Path Double Covering Number of a Bicyclic Graphs

T. Gayathri

Department of Mathematics
Sri Manakula Vinayagar Engineering College
Puducherry-605 107, India

S. Meena

Department of Mathematics
Government Arts and Science College
Chidambaram-608 102, India

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Abstract

A path double cover of a graph G is a collection $\mathcal{P}$ of paths in G such that every edge of G belongs to exactly two paths in $\mathcal{P}$. The minimum cardinality of a path double cover is called the path double covering number of G and is denoted by $\eta_{PD}(G)$. In this paper we determine the exact value of this parameter for several classes of graphs.

Keywords: Graphoidal covers, path double covers, path double covering number of a graph, bicyclic graphs

1 Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here we consider only nontrivial, finite, connected undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [6]. The concept of graphoidal cover was introduced by B.D Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4]. The reader may refer [6] and [2] for the terms not defined here.

Let $p = (v_1, v_2, v_3, \dots, v_r)$ be a path or a cycle in a graph $G = (V, E)$. Then vertices $(v_2, v_3, \dots, v_{r-1})$ are called internal vertices of P and v_1 and v_r are called external vertices of P . Two paths P and Q of a graph G are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q . If $P = (v_0, v_1, v_2, \dots, v_r)$ and $Q = (v_r = w_0, w_1, w_2, \dots, w_r)$ are two paths in G then the walk obtained by concatenating P and Q at v_r is denoted by $P \circ Q$ and the path $(v_r, v_{r-1}, \dots, v_0)$ is denoted by P^{-1} . [3]. Bondy [5] introduced the concept of path double cover of a graph. This was further studied by Hao Li [7].

Definition 1.1[3]: A path double cover (PDC) of a graph G is a collection $\mathcal{P}$ of paths in G such that every edge of G belongs to exactly two paths in $\mathcal{P}$.

The collection $\mathcal{P}$ may not necessarily consist of distinct paths in G and hence it cannot be treated as a set in the standard sense. For any graph $G = (V, E)$, let $\mathcal{P}$ denote the collection of all paths of length one each path appearing twice in the

collection. Clearly $\mathcal{P}$ is a path double of G and hence the set of all path double covers of G is non-empty.

Arumugam and Meena [3] introduced the concept of path double covering number of a graph G .

Definition 1.2[3]: The minimum cardinality of a path double cover of a graph G is called path double covering number of G and is denoted by $\eta_{PD}(G)$

In [9] it has been observed that for any graph G $\eta_{PD}(G) \leq 2q$ and equality holds if and only if G is isomorphic to qK_2 and the following results have been proved .

Theorem1.3[3]: Let $\mathcal{P}$ be any path double cover of a graph G . Then $|\mathcal{P}| = 2q - ip$ where $ip = \sum_{p \in \mathcal{P}} i(p)$ where $i(p)$ is the number of internal vertices of $\mathcal{P}$.

Theorem1.4 [3]: $\eta_{PD} = 2q - i$ where $i = \max_{\mathcal{P}} i$ the maximum being taken over all path double covers $\mathcal{P}$ of G

Theorem 1.5 [3]: Let G be a graph with $\delta = 1$, if there exists a path double cover $\mathcal{P}$ such that every non pendant vertex of G is an internal vertex of $d(v)$ paths in $\mathcal{P}$ then $\mathcal{P}$ is minimum path double cover and $\eta_{PD} = |\mathcal{P}|$

Theorem 1.6 [3]: For any tree T , $\eta_{PD}(T) = n$ where n is the number of pendent vertices of T .

Theorem 1.7 [3]: For any graph G , $\eta_{PD}(T) \geq \Delta$. Further for any tree T , $\eta_{PD}(T) = \Delta$ if and only if T is homeomorphic to a star.

Definition 1.8 [8]: A connected $(p, p+1)$ graph G is called a bicyclic graph.

