The Effect of Occasional Smokers on the Dynamics of a Smoking Model

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Abstract. We present a non-linear mathematical model which analyzes the spread of smoking in a population. In this paper, the population is divided into five classes: potential smokers, occasional smokers, heavy smokers, temporary quitters and permanent quitters. We study the effect of considering the class of occasional smokers and the impact of adding this class to the smoking model in [1] on the stability of its equilibria. Numerical results are also given to support our results.

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1 Introduction

The World Health Organization estimates that tobacco causes approximately 5 million deaths annually worldwide, and this number is expected to double by 2025. The reason for that high number is that tobacco use is a major cause of many of the world’s top killer diseases including cardiovascular disease, chronic lung disease and lung cancer. Smoking is often the hidden cause of many killing diseases. In Saudi Arabia, the prevalence of current smoking ranges from 2.4-52.3% (median = 17.5%) depending on the age group. The results of a Saudi modern study predicted an increase of smokers number in the country to 10 million smokers by 2020. The current number of smokers in Saudi Arabia is approximately 6 million, and they spend around 21 billion Saudi Riyal on smoking annually. Clearly smoking is a prevalent problem among Saudis that requires intervention for eradication. Persistent education of the health hazards related to smoking is recommended particularly at early ages in order to prevent initiation of smoking [3, 17]. Tobacco use is considered a disease that can spread through social contact in a way very similar to the spread of infectious diseases.

Like many infectious diseases, mathematical models can be used to understand the spread of smoking and to predict the impact of smokers on the community in order to help reducing the number of smokers. Castillo-Garsow et al. [7] presented a general epidemiological model to describe the dynamics of Tobacco use and they considered the effect of peer pressure, relapse, counselling and treament. In their model the population was divided into non-smokers, smokers and smokers who quit smoking. Later, this mathematical model was refined by Sharomi and Gumel [14], they introduced a new class $Q_t$ of smokers who temporarily quit smoking. They concluded that the smoking-free equilibrium is globally-asymptotically stable whenever a certain threshold, known as the smokers-generation number, is less than unity, and unstable if this threshold is greater than unity. The public health implication of this result is that the number of smokers in the community will be effectively controlled (or eliminated) at equilibrium point if the threshold is made to be less than unity. Such a control is not feasible if the threshold exceeds unity. Later, Lahrouz et al. [11] proved the global stability of the unique smoking-present equilibrium state of the mathematical model developed by Sharomi and Gumel. Zaman [18] derived and analyzed a smoking model taking into account the occasional smokers compartment, and later [19] he extended the model to consider the possibility of quitters becoming smokers again. Erturk et al [8] introduced fractional derivatives into the model and studied it numerically. Zeb et al.[20] presented a new giving up smoking model based on the model in [18] for which the interaction term is the square root of potential and occasional smokers. Van Voorn and Kooi [16] presented a three compartment
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smoking model which was studied using brute force simulations for the short term dynamics and bifurcation analysis for the long-term dynamics. In 2013 [1], we adopted the model developed and studied in [11, 14] and considered the effect of peer pressure on temporarily quitters. By this we mean the effect of smokers on temporarily quitters which is considered one of the main causes of their relapse.

In this paper, we introduce a new model based on the model in [1] by dividing the smokers into two subclasses: occasional smokers and heavy smokers, and the impact of these two subclasses on the existence and stability of equilibrium points. The aim of this work, is to analyze the model by using stability theory of non-linear differential equations and supporting the results with numerical simulation. The paper is organized as follows: In section 2, we explain the formulation of the model. In section 3, the smoking generation number is calculated and the equilibria are found. The stability of the equilibria is investigated in section 4. In section 5, we present numerical simulations to support our results. Finally, we end the paper with a conclusion.

