Extensions of Laplace Type Problems in the Euclidean Space

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Abstract

The classical Buffon needle problem is to find the probability that a needle of length n when dropped on a floor made of boards of width b will cross a crack between the boards. This problem can be solved by evaluating a simple single integral. In his extension of the problem, Laplace considered a floor tiled by congruent rectangles and considered the probability of the needle crossing one or two of the cracks between the rectangles. In 1974 M. Stoka studies an extension of the Buffon-Laplace needle problem in the space R^n . In this paper we consider two extension of the Laplace problem in E_3

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1 The Buffon-Laplace needle problem

In Buffon's original work the particle was considered to have length l and the mesh separation distance d [2, 3]. Henceforth all lengths are in units of the line spacing d . The solution is found by considering only two lines, one at

$y = 0$ and the other at $y = 1$, and the needle's center of mass may lie at any height between 0 and 1. The needle orientation takes any angle $\theta \in \pm\pi/2$ with uniform probability. An intersection is avoided when the center of mass position is within $y \in [l|\sin\theta|/2, 1 - l|\sin\theta|/2]$. Averaging this range over all orientations results in the intersection probability

$$P_{int}(l > 1) = 1 - \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (1 - l|\sin\theta|) d\theta = \frac{2l}{\pi}. \quad (1)$$

When the needle length equals the line spacing ($l = 1$) the intersection probability is $2/\pi = 0.637$, indicating non-intersecting orientations about $\theta = 0$. The solution for $l > 1$ is found by restricting the range in θ to $\theta \in \pm\arcsin(1/l)$, with intersection probability

$$P_{int}(l > 1) = 1 - \frac{1}{\pi} \int_{-\arcsin(1/l)}^{\arcsin(1/l)} (1 - l|\sin\theta|) d\theta = \quad (2)$$

$$\frac{2}{\pi} \left(l - \sqrt{l^2 - 1} + \sec^{-1} l \right). \quad (3)$$

This expression has the correct large l asymptote of $P(l \rightarrow \infty) = 1$. This analysis has been used to find experimental values for π by dropping needles and counting the intersections, with at least one suspected case of fraud [1].

While Laplace's generalization of Buffon's work [5, 6] was rectangular meshes of arbitrary length and width, we only consider the square mesh of side-length. The center of mass samples an allowable $A(\theta) = (1 - l\cos\theta)(1 - l|\sin\theta|)$, with the result that

$$P_{int}(l > 1) = 1 - \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} A(\theta) d\theta = \frac{l(4-l)}{\pi}. \quad (4)$$

It does not appear that Laplace calculated the probabilities for longer lengths, although it is clear that not until $l = \sqrt{2}$ is an intersection guaranteed. Changing the limits in the integral in equation (4) to $\pm\arcsin(1/l)$ (as in equation (2)) results in an intersection probability of

$$P_{int}(l > 1) = 1 - \frac{1}{\pi} \int_{-\arcsin(1/l)}^{\arcsin(1/l)} A(\theta) d\theta =$$

$$1 - \frac{1}{\pi} \left[\arcsin(1/l) - \arccos(1/l) + 2\sqrt{l^2 - 1} - \frac{l^2}{2} - 1 \right]. \quad (5)$$

This expression has the correct limit that $P_{int}(l = \sqrt{2}) = 1$.

To make the Buffon–Laplace results applicable to actual particles, we first generalize the analysis to spherocylinders [7] - cylinders (dimensionless diameter σ) with hemispherical endcaps. All points on the surface of the spherocylinder are a distance $\sigma/2$ from the central line segment. The total length of the spherocylinder is $l + \sigma$ and its aspect ratio $1 + l/\sigma$, with $l = 0$ corresponding to a sphere. The finite particle width reduces the total area available to the center of mass to

$$A(\theta, \sigma) = (1 - \sigma - l \cos \theta)(1 - \sigma - l |\sin \theta|), \quad (6)$$

and the intersection probability is found to be

$$P_{int}(l < 1) = \frac{4l - l^2}{\pi} + \frac{\sigma}{\pi} [(2\pi - 4) - \sigma(\pi - 3) - 2l]. \quad (7)$$

Now we considering the Stoka paper [8] we prove two others extensions.

