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A Fixed Point Theorem of Caccioppoli - Kannan Type on a Class of Generalized Metric Spaces

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Abstract

In this paper we prove a Caccioppoli type Kannan's fixed point theorem in g.m.s. that includes Kannan's fixed point theorem in g.m.s. of P.K. Saha (Th. 2.1[4]) and of Dorel Mihet (Th. 2.1[5]).

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1 Introduction

Recently Branciari [1] introduced the concept of a generalized metric space where the triangular inequality of a metric space has been replaced by a more general inequality involving four points instead of three. In this paper we generalized Caccioppoli-type fixed point theorem of Hiranmay Dasgupta [2] in generalized metric spaces (Th.3.1) and it is proved (Th.3.2) that the Kannan's fixed point theorem in generalized metric space ([4],[5]) follows from our theorem (Th.3.1). For our study we need the following known definitions and results.

2 Preliminaries

Definition 2.1 [1]. Let X be a non-void set and $d : X \times X \rightarrow R^+$ (non-negative reals) be a mapping such that for $x, y \in X$ and for all distinct points $\xi, \eta \in X$, each of them different from x and y , one has

$$\begin{aligned} d(x, y) &= 0 \text{ if and only if } x = y; \\ d(x, y) &= d(y, x); \\ \text{and } d(x, y) &\leq d(x, \xi) + d(\xi, \eta) + d(\eta, y) \text{ [rectangular property]}, \end{aligned}$$

then (X, d) is called a generalized metric space, (shortly a g.m.s.) and d is called a generalized metric for X .

Definition 2.2 [1]. Let (X, d) be a g.m.s. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer N such that $d(x_n, x_{n+m}) < \epsilon$ whenever $n \geq N$ and $m = 1, 2, \dots$. A g.m.s. (X, d) is called complete if every Cauchy sequence in X is convergent in X .

Definition 2.3 [1]. Let $T : X \rightarrow X$ be a mapping where X is a g.m.s. For each $x \in X$, let $O(x, \infty) = \{x, Tx, T^2x, T^3x, \dots\}$. Then X is said to be T -orbitally complete if and only if every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in X .

Remark 2.4 Every complete g.m.s. X (where a mapping $T : X \rightarrow X$ is defined) is T -orbitally complete but a T -orbitally complete g.m.s. X need not be complete ([3], Note, p 590).

Theorem 2.5 Caccioppoli type fixed point theorem[2]: Let (X, ρ) be a complete metric space and let T be a mapping of X into itself. Suppose that for each positive integer n , $\rho(T^n x, T^n y) \leq a_n[\rho(x, Tx) + \rho(y, Ty)]$, for all $x, y \in X$ where $a_n(> 0)$ is independent of x, y and $0 < a_1 < 1$. Then if the series $\sum_{i=1}^{\infty} a_n$ is convergent, T has a unique fixed point in X .

3 New Results

Theorem 3.1 Let (X, d) be a generalized metric space and $T : X \rightarrow X$ be a mapping such that (A): $d(T^n x, T^n y) \leq a_n[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$, $a_n(> 0)$ is independent of x, y and $0 < a_1 < \frac{1}{2}$. If X is T -orbitally complete with respect to the generalized metric d , and $\sum_{i=1}^{\infty} a_n$ is convergent, then T has a unique fixed point in X .

Proof. We note that T satisfies the following:

(B): $d(T^n x, T^{n+1} x) \leq (\frac{a_1}{1-a_1})^n d(x, Tx)$, $x \in X, n \in \mathbb{N}$ where $0 < \frac{a_1}{1-a_1} < 1$, which easily follows from (A) using method of induction. We also observe that if T has a fixed point it is unique by the condition (A) of the theorem.