2 Formulation of the model

Let the total population size at time \( t \) be denoted by \( N(t) \). We divide the population \( N(t) \) into five subclasses, potential smokers (non-smoker) \( P(t) \), occasional (Light) smokers \( L(t) \), heavy smokers \( S(t) \), smokers who temporary quit smoking \( Q_t(t) \) and smokers who permanently quit smoking \( Q_p(t) \) such that \( N(t) = P(t) + L(t) + S(t) + Q_t(t) + Q_p(t) \). In [1] we studied the dynamics of smoking by the following four non-linear differential equations:

\[
\begin{align*}
\frac{dP}{dt} &= \mu - \mu P - \beta PS, \\
\frac{dS}{dt} &= -(\mu + \gamma)S + \beta PS + \alpha SQ_t, \\
\frac{dQ_t}{dt} &= -(\mu + \alpha S)Q_t + \gamma(1 - \sigma)S, \\
\frac{dQ_p}{dt} &= -\mu Q_p + \sigma \gamma S.
\end{align*}
\]

where \( \beta \) is the contact rate between potential smokers and smokers, \( \mu \) is the rate of natural death, \( \alpha \) is the contact rate between smokers and temporary quitters who revert back to smoking, \( \gamma \) is the rate of quitting smoking, \( (1 - \sigma) \) is the fraction of smokers who temporarily quit smoking (at a rate \( \gamma \)), \( \sigma \) is the remaining fraction of smokers who permanently quit smoking (at a rate \( \gamma \)).
In this paper, we treat occasional smokers and heavy smokers as two different subclasses in order to make the model more accurate. So we consider instead the following five non-linear differential equations:

\[
\begin{align*}
\frac{dP}{dt} &= \mu - \mu P - \beta_1 PL, \\
\frac{dL}{dt} &= -\mu L + \beta_1 PL - \beta_2 LS, \\
\frac{dS}{dt} &= -(\mu + \gamma)S + \beta_2 LS + \alpha Q_t, \\
\frac{dQ_t}{dt} &= -(\mu + \alpha)Q_t + \gamma(1 - \sigma)S, \\
\frac{dQ_p}{dt} &= -\mu Q_p + \sigma \gamma S.
\end{align*}
\]

where \(\beta_1\) is the contact rate between potential smokers and occasional smokers, \(\beta_2\) is the contact rate between occasional smokers and heavy smokers.

We assume that the class of potential smokers is increased by the recruitment of individuals at a rate \(\mu\). In system (2), the total population is supposed constant and \(P(t), L(t), S(t), Q_t(t), Q_p(t)\) are respectively the proportions of potential smokers, occasional smokers, heavy smokers, temporarily quitters and permanent quitters at time \(t\). Then \(P(t) + L(t) + S(t) + Q_t(t) + Q_p(t) = 1\). Since the variable \(Q_p\) of system (2) does not appear in the first three equations, we will only consider the subsystem:

\[
\begin{align*}
\frac{dP}{dt} &= \mu - \mu P - \beta_1 PL \\
\frac{dL}{dt} &= -\mu L + \beta_1 PL - \beta_2 LS \\
\frac{dS}{dt} &= -(\mu + \gamma)S + \beta_2 LS + \alpha Q_t \\
\frac{dQ_t}{dt} &= -(\mu + \alpha)Q_t + \gamma(1 - \sigma)S
\end{align*}
\]

From system (3) we find that

\[
\frac{dP}{dt} + \frac{dL}{dt} + \frac{dS}{dt} + \frac{dQ_t}{dt} \leq \mu - \mu(P + L + S + Q_t).
\]

Let \(N_1 = P + L + S + Q_t\) then \(N_1^t \leq \mu - \mu N_1\). The initial value problem \(\Phi = \mu - \mu \Phi\), with \(\Phi(0) = N_1(0)\), has the solution \(\Phi(t) = 1 - e^{-\mu t}\), and \(\lim_{t \to \infty} \Phi(t) = 1\). Therefore, \(N_1(t) \leq \Phi(t)\), which implies that \(\lim_{t \to \infty} \sup_{t \to \infty} N_1(t) \leq 1\). Thus, the considered region for system (3) is:

\[
\Gamma = \{(P, L, S, Q_t) : P + L + S + Q_t \leq 1, P > 0, L > 0, S > 0, Q_t \geq 0\}
\]
All solutions of system (3) are bounded and enter the region Γ. Hence, Γ is positively invariant, i.e. every solution with initial conditions in Γ remains there for all \( t > 0 \).