2 First extension

We consider a lattice $\mathfrak{R}(a_1, a_2, a_3; \lambda)$ with fundamental cell $\mathcal{P}_0$ a parallelepiped with sides a_1, a_2, a_3 e angolo $\lambda \in]0, \frac{\pi}{2}]$.

We have that:

$$\text{vol} \mathcal{P}_0 = a_1 a_2 a_3 \sin \lambda. \quad (8)$$

Let s a random segment of constant lenght l with $l < \inf(a_1, a_2, a_3)$.

Our aim is to compute the probability that this segment intersects a side of lattice $\mathfrak{R}$. To be more specific, we denote by O the center of s , the position of the segment is determined by the center and the direction cosines of the line support:

$$\alpha_1 = \cos \theta, \quad \alpha_2 = \sin \theta \cos \varphi, \quad \alpha_3 = \sin \theta \sin \varphi.$$

To compute the probability P_{int} we consider the parallelepiped $\mathcal{P}_0^*$ with a vertex in $A(\frac{l}{2} \cos \theta, \frac{l}{2} \sin \theta \cos \varphi, \frac{l}{2} \sin \theta \sin \varphi)$, with the sides parallel to the sides of $\mathcal{P}_0$ and with sides

$$a_1 - l \cos \theta, \quad a_2 - l \sin \theta \cos \varphi, \quad a_3 - l \sin \theta \sin \varphi,$$

then

$$\text{vol}\mathcal{P}_0^* = (a_1 - l \cos \theta)(a_2 - l \sin \theta \cos \varphi)(a_3 - l \sin \theta \sin \varphi) \sin \lambda. \quad (9)$$

Denoting by M_1 the set of all segments s which have their center in $\mathcal{P}_0$. Denote likewise by N_1 the set of all segments s in completely contained in P_0 . In view of [8]:

$$P_{int} = 1 - \frac{\mu(\mathcal{N}_1)}{\mu(\mathcal{M}_1)}, \quad (10)$$

where μ is the Lebesgue measure in the space E_3 .

We compute the measures $\mu(\mathcal{M}_1)$ and $\mu(\mathcal{N}_1)$ using the cinematic measure of Blaschke in the space $E_3[2]$.

$$dk = d\psi \wedge d\varphi \wedge \sin \theta \wedge d\theta \wedge dx \wedge dy \wedge dz, \quad (11)$$

where x, y, z are the coordinates of the center of s , φ and θ are the angles and ψ the rotation angle.

We have

$$0 \leq \varphi, \theta \leq \frac{\pi}{2}, \quad 0 \leq \psi \leq 2\pi. \quad (12)$$

By (8), (11) and (12) we have that

$$\mu(\mathcal{M}_1) = \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_{\{(x,y,z) \in \mathcal{P}_0\}} dx dy dz = \pi^2 a_1 a_2 a_3 \sin \lambda. \quad (13)$$

Considering (9), we have

$$\begin{aligned} \mu(\mathcal{N}_1) &= \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_{\{(x,y,z) \in \mathcal{P}_0^*\}} dx dy dz = \\ &= 2\pi \sin \lambda \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \theta (a_1 - l \cos \theta)(a_2 - l \sin \theta \cos \varphi)(a_3 - l \sin \theta \sin \varphi) d\theta d\varphi = \\ &= 2\pi \sin \lambda \left[\frac{\pi}{2} a_1 a_2 a_3 - \frac{\pi l}{4} (a_1 a_2 + a_1 a_3 + a_2 a_3) + \frac{l^2}{3} (a_1 + a_2 + a_3) - \frac{l^3}{8} \right]. \quad (14) \end{aligned}$$

By cobining (10), with (13) and (14) we obtain that

$$P_{int} = \frac{l}{\pi a_1 a_2 a_3} \left[\frac{\pi}{2} (a_1 a_2 + a_1 a_3 + a_2 a_3) - \frac{l}{3} (a_1 + a_2 + a_3) - \frac{l^2}{8} \right]. \quad (15)$$

This probability is indipendent by λ , i.e. is equal with the probability for a right parallelepiped, that was find in a previous paper [8].