We now prove the theorem considering the following two cases:

Case P. In this case we assume that $T^m x \neq T^n x$ for $m, n \in \mathbb{N}$, $m \neq n$ and $x \in X$. For $x, y \in X$, we define $\rho(x, y) = \begin{cases} d(x, Tx) + d(y, Ty), & \text{for } x \neq y; \\ 0, & \text{for } x = y. \end{cases}$

So for $x \neq y$, $\rho(x, y) \neq 0$ as T cannot have more than one fixed point. Again since $\rho(x, y) = d(x, Tx) + d(y, Ty) \leq d(x, Tx) + 2d(z, Tz) + d(y, Ty) = \rho(x, z) + \rho(z, y)$, for $x, y, z \in X$ with $x \neq y$, ρ is a metric on X . In this metric space (X, ρ) it is verified that T satisfies the results (C) and (D), where

(C): $\rho(T^n x, T^n y) \leq b_n \rho(x, y)$ for $n \in \mathbb{N}$, $x, y \in X$, $b_n = \frac{a_n}{1-a_1} (> 0)$ and b_n is independent of x, y , and

(D): $d(T^n x, T^m x) \leq 2\rho(T^n x, T^m x)$ for any $x \in X$ and $m, n \in \mathbb{N}, m \geq n$.

We now show that $\{T^n x\}$ is a ρ -Cauchy sequence in the metric space (X, ρ) .

For, if $m, n \in \mathbb{N}$ with $m > n$, then $\rho(T^n x, T^m x) \leq \sum_{k=n}^{m-1} \rho(T^k x, T^{k+1} x)$
 $= \sum_{k=n}^{m-1} \rho(T^k x, T^k(Tx)) \leq \sum_{k=n}^{m-1} b_k \rho(x, Tx)$, using (C). As the series $\sum_{n=1}^{\infty} a_n$ is convergent, $\sum_{n=1}^{\infty} b_n$ is also convergent. Let λ be the positive integer such that $\lambda > \rho(x, Tx)$. Then for an arbitrary $\epsilon (> 0)$, there exists a positive integer p such that $\sum_{k=n}^{m-1} b_k < \frac{\epsilon}{\lambda}$ if $m > n \geq p$. Then for $m > n \geq p$, we have

$\rho(T^n x, T^m x) < \frac{\epsilon}{\lambda} \rho(x, Tx) < \epsilon$. Thus $\{T^n x\}$ is a ρ -Cauchy sequence in (X, ρ) contained in $O(x, \infty)$. Hence from (D) it follows that $\{T^n x\}$ is also a d -Cauchy sequence in (X, d) contained in $O(x, \infty)$. As (X, d) is T -orbitally complete, $d(T^n x, u) \rightarrow 0$ as $n \rightarrow \infty$ for some $u \in X$. We now show that $Tu = u$. We observe that $T^n x \neq u, Tu$ for any $n \in \mathbb{N}$; for if $T^k x = u$ or $T^k x = Tu$, for some $k \in \mathbb{N}$, $x \in X$, then it follows that $\{T^n u\}$ is a sequence with the following properties in (X, d) :

- (i) $\lim_{n \rightarrow \infty} T^n u = u$, (ii) $T^n u \neq u$ for any $n \in \mathbb{N}$
- (iii) $T^p u \neq T^r u$ for $p, r \in \mathbb{N}, p \neq r$.

So $d(T^{n+1}u, Tu) \leq a_1[d(T^n u, T^{n+1}u) + d(u, Tu)] \leq a_1[(\frac{a_1}{1-a_1})^n d(u, Tu) + d(u, Tu)]$, using (B) and letting $n \rightarrow \infty$, we get $d(u, Tu) \leq a_1 d(u, Tu)$ which implies $Tu = u$ as $a_1 < 1$, which contradicts (ii). Thus $T^n x \neq u, Tu$ for any $n \in \mathbb{N}$. Now using the rectangular inequality and (A), we obtain,

$$d(u, Tu) \leq d(u, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tu)$$

$$\leq d(u, T^n x) + d(T^n x, T^{n+1}x) + a_1[d(T^n x, T^{n+1}x) + d(u, Tu)].$$
So, $d(u, Tu) \leq \frac{1}{1-a_1}[d(u, T^n x) + (1+a_1)d(T^n x, T^{n+1}x)]$

$$\leq \frac{1}{1-a_1}[d(u, T^n x) + (1+a_1)(\frac{a_1}{1-a_1})^n d(x, Tx)],$$
 using (B), and letting $n \rightarrow \infty$ we get $d(u, Tu) = 0$, i.e., $Tu = u$.

