Graceful Labeling for Some Star Related Graphs

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Abstract

We have investigate graceful labeling and cordial labeling for the graphs obtained by joining apex vertices of some star graphs to a new vertex.

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1 INTRODUCTION :

We begin with the simple, connected, undirected and finite graph $G = (V, E)$. For all standard terminology and notations we follow Harary [4]. We will give brief summery of definitions which are used in this paper.

Definition–1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Definition–1.2: Given a graph $G = (V, E)$, the set N of non–negative integers and a commutative binary operation $*$: $N \times N \rightarrow N$, every vertex $f : V \rightarrow N$ induces an edge function $f^* : E \rightarrow N$ such that $f^*(uv) = |f(u) - f(v)|$, for all $uv \in E$. A function f is called *graceful labeling* of a graph G if $f : V \rightarrow \{0, 1, 2, \dots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \dots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called *graceful graph*.

Definition–1.3: A function $f : V \rightarrow \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex v* of G under f .

For an edge $e = uv$, the induced function $f^* : E \rightarrow \{0, 1\}$ is given as $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^* .

A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph*.

Definition–1.4: Let $G = (V, E)$ be a graph. A mapping $f : V \rightarrow \{0, 1, 2\}$ is called *ternary vertex labeling* of G and $f(v)$ is called *label of the vertex v* of G under f .

For an edge $e = uv$ the induced edge labeling $f^* : E \rightarrow \{0, 1, 2\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1), v_f(2)$ be the number of vertices of G having labels 0, 1 and 2 respectively under f and let $e_f(0), e_f(1), e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under f^* .

A ternary vertex labeling of a graph G is called *3–equitable labeling* if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, $0 \leq i, j \leq 2$. A graph G is called *3–equitable graph* if it admits 3–equitable labeling.

Definition–1.5: Consider t copies of stars namely $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t}$. Then the $G = \langle K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t} \rangle$ is the graph obtained by joining apex vertices of each K_{1,n_i} and $K_{1,n_{i+1}}$ to a new vertex u_i , where $1 \leq i \leq t-1$.

For detail survey of graph labeling one can refer Gallian [3]. Vast amount of literature is available on different type of graph labeling. Labeled graphs have variety of applications in coding theory particularly missile guidance codes. A detail study of variety of applications of graph labeling is given by bloom and Golomb [1].

The graceful labeling was introduced by A. Rosa [5] during 1967. He proved that the cycle C_n is graceful, when $n \equiv 0, 3 \pmod{4}$. Rosa [5] and Golomb [2] proved that the complete graph bipartite graphs $K_{m,n}$ are graceful. Vaidya et al. [6] proved a graph obtained by some copies of star by joining their apex with a new vertex is cordial as well as 3-equitable.

2 MAIN RESULTS :

Theorem-2.1 : The graph G obtained by joining some stars $\langle K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t} \rangle$ is graceful.

Proof : Let $v_{i,0}, v_{i,1}, \dots, v_{i,n_i}$ $i = 1, 2, \dots, t$ be vertices of star graphs K_{1,n_i} , $i = 1, 2, \dots, t$. We shall join these graphs K_{1,n_i} and $K_{1,n_{i+1}}$ by new vertex u_i , where $1 \leq i \leq t-1$ by their apex vertices. We define labeling function $f : V \rightarrow \{0, 1, \dots, q\}$, where $q = \sum_{i=1}^t n_i + (2t - 2)$ as follows:

$$\begin{aligned} f(v_{i,0}) &= i - 1, & \forall i = 1, 2, \dots, t. \\ f(v_{1,j}) &= q - (j - 1), & \forall j = 1, 2, \dots, n_1. \\ f(u_j) &= q - \sum_{l=1}^j n_l - (j - 1) \\ f(v_{j,i}) &= q - \sum_{l=1}^{j-1} n_l - (j + i - 2), & \forall i = 1, 2, \dots, n_j; \\ & & \forall j = 2, 3, \dots, t. \end{aligned}$$

The above labeling pattern give rise graceful labeling to the given graph G .

Illustration-2.2: A graph $G = \langle K_{1,6}, K_{1,5}, K_{1,3} \rangle$ and its graceful labeling is shown in following Figure-1.

Figure-1

Let $v_f(\text{odd})$, $v_f(\text{even})$ be the number of vertices of a graceful graph G with labeling function f having odd labels and even labels respectively under f . Also $e_f(\text{even})$ = the number of even edge label of the graph G , $e_f(\text{odd})$ = the number of odd edge label of the graph G under the corresponding function f^* . We know that for a graceful graph G with graceful labeling f , $|e_f(\text{odd}) - e_f(\text{even})| \leq 1$. This property will help for following result.

Theorem-2.3 : For a graceful graph G with graceful labeling f , if it satisfies the condition $|v_f(\text{odd}) - v_f(\text{even})| \leq 1$ then G is a cordial graph.

Proof : We define $g : V \rightarrow \{0, 1\}$, the binary vertex labeling for G as follows:

$f(u) = 1$ if u has odd vertex label and

$f(u) = 0$ if u has even vertex label under graceful labeling f .

Then we see that $|v_g(0) - v_g(1)| \leq 1$ and

$|e_g(0) - e_g(1)| \leq 1$, because $e_g(0) = e_f(\text{even})$ and $e_g(1) = e_f(\text{odd})$, where $v_g(0)$, $v_g(1)$ be number of vertices of G having labels 0 and 1 respectively under g and let $e_g(0)$, $e_g(1)$ be number of edges of G having labels 0 and 1 respectively under the induced function g^* by g .

Therefore G admits a cordial labeling g and so G is a cordial graph.

Illustration-2.4: $K_{4,3}$ with its graceful labeling and the corresponding cordial labeling is shown in following Figure-2.

Figure-2

Remarks :- Every graceful graph does not have the property $|v_f(\text{odd}) - v_f(\text{even})| \leq 1$, where f is graceful labeling for G and so every graceful graph may not be cordial. We know that K_4 is graceful graph, while it is not a cordial graph as in K_4 , $v_f(\text{odd}) = 1$ and $v_f(\text{even}) = 3$.

Corollary–2.5 : The graph $G = \langle K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_t} \rangle$ is cordial.

Proof : Above graph G is a tree and so the number of vertices for G is $q + 1$, where q = the number of edges for G , which is here $\sum_{i=1}^t n_i + 2(t - 1)$.

So the graceful labeling f for G is bijective map. Hence it satisfies $|v_f(\text{odd}) - v_f(\text{even})| \leq 1$. Therefore by *Theorem–2.3* G is a cordial graph.

3 CONCLUDING REMARKS :

Labeled graph is the topics of current interest. We discussed here graceful labeling and cordial labeling of some star related graphs. In [6] vaidya et al. proved that the graph obtained by $\langle K_{1,n}^1; K_{1,n}^2; \dots; K_{1,n}^k \rangle$ (k copies of $K_{1,n}$) is cordial graph, which is particular case of our *Corollary–2.5* by taking $n_1 = n_2 = \dots = n_t = n$ and $t = k$.

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