Now we want to show that if the sides a_1 , a_2 , a_3 and l multiply with the same factor μ , the probability P_{int} is unchanged.

In fact, if replacing

$$a_1, a_2, a_3, l$$

with

$$\mu a_1, \mu a_2, \mu a_3, \mu l, \quad (\mu > 0),$$

the probability (15) remains unchanged.

3 Second extension

Denoting by S_0 the lateral surface area L_0 of the parallelepiped P_0 we have that

$$S_0 = 2 (a_1 a_2 + a_2 a_3 + a_1 a_3 \sin \lambda) \quad (16)$$

and if S_0^* is the lateral surface area L_0^* of the parallelepiped P_0^* , we have that

$$\begin{aligned} S_0^* = & [2 (a_1 - l \cos \theta) (a_2 - l \sin \theta \cos \varphi) + \\ & (a_2 - l \sin \theta \cos \varphi) (a_3 - l \sin \theta \sin \varphi) + \\ & (a_1 - l \cos \theta) (a_3 - l \sin \theta \sin \varphi) \sin \lambda]. \end{aligned} \quad (17)$$

Denoting by M_2 the set of all segments s which are in one of the two planes which dtermine the parallelepiped P_0 and which have the center in one of these piani. Denote likewise by N_2 the set of all segments s in completely contained in lateral surface of P_0 . In view of [8]:

$$P_{int} = 1 - \frac{\mu(N_2)}{\mu(M_2)}. \quad (18)$$

By (11), (12) and (16) we have that

$$\mu(M_2) = \int_0^{2\pi} d\psi \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \sin \theta d\theta \iiint_{\{(x,y,z) \in L_0\}} dx dy dz =$$

$$2\pi \int_0^{\pi/2} d\varphi \int_0^{\pi/2} (S_0 \sin \theta) d\theta = 2\pi^2 (a_1 a_2 + a_2 a_3 + a_1 a_3 \sin \lambda). \quad (19)$$

In the same way and by (17) we have that:

$$\mu(N_2) = \int_0^{2\pi} d\psi \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \sin \theta d\theta \iiint_{\{(x,y,z) \in L_0^*\}} dx dy dz = 2\pi \int_0^{\pi/2} d\varphi \int_0^{\pi/2} (S_0^* \sin \theta) d\theta =$$

$$2\pi^2 (a_1 a_2 + a_2 a_3 + a_1 a_3 \sin \lambda) -$$

$$l \left[a_1 (1 + \sin \lambda) + a_2 \left(1 + \frac{\pi}{2} \right) + \frac{\pi}{2} a_3 (1 + \sin \lambda) \right] - \frac{l^2}{2} (2 - \sin \lambda). \quad (20)$$

By the (18), (19) and (20) we obtain that:

$$P_{int} = \frac{l}{2\pi^2 (a_1 a_2 + a_2 a_3 + a_1 a_3 \sin \lambda)}.$$

$$\left[a_1 (1 + \sin \lambda) + a_2 \left(1 + \frac{\pi}{2} \right) + \frac{\pi}{2} a_3 (1 + \sin \lambda) + \frac{l}{2} (2 - \sin \lambda) \right]. \quad (21)$$

Replacing a_1 , a_2 , a_3 , with μa_1 , μa_2 , μa_3 , μl , ($\mu > 0$), the probability (21) remains unchanged.

Finally, for $\lambda = \pi/2$, i.e. for a right parallelepiped, the probability (21) became

$$P_{int} = \frac{l}{2\pi^2 (a_1 a_2 + a_2 a_3 + a_1 a_3)} \left[2a_1 + a_2 \left(1 + \frac{\pi}{2} \right) + \pi a_3 + \frac{l}{2} \right].$$

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