Definition 1.9 [8]: A one – point union of two cycles is a simple graph obtained from two cycles, say C_l and C_m where $l, m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we may assume the l -cycle to be $u_0u_1 \dots u_{l-1}u_0$ and the m -cycle to be $u_0u_lu_{l+1} \dots u_{m+l-2}u_0$. We denote this graph by $U(l;m)$

Definition 1.10 [8]: A long dumbbell graph is a simple graph obtained by joining two cycles C_l and C_m where $l, m \geq 3$, with a path of length i , $i \geq 1$. Without loss of generality, we may assume $C_l = u_0u_1 \dots u_{l-1}u_0$, $P_i = u_{l-1}u_lu_{l+1} \dots u_{l+i-1}$ and $C_m = u_{l+i-1}u_{l+i} \dots u_{l+m+i-2}u_{l+i-1}$. We denote this graph by $D(l,m,i)$

Definition 1.11 [8]: A cycle with a long chord is a simple graph obtained from an m -cycle, $m \geq 4$, by adding a chord of length l where $l \geq 1$. Let the m -cycle be $u_0u_1 \dots u_{m-1}u_0$. Without loss of generality, we may assume the chord joins u_0 with u_i , where $2 \leq i \leq m-2$. That is, $u_0u_mu_{m+1} \dots u_{l+m-2}u_i$ is the chord. We denote this graph by $C_m(i;l)$.

2 Main Results

In this paper we determine the value of η_{PD} for $U(l;m)$, $D(l,m,i)$ and $C_m(i;l)$ and for some classes of graphs.

Theorem 2.1

Let G be a bicyclic graph with n pendant vertices and G containing a $U(l,m)$ and j be the number of vertices greater than 2 on $U(l,m)$ other than u_0 . Let j_1 and j_2 are the number of vertices of degree greater than 2 on C_l and C_m respectively.

Then the path double covering number of G is

$$\eta_{PD} = \begin{cases} 4 & \text{if } G = U(l, m) \\ n+3 & \text{if } j_1 = 0 \text{ or } j_2 = 0 \text{ and } j = 1 \text{ \& } n \geq j \\ n+2 & \text{if } \begin{cases} j_1 = 0 \text{ or } j_2 = 0 \text{ and } j \geq 2 \text{ \& } n > j \\ j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2, 3 \text{ \& } n = j \\ j_1 \geq 1 \text{ or } j_2 \geq 1 \text{ and } j = 2 \text{ \& } n > j \end{cases} \\ n+1 & \text{if } j = 4 \text{ and } n = j \\ n & \text{otherwise} \end{cases}$$

Proof: Let $V(U(l, m)) = \{u_0, u_1, u_2, u_3, \dots, u_{l-1}, u_l, u_{l+1}, \dots, u_{l+m-2}\}$

And let $j = j_1 + j_2$ Where j_1 and j_2 are the number of vertices of degree greater than 2 on C_l and C_m respectively.

Case 1: $U(l, m)$ The path double covering of G is as follows

$$P_1 = \{u_1, u_2, u_3, \dots, u_{l-1}, u_0, u_l, u_{l+1}, u_{l+2}, \dots, u_{l+m-2}\}$$

$$P_2 = \{u_{l-1}, u_{l-2}, \dots, u_2, u_1, u_0, u_{l+m-2}, u_{l+m-1}, \dots, u_{l+1}, u_l\}$$

$$P_3 = \{u_1, u_0, u_{l+m-2}\}$$

$$P_4 = \{u_{l-1}, u_0, u_l\}$$

$$\eta_{PD}(G) = 4 = \Delta$$

Case 2: Either $j_1 = 0$ or $j_2 = 0$ and $j = 1$

We prove this by induction on n .

When $j = 1$ and $n = 1$

G is isomorphic to the graph consisting of $U(l, m)$ together with a path

$$P = \{w_1, w_2, \dots, w_r (= u_i)\} \text{ where } u_i \text{ on } C_l$$

The path double cover is as follows

$$P_1 = \{w_1, w_2, w_3, \dots, w_r = u_i, u_{i+1}, u_{i+2}, \dots, u_{l-1}, u_0, u_l, \dots, u_{l+m-2}\}$$

$$P_2 = \{u_l, u_{l+1}, \dots, u_{l+m-2}, u_0, u_1, u_2, \dots, u_i = w_r, w_{r-1}, w_{r-2}, \dots, w_1\}$$

$$P_3 = \{u_{l-1}, u_0, u_l\}$$

$$P_4 = \{u_{l+m-2}, u_0, u_1, \dots, u_i, \dots, u_{l-1}\}$$

$$\Rightarrow \eta_{PD} \leq 4$$

Since $\eta_{PD} \geq \Delta = 4$

$$\therefore \eta_{PD} = 4 = n + 3$$

Now assume that the result is true for all bicyclic graphs containing a $U(l, m)$

with $j=1$ and having less than n pendant vertices and $n \geq 2$

Let G be any bicyclic graph containing a $U(l, m)$ with n pendant vertices and $j=1$

Let w be any pendant vertex of G .