### 3 Smoking generation number and equilibria

Setting the right hand side of the equations in system (3) to zero, we find three equilibria: the smoking-free equilibrium \( E_0 = (1, 0, 0, 0) \), the boundary equilibrium \( E_1 = (\frac{\mu}{\beta_1}, \frac{\beta_1 - \mu}{\beta_1}, 0, 0) \) and the smoking-present equilibrium \( E^* = (P^*, L^*, S^*, Q^*_t) \), where

\[
\begin{align*}
P^* &= \frac{\mu \beta_2 (\mu + \alpha)}{\mu \beta_2 (\mu + \alpha) + \beta_1 (\sigma \alpha \gamma + \mu (\mu + \gamma) + \gamma \mu)}, \\
L^* &= \frac{\alpha \sigma \gamma + \mu (\mu + \alpha) + \gamma \mu}{\beta_2 (\mu + \alpha)}, \\
S^* &= \frac{\mu \beta_2 (\mu + \alpha) [\beta_1 \beta_2 (\mu + \alpha) + \beta_1 (\alpha \gamma \sigma + \mu (\mu + \gamma) + \gamma \mu) - 1]}{\mu + \alpha}, \\
Q^*_t &= \frac{\gamma (1 - \sigma) S^*}{\mu + \alpha}.
\end{align*}
\]

We will find the smoking generation number \( R_0 \) by the method of next generation matrix [15, 4]. Let \( X = (L, S, Q_t, P) \), then system (3) can be rewritten as: \( X' = \mathfrak{F}(X) - \mathfrak{V}(X) \) such that

\[
\mathfrak{F}(X) = \begin{bmatrix}
\beta_1 PL \\
\beta_2 LS \\
0 \\
0
\end{bmatrix}, \quad \mathfrak{V}(X) = \begin{bmatrix}
\beta_2 LS + \mu L \\
(\mu + \gamma) S - \alpha Q_t \\
(\mu + \alpha) Q_t - \gamma (1 - \sigma) S \\
-\mu + \mu P + \beta_1 P L
\end{bmatrix}.
\]

By calculating the Jacobian matrices at \( E_0 \), we find that \( D(\mathfrak{F}(E_0)) = F \begin{bmatrix}
0 \\
0
\end{bmatrix} \) and \( D(\mathfrak{V}(E_0)) = \begin{bmatrix}
V & 0 \\
J_1 & J_2
\end{bmatrix} \)

where \( F = \begin{bmatrix}
\beta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \) and \( V = \begin{bmatrix}
\mu & 0 & 0 \\
0 & \mu + \gamma & -\alpha \\
0 & -\gamma (1 - \sigma) & \mu + \alpha
\end{bmatrix} \).

Thus, the next generation matrix is \( FV^{-1} = \begin{bmatrix}
\beta_1 & 0 & 0 \\
\mu & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \) and the smoking generation number \( R_0 \) is the spectral radius \( \rho(FV^{-1}) = \beta_1/\mu \).

Next we explore the existence of positive smoking-present equilibrium of system (3).
Let $R_1 = \frac{\beta_1 \beta_2 (\mu + \alpha)}{\mu \beta_2 (\mu + \alpha) + \beta_1 (\alpha \gamma \sigma + \mu (\mu + \gamma) + \gamma \mu)}$

If $R_1 > 1$, we have a unique positive equilibrium point.

If $R_1 = 1$, then we get the equilibrium point $E_1 = (\frac{\mu}{\beta_1}, \frac{\beta_1 - \mu}{\beta_1}, 0, 0)$.

If $R_1 < 1$, then there are no positive solutions. These results are summarized below.

**Theorem 1** System (3) always has the smoking-free equilibrium point $E_0 = (1, 0, 0, 0)$. As for the existence of a smoking-present equilibrium point, we have three cases:

(i) If $R_1 < 1$, we have no positive equilibrium point.

(ii) If $R_1 > 1$, we have a unique positive equilibrium point $E^*$.

(iii) If $R_1 = 1$ and $R_0 > 1$, we have the equilibrium point $E_1 = (\frac{\mu}{\beta_1}, \frac{\beta_1 - \mu}{\beta_1}, 0, 0)$.

### 4 Stability analysis

#### 4.1 Local stability

First, we give our first result about the local stability of $E_0$:

**Theorem 2** *(Local stability of $E_0$)* If $R_0 < 1$, the smoking-free equilibrium point $E_0$ is locally asymptotically stable. If $R_0 = 1$, $E_0$ is locally stable. If $R_0 > 1$, $E_0$ is unstable.

