16\|x\| + 30\|y\|)x_0\|^*$$

for all $x, y \in X$. Let $\phi(x, y) = (16\|x\| + 30\|y\|)x_0$ for all $x, y \in X$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\|\phi(2^n x, 2^n y)\|_\alpha^{*1}}{|8^n|} &= \lim_{n \rightarrow \infty} \frac{\|(16\|2^n x\| + 30\|2^n y\|)x_0\|}{|8^n|} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\|\phi(2^n x, 2^n y)\|_\alpha^{*2}}{|8^n|} &= \lim_{n \rightarrow \infty} \frac{\|(16\|2^n x\| + 30\|2^n y\|)x_0\|}{\alpha|8^n|} \\ &= 0 \end{aligned}$$

Moreover the above conditions hold for all values of t . Therefore all the conditions of Theorem 3.1 hold and so there exists a unique cubic mapping $C : X \rightarrow X$ such that

$$\|C(x) - f(x)\|^* \preceq \left\| \frac{71}{9} \|x\|x_0 \right\|^\sim \preceq \|8\|x\|x_0\|^\sim$$

for all $x \in X$.

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