Choose a vertex v such that $\deg v \geq 5$ and $d(w, v)$ is maximum.

Let Q denote the (w, v) path, Since $\deg v \geq 5$ there exist pendant vertices w_1, w_2 such that (w, w_1) path Q_1 and (w, w_2) path Q_2 both contain Q .

Now let P_1 and P_2 denote the (w_1, v) section of Q_1 and (w_2, v) section of Q_2 respectively.

$$\text{Let } P = P_1 \circ P_2^{-1}$$

Clearly v is the only vertex of degree greater than two on P .

$$\text{let } P = (w_1 = v_0, v_1, v_2, \dots, v_r = v, \dots, v_k = w_2)$$

Consider the graph $G_1 = G - P$

$$\text{Using induction hypothesis } \eta_{PD}(G_1) = (n-2) + 3 = n+1$$

Let $\mathbb{P}_i$ be a minimum path double covering of G_1

Let $\mathbb{P} = \mathbb{P}_i \cup (P, P)$ be a minimum path double cover of G

$$\eta_{PD}(G) = (n+1) + 2 = n+3$$

Case 3:

Sub case 3a: Either $j_1=0$ or $j_2=0$ and $n > j$ and $j \geq 2$

We prove by induction on n

Let $j = 2, n = 3$ (since $j \geq 2$ and $n > j$)

Without loss of generality assume that u_k on C_l is incident with the the path has a pendant vertex w_1 and u_i on C_l incident with the path has pendant vertices w_2 and w_3 respectively. ($i < k$)

The path double cover is as follows

$$P_1 = \{(w_2 =)w_{21}, w_{22}, w_{23}, \dots, w_{2n}(=u_i), u_{i-1}, u_{i-2}, \dots, u_0, u_{l+m-2}, \dots, u_l\}$$

$$P_2 = \{(w_3 =)w_{31}, w_{32}, w_{33}, \dots, w_{3n}(=u_i), u_{i+1}, \dots, u_k, \dots, u_{l-1}, u_0, u_l, \dots, u_{l+m-2}\}$$

$$P_3 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \dots, w_{1n}(=u_k), u_{k+1}, \dots, u_{l-1}, u_0, u_l\}$$

$$P_4 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \dots, w_{1n}(=u_k), u_{k-1}, \dots, u_i, \dots, u_1, u_0, u_{l+m-2}, \dots, u_l\}$$

$$P_5 = \{(w_2 =)w_{21}, w_{22}, w_{23}, \dots, w_{2n}(=u_i), (w_3 =)w_{31}, w_{32}, w_{33}, \dots, w_{3n}(=u_i)\}$$

$$\therefore \eta_{PD} = n + 2$$

Let us assume that the result is true for all bicyclic graph containing a $U(l, m)$ with $n-1$ pendant vertices, $j_1 = 0$ or $j_2 = 0$ and $n > j$ and $j \geq 2$ and

Now let G be a bicyclic graph containing a $U(l, m)$ with n pendant vertices $(w_1, w_2, \dots, w_n)$ and $j_1 = 0$ or $j_2 = 0$ and $n > j$ and $j \geq 2$

As before the mentioned paths P_1 to P_5 covers the bicyclic as well as three pendant vertices w_1, w_2, w_3

Let $G_1 = G - \{P_1, P_2, P_3, P_4, P_5\}$ is a tree with $n-3$ pendant vertices.

$$\Rightarrow \eta_{PD}(G_1) = n - 3$$

$$\therefore \eta_{PD}(G) = n - 3 + 5 = n + 2$$

Sub case 3b: Let $j_1 \geq 1, j_2 \geq 1, j = 2, 3$ and $j = n$

Let $j = 2$ and $n = j$

Since $j = 2$, clearly $j_1 = 1$ and $j_2 = 1$

Let u_i be the vertex on C_l and u_k be the vertex on C_m are of degree greater than 2.

u_i and u_k is incident with the paths has pendant vertices w_1 and w_2 respectively.