**Proof.** Evaluating the Jacobian matrix of system (3) at $E_0$ (using linearization method [13]) gives

$$J(E_0) = \begin{bmatrix}
-\mu & -\beta_1 & 0 & 0 \\
0 & -\mu + \beta_1 & 0 & 0 \\
0 & 0 & -(\mu + \gamma) & \alpha \\
0 & 0 & \gamma (1 - \sigma) & -(\mu + \alpha)
\end{bmatrix}$$

The eigenvalues are, $\lambda_1 = -\mu < 0$, $\lambda_2 = -\mu + \beta_1 < 0$ if $R_0 < 1$ and $\lambda_3, \lambda_4$ satisfy the equation $\lambda^2 + a_1 \lambda + a_2 = 0$, where

$a_1 = 2 \mu + \alpha + \gamma > 0$

$a_2 = \alpha \gamma \sigma + \mu (\mu + \alpha) + \gamma \mu > 0$

Hence, by the Routh-Hurwitz criterion, $\lambda_3$ and $\lambda_4$ have negative real parts. So all eigenvalues are negative if $R_0 < 1$, and hence, $E_0$ is locally asymptotically stable.

If $R_0 = 1$, then $\lambda_2 = 0$ and $E_0$ is locally stable.

If $R_0 > 1$, then $\lambda_2 > 0$ which means that there exist a positive eigenvalue. So, $E_0$ is unstable.

Next, the local stability of $E_1$ is studied in the next theorem:
Theorem 3 (Local stability of $E_1$) The equilibrium point $E_1$ is locally asymptotically stable if $\mu \geq \beta_2$.

Proof. The Jacobian matrix at the equilibrium $E_1 = (\frac{\mu}{\beta_1}, \frac{\beta_1 - \mu}{\beta_1}, 0, 0)$ gives:

$$J(E_1) = \begin{bmatrix} -\beta_1 & -\mu & 0 & 0 \\ \beta_1 - \mu & 0 & -\frac{\beta_2(\beta_1 - \mu)}{\beta_1} & 0 \\ 0 & 0 & -(\mu + \gamma) + \frac{\beta_2(\beta_1 - \mu)}{\beta_1} & \alpha \\ 0 & 0 & \gamma(1 - \sigma) & -(\mu + \alpha) \end{bmatrix}$$

The eigenvalues are, $\lambda_1 = -\mu < 0$, $\lambda_2 = \mu - \beta_1 < 0$, since $R_0 > 1$, and $\lambda_3, \lambda_4$ satisfy the equation $\lambda^2 + a_1 \lambda + a_2 = 0$, where

$$a_1 = \mu + \alpha + \gamma + \mu \frac{\beta_2}{\beta_1} + (\mu - \beta_2)$$

$$a_2 = \alpha \gamma \sigma + (\mu - \beta_2)(\mu + \alpha) + \gamma \mu + \mu \frac{\beta_2}{\beta_1}(\mu + \alpha) > 0$$

Hence, by the Routh-Hurwitz criterion, $\lambda_3$ and $\lambda_4$ have negative real parts if and only if $\mu \geq \beta_2$. So all eigenvalues are negative if $\mu \geq \beta_2$, and hence, $E_1$ is locally asymptotically stable. ■

Now we investigate the local stability of $E^*$

Theorem 4 (Local stability of $E^*$) The smoking-present equilibrium point $E^*$ is locally asymptotically stable if $R_1 > 1$.

Proof. Linearizing system (3) at the equilibrium $E^* = (P^*, L^*, S^*, Q_t^*)$ gives:

$$J(E^*) = \begin{bmatrix} -\mu - \beta_1 L^* & -\beta_1 P^* & 0 & 0 \\ \beta_1 L^* & 0 & -\beta_2 L^* & 0 \\ 0 & \beta_2 S^* & -\frac{\alpha Q_t^*}{S^*} & \alpha \\ 0 & 0 & (\mu + \alpha) \frac{Q_t^*}{S^*} & -(\mu + \alpha) \end{bmatrix}$$

The characteristic equation about $E^*$ is given by:

$$\lambda^4 + (2\mu + \beta_1 L^* + \alpha + \alpha \frac{Q_t^*}{S^*})\lambda^3 + (\mu^2 + \mu \beta_1 L^* + \alpha \mu + \alpha \frac{Q_t^*}{S^*} \mu + P^* \beta_2^2 L^* + \alpha \beta_1 L^* + \alpha \frac{Q_t^*}{S^*} \beta_1 L^* + \beta_2^2 L^* S^*) \lambda^2 + (P^* \mu \beta_1 L^* + 2\mu \beta_2^2 L^* S^* + \alpha \mu \beta_1^2 L^* L^* + \alpha \frac{Q_t^*}{S^*} P^* \beta_1^2 L^* + \beta_1 \beta_2^2(L^*)^2 S^* + \alpha \beta_1^2 \beta_2^2(L^*) S^* + \alpha \beta_1 \beta_2^2(L^*)^2 S^* + \alpha \mu \beta_1 \beta_2^2(L^*)^2 S^* = 0$$