Case Q. In this case we assume that $T^m x = T^n x$, for some $m, n \in \mathbb{N}, m \neq n$. Let $m > n$, then $T^{m-n}(T^n x) = T^n x$, i.e., $T^k y = y$ where $k = m - n \geq 1, y = T^n x$. Now for $k = 1, Ty = y$ and for $k > 1, d(y, Ty) = d(T^k y, T^{k+1}y) \leq (\frac{a_1}{1-a_1})^k d(y, Ty)$, using (B) and so $Ty = y$ as $0 < \frac{a_1}{1-a_1} < 1$. Thus from cases P and Q we see that T has a fixed point in X . This proves the theorem.

We now show that Kannan's fixed point theorem in generalized metric space follows from our Theorem-3.1.

Theorem 3.2 (Kannan's fixed point theorem in g.m.s. ([4], [5])).

Let (X, d) be a generalized metric space and T be a mapping such that $T : X \rightarrow X$ and T satisfies the condition: $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$, where $0 < \beta < 1/2$ and $x, y \in X$. If X is T -orbitally complete in (X, d) then T has a unique fixed point in X .

Proof. We have, $d(Tx, Ty) \leq \frac{\beta}{\alpha} \alpha[d(x, Tx) + d(y, Ty)]$, where $0 < \alpha = \frac{\beta}{1-\beta} < 1$. Again $d(T^2x, T^2y) \leq \beta[d(Tx, T^2x) + d(Ty, T^2y)]$. Now $d(Tx, T^2x) = d(Tx, T(Tx)) \leq \beta[d(x, Tx) + d(Tx, T^2x)]$, which gives $d(Tx, T^2x) \leq \frac{\beta}{1-\beta} d(x, Tx) = \alpha d(x, Tx)$. Similarly $d(Ty, T^2y) \leq \alpha d(y, Ty)$. Hence, $d(T^2x, T^2y) \leq \beta \alpha[d(x, Tx) + d(y, Ty)] = (\frac{\beta}{\alpha}) \alpha^2[d(x, Tx) + d(y, Ty)]$.

Hence by induction, $d(T^n x, T^n y) \leq \frac{\beta}{\alpha} \alpha^n [d(x, Tx) + d(y, Ty)]$
i.e., $d(T^n x, T^n y) \leq a_n [d(x, Tx) + d(y, Ty)]$ where $a_n = (\frac{\beta}{\alpha}) \alpha^n > 0$,
which is independent of $x, y \in X$ and $a_1 = \frac{\beta}{\alpha} \alpha = \beta < \frac{1}{2}$. Now the series $\sum_{n=1}^{\infty} a_n$
is convergent. Hence by Theorem 3.1, T has a unique fixed point in X .

References

- [1] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, (57) 1-2 (2000), pp 31-37.
- [2] Hiranmay Dasgupta, A Caccioppoli type fixed point theorem that includes Kannan's fixed point theorem, *Proc. National Symposium, Math. Dept., Kalyani Univ.* (West Bengal, India) 26-27 March (1999).
- [3] B. K. Lahiri and Pratulananda Das, Fixed point of a LjubomirĆirić's quasi - contraction mapping in a generalized metric space, *Publ. Math. Debrecen*, 61, 3-4(2002), pp 589-594.
- [4] P. K. Saha and R. Tiwari, An alternative proof of Kannan's fixed point theorem in generalized metric spaces, *News Bull. Cal. Math. Soc.*, 31, (4-6)(2008) 15-18.
- [5] Dorel Mihet, On Kannan fixed point principle in generalized metric spaces, *J. Nonlinear Sci. Appl.* 2(2009), no.2, pp 92-96.

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