The path double cover is as follows

$$P_1 = \{u_1, u_2, \dots, u_i, \dots, u_0, u_l, \dots, u_{l+m-2}\}$$

$$P_2 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \dots, w_{1n}(=u_i), u_{i+1}, \dots, u_{l-1}, u_0, u_l, \dots, u_k, w_{2n}(=u_k), w_{2(n-1)}, \dots, (w_2 =)w_{21}\}$$

$$P_3 = \{(w_1 =)w_{11}, w_{12}, w_{13}, \dots, w_{1n}(=u_i), u_{i-1}, \dots, u_{l-1}, u_0, u_{l+m-2}, \dots, u_k, w_{2n}(=u_k), w_{2(n-1)}, \dots, (w_2 =)w_{21}\}$$

$$P_4 = \{u_1, u_0, \dots, u_{l+m-2}\}$$

$$\Rightarrow \eta_{PD}(G) \leq 4 = n + 2 = \Delta$$

We know that $\eta_{PD}(G) \geq \Delta$

$$\therefore \eta_{PD}(G) = 4 = n + 2 = \Delta$$

Sub case 3c: $j = 2$ and $n > j$

Let $j = 2$

Without loss of generality assume that $n = j + 2 = 4$

u_i is incident with the pendant vertices w_1 and w_2 on C_l

u_k is incident with the pendant vertices w_3 and w_4 on C_m

The path double cover is as follows

$$P_1 = \{w_1, u_i, u_{i+1}, u_{l-1}, u_0, u_l, \dots, u_k, w_4\}$$

$$P_2 = \{w_4, u_k, u_{k-1}, \dots, u_l, u_0, u_{l-1}, \dots, u_i, w_1\}$$

$$P_3 = \{w_2, u_i, \dots, u_1, u_0, u_{l+m-2}, \dots, u_k, w_3\}$$

$$P_4 = \{w_2, u_i, u_{i+1}, \dots, u_0, u_{l+m-2}, \dots, u_k, w_3\}$$

$$\therefore \eta_{PD}(G) = 4 = \Delta$$

Similarly we can find the path double covering for the case $j = 2, n > j$ and for the case $j = 3$

So that $\eta_{PD}(G) = n + 2 = \Delta$

Case 4: Let $j = 4, n = j = 4$

Let u_i, u_k are the two vertices on C_l are of degree greater than two incident with the pendant vertices w_1 and w_2 respectively.

Let u_m, u_n are the two vertices on C_m are of degree greater than two incident with the pendant vertices w_3 and w_4 respectively.

The path double cover is as follows

$$P_1 = \{w_1, u_i, u_{i+1}, u_{l-1}, u_0, u_l, \dots, u_m, w_3\}$$

$$P_2 = \{w_2, u_k, \dots, u_i, u_{i+1}, \dots, u_{l-1}, u_0, u_l, \dots, u_{m+1}, \dots, u_n, w_4\}$$

$$P_3 = \{w_1, u_i, \dots, u_k, w_2\}$$

$$P_4 = \{u_k, u_{k-1}, \dots, u_0, u_{l+m-2}, \dots, u_n, w_4\}$$

$$P_5 = \{u_k, \dots, u_l, u_0, \dots, u_{l+m-2}, \dots, u_m, w_3\}$$

$$\therefore \eta_{PD}(G) = n + 1$$

Similarly we can find the path double covering for the case $j=4, n > j$

So that $\eta_{PD}(G) = n + 1$

Case 5: Let $j > 5, n > j$

Here the number of pendant vertices is greater than 5, we can find the path double covering such that all the vertices are internal vertices except the pendant vertices by using above proof technique.