Which can be simplified as $\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0$ where

$$a_1 = 2\mu + \beta_1 L^* + \alpha + \alpha \frac{Q_t^*}{S^*} > 0,$$

$$a_2 = \mu^2 + \mu \beta_1 L^* + \alpha \mu + \alpha \frac{Q_t^*}{S^*} + \beta_1^2 \beta_2^2 P^* L^* + \alpha \beta_1 L^* + \alpha \beta_1 \frac{Q_t^*}{S^*} L^* + \beta_2^2 L^* S^* > 0,$$
a_3 = \mu \beta_2^2 P^* L^* + 2 \mu \beta_2^2 L^* S^* + \alpha \beta_1^2 P^* L^* + \alpha \beta_1^2 Q^*_t P^* L^* + \beta_1 \beta_2^2 (L^*)^2 S^* + \alpha \beta_2^2 L^* S^* > 0,

a_4 = \mu \beta_2^2 P^* L^* + \beta_1 \beta_2^2 (L^*)^2 S^* + \alpha \mu \beta_2^2 L^* S^* + \alpha \beta_1 \beta_2^2 (L^*)^2 S^* > 0.

Also by using mathematica we can verify that \( a_1 a_2 a_3 - (a_3^2 + a_1^2 a_4) > 0 \). Hence, by using Routh-Herwitz criteria, all eigenvalues of \( J(E^*) \) have negative real parts. Thus, \( E^* \) is locally asymptotically stable. ■

4.2 Global stability

We will investigate the global stability of \( E_0 \) when \( R_0 \leq 1 \).

**Theorem 5 (Global stability of \( E_0 \))** If \( R_0 \leq 1 \), then \( E_0 \) is globally asymptotically stable in \( \Gamma \).

**Proof.** First, it should be noted that \( P < 1 \) in \( \Gamma \) for all \( t > 0 \). Consider the following Lyapunov function [6],

\[
V = L + S + Q_t,
\]

\[
\frac{dV}{dt} = \frac{d}{dt}(L + S + Q_t),
\]

\[
= -\mu L + \beta_1 P L - \beta_2 S L - (\mu + \gamma) S + \beta_2 S L + \alpha Q_t - (\mu + \alpha) Q_t + \gamma (1 - \sigma) S
\]

\[
\leq (-\mu + \beta_1) L - \mu (S + Q_t) - \gamma \sigma S
\]

\[
\frac{dV}{dt} \leq 0 \text{ for } R_0 \leq 1, \text{ and } \frac{dV}{dt} = 0 \text{ only if } L = 0, S = 0 \text{ and } Q_t = 0. \text{ Therefore, the only trajectory of the system on which } \frac{dV}{dt} = 0 \text{ is } E_0. \text{ Hence, by Lasalle’s invariance principle [6], } E_0 \text{ is globally asymptotically stable in } \Gamma. \text{ ■}

Our next theorem states the condition for global stability of \( E_1 \)

**Theorem 6 (Global stability of \( E_1 \))** The equilibrium point \( E_1 \) is globally asymptotically stable if \( \mu \geq \beta_2 \).

**Proof.** Consider the Lyapunov function,

\[
V = (P - P_1 - P_1 \log \frac{P}{P_1}) + (L - L_1 - L_1 \log \frac{L}{L_1}) + S + Q_t
\]
Theorem 7 (Global stability of $E^*$) If $R_1 > 1$, then the smoking-present equilibrium point $E^*$ is globally asymptotically stable.