$$\therefore \eta_{PD}(G) = n$$

Theorem 2.2

Let G be a bicyclic graph with n pendant vertices and G containing a $D(l, m, i)$ is the unique bicycle in G and let j be the number of vertices of degree greater than 2 on $D(l, m, i)$ except u_{l-1} & u_{l+i-1} . Then the path double covering of G is

$$\eta_{PD} = \begin{cases} 4 & \text{if } G = D(l, m, i) \\ n+4 & \text{if } \begin{cases} j_1 = j_2 = 1 \text{ and } j_3 = 0 \\ j_1 = j_2 = 0 \text{ and } j_3 \geq 1 \end{cases} \\ n+3 & \text{if } \begin{cases} j_1 = 1, 2, j_2 = j_3 = 0 \\ j_1 = 2 \text{ or } j_2 = 1 \text{ and } j_3 = 0 \\ j_1 = j_2 = j_3 = 1 \\ j_1 \text{ or } j_2 = 1, j_3 = 0 \end{cases} \\ n+2 & \text{if } \begin{cases} j_1 \geq 3, j_2 = 0, 1, j_3 = 0 \\ j_1 = j_2 = 2, j_3 = 0 \text{ or } 2 \\ j_1 = j_2 = 1, j_3 = 0 \end{cases} \\ n+1 & \text{if } j_1 \geq 3, j_2 = 2, j_3 = 0 \\ n & \text{otherwise} \end{cases}$$

Where j_1, j_2 and j_3 are the number of vertices greater than two on C_l, C_m and on the path respectively.

Proof

Let $V(D(l, m, i)) = \{u_0, u_1, u_2, u_3, \dots, u_{l-1}, u_l, u_{l+i-1}, u_{l+i}, \dots, u_{l+m+i-2}\}$

$j = j_1 + j_2 + j_3$ Where j_1 and j_2 are the number of vertices of degree greater than 2 on C_l and C_m and j_3 is the number of vertices of degree greater than 2 on the path respectively.

Case 1: $G = D(l, m, i)$

The path double covering of G is as follows

$$P_1 = \{u_0, u_1, u_2, \dots, u_{l-1}, u_l, \dots, u_{l+i-1}, u_{l+i}, \dots, u_{l+m+i-2}\}$$

$$P_2 = \{u_{l+i}, u_{l+i+1}, \dots, u_{l+m+i-2}, u_{l+i-1}, u_l, u_{l-1}, u_0, u_1, u_2, \dots, u_{l-2}\}$$

$$P_3 = \{u_0, u_{l-1}, u_{l-2}\}$$

$$P_4 = \{u_{l+i}, u_{l+i-1}, u_{l+m+i-2}\}$$

$$\therefore \eta_{PD}(G) = 4$$

Case 2:

Sub Case 2a: Let $j_1 = j_2 = 1$ and $j_3 = 0$

$$j = j_1 + j_2 = 2$$

We prove by induction on n

Let $n = 2$ and let v_0, w_0 are pendant vertices.

The path double covering of G is as follows

Let $v_0, v_1, v_2, \dots, v_r$ be a path attached with u_i in C_l and $w_0, w_1, w_2, \dots, w_r$ be a path attached with u_k in C_m

$$P_1 = \{v_0, v_1, v_2, \dots, v_r = u_i, u_{i+1}, u_1, u_0, u_{l-1}, u_l, \dots, u_{l+i-1}, u_{l+i}, \dots, w_r = u_r, w_{r-1}, \dots, w_0\}$$

$$P_2 = \{v_0, v_1, v_2, \dots, v_r = u_i, u_{i-1}, \dots, u_{l-2}, u_{l-1}, u_l, \dots, u_{l+i-1}, u_{l+m+i-2}, w_r = u_r, w_{r-1}, \dots, w_0\}$$

$$P_3 = \{v_r = u_i, u_{i+1}, \dots, u_{l-1}\}$$

$$P_4 = \{v_r = u_i, u_{i-1}, \dots, u_{l-1}\}$$

$$P_5 = \{w_r = u_r, u_{r-1}, \dots, u_{l+i-1}\}$$

$$P_6 = \{w_r = u_r, u_{r+1}, \dots, u_{l+i-1}\}$$

Here the two pendant vertices are exterior vertices and exactly two vertices on both C_l and C_m are exterior vertices.

$$\therefore \eta_{PD}(G) = n + 4$$

Assume that the result is true for all bicyclic graph with $n - 1$ pendant vertices containing a $D(l, m, i)$

Let G be a graph bicyclic graph with n pendant vertices containing a $D(l, m, i)$

Let $j = 2$

The proof is similar to case 2 in Theorem 2.1

Sub Case 2b: Let $j_1 = j_2 = 0$ and $j_3 \geq 1$

Let $j = n = 1$

G is isomorphic to the graph consisting of $D(l, m, i)$ together with a path

$$P = \{p_0, p_1, \dots, p_i (= u_l)\}$$
 attached with the vertex u_l

The path double cover is as follows

$$P_1 = \{u_{l-2}, \dots, u_2, u_1, u_0, u_{l-1}, u_l, \dots, u_{l+i-1}, u_{l+i}, \dots, u_{l+m+i-2}\}$$

$$P_2 = \{p_0, p_1, \dots, p_i (= u_l), u_{l+1}, \dots, u_l, \dots, u_{l+i-1}, u_{l+m+i+2}, \dots, u_{l+i}\}$$

$$P_3 = \{p_0, p_1, \dots, p_i (= u_l), u_{l-1}, u_{l-2}, \dots, u_1, u_0\}$$

$$P_4 = \{u_0, u_{l-1}, u_{l-2}\}$$

$$P_5 = \{u_{l+i}, u_{l+i-1}, u_{l+m+i-2}\}$$

Here one pendant vertex and two vertices in each cycles C_l and C_m are exterior vertices.

$$\therefore \eta_{PD} = n + 4$$

If $j=1$ and $n > 1$ we prove this by using the technique in Theorem 2.1 case 2.

Case 3:

Sub case 3a:

$$(i) j_1 = 1, 2; j_2 = j_3 = 0; j = n \text{ or } j_2 = 1, 2; j_1 = j_3 = 0, j = n$$

$$(ii) j_1 = j_2 = j_3 = 1; j = n$$

$$(iii) j_1 = 1, j_2 = j_3 = 0; j < n$$

In the above cases we can find the path double covering such that the n pendant vertices and three vertices on both C_l on C_m are exterior points.

$$\therefore \eta_{PD} = n + 3$$

In the remaining cases we can find the path double cover such that all the vertices are internal vertices except the pendant vertices

$$\therefore \eta_{PD} = n$$

Theorem 2.3

Let G be a unicyclic graph with n pendant vertices and G containing a $C_m(i, l)$ is the unique cycle in G and let j be the number of vertices greater than 3 on $C_m(i, l)$. Then the path double covering of G is

$$\eta_{PD} = \begin{cases} 3 & \text{if } G = C_m(i, l) \\ n + 3 & \text{if } j = 1 \text{ and } \deg u_i \geq 6 \text{ for some } i \\ n + 2 & \text{if } j_1 \geq 1, j_2 = 0 \text{ or } j_2 \geq 1, j_1 = 0 \\ n & \text{otherwise} \end{cases},$$

where j_1 and j_2 are the number of vertices of degree greater than 2 on the upper half and lower half on $C_m(i, l)$ respectively and $j = j_1 + j_2$

Assume that if $\deg u_i$ or $\deg u_0 \geq 4$ this will increase the number of j_1

Proof

Let $V(C_m(i, l)) = \{u_0, u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_{m-1}, u_m, u_{m+1}, \dots, u_{l+m-2}\}$

$$j = j_1 + j_2$$

Case 1: $G = C_m(i, l)$

The path double covering of G is as follows

$$P_1 = \{u_{l+m-2}, \dots, u_{m+1}, u_m, u_0, u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_{m-1}\}$$

$$P_2 = \{u_m, u_{m+1}, \dots, u_{l+m-2}, u_i, u_{i+1}, \dots, u_{m-1}, u_0, u_1, u_2, \dots, u_{i-1}\}$$

$$P_3 = \{u_m, u_0, u_{m-1}\}$$

$$P_4 = \{u_{l+m-2}, u_i, u_{i-1}\}$$

$$\therefore \eta_{PD}(G) = 4$$

Case 2: If $j = 1$ and $\deg u_m \geq 6$ for some m

Without loss of generality assume that $\deg u_i = 7$

The path double covering of G is as follows

$$P_1 = \{u_1, u_2, \dots, u_{i-1}, u_i, u_{l+m-2}, u_{m+1}, u_m, u_0, u_{m-1}, \dots, u_{i+1}\}$$