Proof. Consider the Lyapunov function:

$$V = (P - P^* - P^* \log \frac{P}{P^*}) + (L - L^* - L^* \log \frac{L}{L^*}) + (S - S^* - S^* \log \frac{S}{S^*}) + \frac{\alpha}{\mu + \alpha} (Q_t - Q_t^* - Q_t^* \log \frac{Q_t}{Q_t^*})$$

$$\frac{dV}{dt} = (1 - \frac{P^*}{P}) \frac{dP}{dt} + (1 - \frac{L^*}{L}) \frac{dL}{dt} + (1 - \frac{S^*}{S}) \frac{dS}{dt} + \frac{\alpha}{\mu + \alpha} (1 - \frac{Q_t^*}{Q_t}) \frac{dQ_t}{dt},$$

$$= (1 - \frac{P^*}{P})(\mu - \mu P - \beta_1 PL) + (1 - \frac{L^*}{L})(-\mu L + \beta_1 PL - \beta_2 LS) + (1 - \frac{S^*}{S})$$

$$(-\mu + \gamma)S + \beta_2 LS + \alpha Q_t) + \frac{\alpha}{\mu + \alpha} (1 - \frac{Q_t^*}{Q_t})(-(\mu + \alpha)Q_t + \gamma(1 - \sigma)S),$$

$$= -\mu(\frac{P - P^*}{P})^2 + \beta_1 P^* L^*(1 - \frac{P^*}{P} - \frac{P}{P^*}) + (-\mu L^* + \beta_1 P^* L^* - \beta_2 L^* S^*) \frac{L}{L^*}$$

$$+ (\frac{S^*}{S})^2 + \alpha Q_t^*(1 - \frac{Q_t S}{Q_t S^*}) + \frac{Q_t S}{Q_t S^*} + \beta P^* L^* + \alpha Q_t^*,$$

$$= -\mu(\frac{P - P^*}{P})^2 + \beta_1 P^* L^*(2 - \frac{P^*}{P} - \frac{P}{P^*}) + \alpha Q_t^*(2 - \frac{Q_t S}{Q_t S^*} - \frac{Q_t S}{Q_t S^*}),$$

$$= -(\mu + \beta_1 L^*) \frac{(P - P^*)^2}{P} - \frac{\alpha}{S^*} \frac{(Q_t S - Q_t S^*)^2}{Q_t S}$$
By using $\mu = \mu P^* + \beta_1 P^* L^*$

$$\alpha Q_i^* = \frac{\alpha \gamma (1 - \sigma)}{S^*} \frac{\alpha \gamma (1 - \sigma)}{S^*}$$

$$\mu L^* = \beta_1 P^* L^* - \beta_2 L^* S^*$$

$$(\mu + \gamma) S^* = \beta_2 L^* S^* + \alpha Q_i^* \frac{dV}{dt} \leq 0, \quad \frac{dV}{dt} = 0 \text{ only if } P = P^*, \quad \frac{S}{S^*} = \frac{Q_t}{Q_t^*} = k.$$ 

Substituting $P = P^*$, $S = k S^*$ within the second equation of system (3), we obtain

$$0 = -\mu + \beta_1 P^* - k \beta_2 S^* = \beta_2 S^* - k \beta_2 S^*$$

Then $k = 1$. Therefore, the only trajectory of the system on which $\frac{dL}{dt} = 0$ is $E^*$. Hence, by Lasalle’s invariance principle, $E^*$ is globally asymptotically stable in $\Gamma$. \square

5 Numerical simulations

In this section, we illustrate some numerical solutions of system (2) for different values of the parameters, and show that these solutions are in agreement with the qualitative behavior of the solutions.

We use the following parameters: $\beta_2 = 0.3$, $\gamma = 0.2$, $\alpha = 0.25$ and $\sigma = 0.4$, with two different values of $\beta_1$ and $\mu$: for $R_0 < 1$, we use $\beta_1 = 0.04$ and $\mu = 0.15$, and for $R_0 > 1$ and $\mu \geq \beta_2$, we use $\beta_1 = 0.23$, $\mu = 0.15$ and $\beta_2 = 0.1$, and for $R_1 > 1$, we use $\beta_1 = 0.23$ and $\mu = 0.04$, and we choose different initial values such that $P + L + S + Q_t + Q_p = 1$ as follows

1$-P(0) = 0.60301, L(0) = 0.24000, S(0) = 0.10628, Q_t(0) = 0.03260, Q_p(0) = 0.01811$,

2$-P(0) = 0.55000, L(0) = 0.20000, S(0) = 0.17272, Q_t(0) = 0.06700, Q_p(0) = 0.01028$,

3$-P(0) = 0.50000, L(0) = 0.15000, S(0) = 0.26200, Q_t(0) = 0.08066, Q_p(0) = 0.00734$,

4$-P(0) = 0.45900, L(0) = 0.10000, S(0) = 0.21900, Q_t(0) = 0.21800, Q_p(0) = 0.00400$.