$$P_2 = \{w_{11}, w_{12}, \dots, u_i, u_{i+1}, \dots, u_0, u_1, u_2, \dots, u_{i-1}\}$$

$$P_3 = \{w_{21}, w_{22}, \dots, u_i, u_{l+m-2}, \dots, u_m, u_0, u_1\}$$

$$P_4 = \{w_{31}, w_{32}, \dots, u_i, u_{i+1}\}$$

$$P_5 = \{w_{41}, w_{42}, \dots, u_i, u_{i-1}\}$$

$$P_6 = \{w_{11}, w_{12}, \dots, w_{1n} = u_i = w_{2n}, \dots, w_{21}\}$$

$$P_7 = \{w_{31}, w_{32}, \dots, w_{3n} = u_i = w_{4n}, \dots, w_{41}\}$$

$$\therefore \eta_{PD}(G) = 7 = n + 3 = \Delta$$

For the different types of paths the proof is similar to case 2 of Theorem 2.1.

If $\deg u_i \geq 7$, then the above path double cover covers four pendant vertices.

Clearly $G_1 = G - \{P_1, P_2, \dots, P_7\}$ is tree with $n - 4$ pendant vertices.

$$\eta_{PD}(G_1) = n - 4$$

$$\therefore \eta_{PD}(G) = n - 4 + 7 = n + 3$$

Case 3: Let $j_1 \geq 1, j_2 = 0$ or $j_2 \geq 1, j_1 = 0$

We prove by induction on j_1

Let $j_1 = 1$ and the number of pendant vertex = 1

Without loss of generality assume that the path $(w_{11}, w_{12}, \dots, w_{1n} = u_1)$ attached with u_1 .

The path double covering of G is as follows

$$P_1 = \{w_{11}, w_{12}, \dots, u_1, u_0, u_m, u_{m+1}, \dots, u_{l+m-2}, u_i, u_{i+1}, \dots, u_{m-1}\}$$

$$P_2 = \{w_{11}, w_{12}, \dots, u_1, u_2, \dots, u_i, u_{l+m-2}, \dots, u_m, u_0, u_{m-1}, \dots, u_{i+1}\}$$

$$P_3 = \{u_{m-1}, u_0, u_1, u_2, \dots, u_i, u_{i+1}\}$$

$$\therefore \eta_{PD}(G) = 3 = \Delta$$

(Note that if $j_1 = 1$ and pendant vertices = n

The proof is similar to the case 2 in Theorem 2.1)

Assume that the result is true for $j_1 \leq n - 1$

Let $j_1 = n$, Since $j_2 = 0$ atleast two vertices among $u_{i+1}, u_{i+2}, \dots, u_{m-1}$ are exterior vertices and the n pendant vertices are exterior points

$$\therefore \eta_{PD}(G) = n + 2$$

Case 4: For the remaining cases we find a path double covering such that all the vertices are internal vertices except the pendant vertices.

$$\therefore \eta_{PD}(G) = n$$

References

- [1] B.D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. Pure Appl. Math.***18** (10) (1987), 882-890.
- [2] S.Arumugam, B.D.Acharya and E.Sampathkumar, Graphoidal covers of a graph: a creative review, in *Proc. National Workshop on Graph theory and its applications*, Manonmaniam Sundaranar University, Tirunelveli, Tata McGraw- Hill, New Delhi, (1997), 1-28.
- [3] S.Arumugam and S.Meena, Paths Double Covers of Graphs, Phd Thesis, Manonmaniam Sundaranar University, August 1999
- [4] S.Arumugam and J.Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.***190**(1998), 67-77
- [5] J. A. Bondy; Perfect path double covers of graphs, *Journal of Graph Theory*, Volume **14** (1990), 259–272.
- [6] F.Harary, *Graph Theory*, Addison-Wesley, Reading , MA, 1969.
- [7] Hao Li, Perfect path double covers in every simple graph, *Journal of Graph Theory*, **14** (6) (1990), 645–650.
- [8] K.Ratan Singh and P.K.Das. On Graphoidal Covers of Bicyclic Graphs. *International Mathematical Forum*, **5**(42), (2010), 2093-2101.

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