In the first five figures 1, we use different initial values and the parameters as presented above for $R_0 = 0.26667 < 1$. Figure 1(a) shows that the number of potential smokers increases and approaches the total population 1. Figure 1(b), 1(c) and 1(d) show that the number of the occasional smokers, the smokers and the temporary quitters decreases and approaches zero. In Figure 1(e), the number of permanent quitters increases at first, after that it decreases and approaches zero. We see from these figures that for any initial value, the solution curves tend to the equilibrium $E_0$ when $R_0 < 1$. Hence, system (2) is locally asymptotically stable about $E_0$ for the above set of parameters.
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Fig. 1. Time plots of system (1) with different initial conditions for $R_0<1$. (a) Potential smokers; (b) Occasional smokers; (c) Smokers; (d) Temporary quitters; (e) Permanent quitters.
In the following figures 2, we use different initial values and the parameters are as presented above for $R_0 = 1.5333 > 1$ and $\mu \geq \beta_2$. Figure 2(a) shows that the number of potential smokers increases at first, then it decreases and approaches $P_1$. Figure 2(b) shows that the number of the occasional smokers decreases at first, then it increases and approaches $L_1$. In Figure 2(c) and 2(d), the number of smokers and permanent quitters decreases and approaches zero. In Figure 2(e), the number of temporary quitters increases at first, after that it decreases and approaches zero. We see from these figures that for any initial value, the solution curves tend to the equilibrium $E_1$, when $\mu \geq \beta_2$. Hence, system (2) is locally asymptotically stable about $E_1$ for the above set of parameters.
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Fig. 2. Time plots of system (1) with different initial conditions for $\mu \geq \beta_2$. (a) Potential smokers; (b) Occasional smokers; (c) Smokers; (d) Temporary quitters; (e) Permanent quitters.

Finally in the following figures 3, we use different initial values and the parameters are as presented above for $R_1 = 1.6499 > 1$. Figure 3(a) shows that the number of potential smokers approaches $P^*$. Figure 3(b) shows that the number of the occasional smokers increases at first, then it decreases and approaches $L^*$. In Figure 3(c) and 3(d), the number of smokers and permanent quitters decreases at first, after that it approaches $S^*$ and $Q_t^*$. In Figure 3(e), the number of temporary quitters increases at first, after that it decreases and approach $Q_p^*$. We see from these figures that for any initial value, the solution curves tend to the equilibrium $E^*$, when $R_1 > 1$. Hence, system (2) is locally asymptotically stable about $E^*$ for the above set of parameters.
Fig. 3. Time plots of system (1) with different initial conditions for $R_1 > 1$. (a) Potential smokers; (b) Occasional smokers; (c) Smokers; (d) Temporary quitters; (e) Permanent quitters.

6 Conclusions

In this paper, a non-linear mathematical model was formulated, which describes the overall smoking population dynamics when the population is assumed to remain constant. We used the stability analysis theory for nonlinear systems to analyze the mathematical smoking models and to study both the local and global behavior of smoking dynamics. Local asymptotic stability for the smoking-free equilibrium $E_0$ can be obtained, if the threshold quantity $R_0 = \frac{\beta_1}{\mu}$ is less than 1 (i.e. when the contact rate $\beta_1$ between potential smokers and occasional smokers is less than natural death rate $\mu$). On the other hand,
if $R_1 > 1$, then the smoking present equilibrium $E^*$ is locally asymptotically stable. Also local asymptotic stability for the equilibrium $E_1$ was obtained, if $\mu \geq \beta_2$. A Liapunov function was used to show global stability. $E_0$ is globally asymptotically stable if $R_0 \leq 1$ (i.e when the contact rate between potential smokers and occasional smokers is less than or equal to the natural death rate $\beta_1 \leq \mu$). This means that the number of smokers can be reduced by reducing the contact rate $\beta_1$ to be less than the natural death rate $\mu$. Also a Liapunov function was used to show global stability of $E^*$, this means that the smoking disease will persist if $R_1 > 1$. This smoking model can be extended and developed to be more realistic for better analysis. We hope to do so in future research papers.

References


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