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On Duality in Linearly Topologized Spaces

Patricia Couto G. Mauro

Centro Interdisciplinar de Ciências da Natureza
Universidade Federal da Integração Latino-Americana
85867-970 Foz do Iguaçu, PR Brasil

Dinamérico P. Pombo Jr

Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21945-970 Rio de Janeiro, RJ Brasil

Abstract

For a weakly continuous linear mapping u, a necessary and sufficient condition for the solubility of the equation $u(x) = y_0$ is obtained. The equivalence between the category of dual systems and the category of linearly topologized spaces whose topologies are weak is also established.

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1 Introduction

The notion of a linearly topologized space over a discrete field has been introduced by Lefschetz [7] in view of applications to Algebraic Topology, and has also been considered in the more general setting in which “field” is replaced by “division ring”. The study of such spaces presents various features, as one may see in [6] and its Bibliography, one of them being the important discussion of duality theory.

The present paper is devoted to certain aspects of duality theory in the context of linearly topologized spaces over an arbitrary discrete division ring, taking as start point the notion of a dual system defined in [4]. The main purpose of our work is a study of the solubility of equations of the form $u(x) = y_0$, where u is a weakly continuous linear mapping, which is strongly related to the concept of orthogonality. The equivalence between the category of dual systems and the category of linearly topologized spaces whose topologies are weak is also proved. For the sake of clarity, a few preliminaries on linearly topologized spaces are included in the paper.

Throughout this paper $\mathbb{K}$ *will denote a discrete division ring.*

2 Preliminaires on linearly topologized spaces

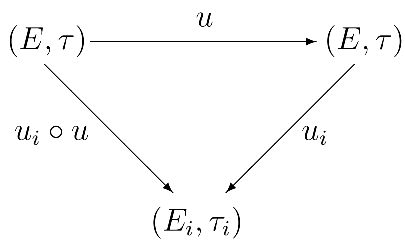
Definition 2.1 *A topology* τ *on a left vector space* E *over* $\mathbb{K}$ *is said to be linear, and* (E, τ) *is said to be a left linearly topologized space over* $\mathbb{K}$*, if* τ *is a Hausdorff translation-invariant topology such that* $0 \in E$ *admits a fundamental system of* τ*-neighborhoods consisting of subspaces of* E *(in this case* (E, τ) *is necessarily a left topological vector space over* $\mathbb{K}$*). Analogously, one defines the notion of a right linearly topologized space over* $\mathbb{K}$*.*

As in the case in which $\mathbb{K}$ is a field ([6], p. 83, (5)), one shows that the unique linear topology on a finite-dimensional left vector space over $\mathbb{K}$ is the discrete.

As we shall see, the next proposition will permit the construction of certain linear topologies from given ones.

Proposition 2.2 *Let* $((E_i, \tau_i))_{i \in I}$ *be a family of left linearly topologized spaces over* $\mathbb{K}$*,* E *a left vector space over* $\mathbb{K}$ *and* $u_i : E \to E_i$ *a* $\mathbb{K}$*-linear mapping* $(i \in I)$*, and assume that for each* $x \in E \backslash \{0\}$ *there exists an* $i \in I$ *such that* $u_i(x) \neq 0$*. If* τ *is the initial topology for the family* $((E_i, \tau_i), u_i)_{i \in I}$*, then* τ *is linear.*

Proof: Let $x_0 \in E$ be arbitrary. We claim that the mapping $u : x \in (E, \tau) \mapsto x + x_0 \in (E, \tau)$ is continuous. In fact, for each $i \in I$ let us consider the diagram



By Proposition 4, p. 30 of [3], u is continuous if and only if $u_i \circ u$ is continuous for all $i \in I$. But, since $(u_i \circ u)(x) = u_i(x + x_0) = u_i(x) + u_i(x_0)$ for $x \in E$, and since u_i is continuous and τ_i is translation-invariant, it follows that $u_i \circ u$ is continuous. Thus u is continuous. And u^{-1} is continuous, because $u^{-1}(x) = x - x_0$ for $x \in E$. Therefore τ is translation-invariant. Moreover, if $x \in E \backslash \{0\}$, there is an $i \in I$ with $u_i(x) \neq 0$. Thus, if U_i is a τ_i- neighborhood of 0 in E_i such that $u_i(x) \notin U_i$, then $u_i^{-1}(U_i)$ is a τ-neighborhood of 0 in E such that $x \notin u_i^{-1}(U_i)$; hence τ is Hausdorff topology.

Finally, if $\mathcal{U}_i$ is a fundamental system of τ_i-neighborhoods of 0 in E_i formed by subspaces of E_i $(i \in I)$, then the set of all finite intersections of sets of the form $u_i^{-1}(U_i)$ $(i \in I, U_i \in \mathcal{U}_i)$ constitutes a fundamental system of τ-neighborhoods of 0 in E formed by subspaces of E.□

Corollary 2.3 *If $((E_i, \tau_i))_{i \in I}$ is a family of left linearly topologized spaces over $\mathbb{K}$, then $(\prod_{i \in I} E_i, \prod_{i \in I} \tau_i)$ is a left linearly topologized space, where $\prod_{i \in I} \tau_i$ is the product topology on the (left) product vector space $\prod_{i \in I} E_i$ over $\mathbb{K}$.*

Proof: Follows immediately from Proposition 2.2. □

Corollary 2.4 *If (E, τ) is a left linearly topologized space over $\mathbb{K}$ and M is a subspace of E, then (M, τ_M) is a left linearly topologized space over $\mathbb{K}$, where τ_M is the topology induced by τ on M.*

Proof: Follows immediately from Proposition 2.2. □

Proposition 2.5 *If (E, τ) is a left linearly topologized space over $\mathbb{K}$ and M is a τ-closed subspace of E, then $(E/M, \tau')$ is a left linearly topologized space, where τ' is the quotient topology on the (left) quotient vector space E/M over $\mathbb{K}$.*

Proof: If $\pi : E \to E/M$ is the canonical surjection, it is easily seen that $\tau' = \{\pi(A) \mid A \in \tau\}$. Consequently, τ' is a linear topology. □

The following linear version of the Hahn-Banach theorem will be important for our purposes.

Proposition 2.6 *Let (E, τ) be a left linearly topologized space over $\mathbb{K}$, M a τ-closed subspace of E and $x \in E \backslash M$. Then there exists a continuous $\mathbb{K}$-linear form φ on E such that $\varphi_{|M} = 0$ and $\varphi(x) = 1$. In particular, for each $x \in E \backslash \{0\}$ there is a continuous $\mathbb{K}$-linear form φ on E such that $\varphi(x) \neq 0$.*

Proof: We shall argue as in the proof of (1'), p. 86 of [6]. Indeed, let U be a τ-neighborhood of 0 in E which is a subspace of E such that $(x + U) \cap M = \emptyset$, and let $\mathcal{B}_1$ be a basis of the subspace $U + M$ of E. Since $x \notin U + M$, the set $\mathcal{B}_1 \cup \{x\}$ is linearly independent, and hence there is a basis $\mathcal{B}$ of E so that $\mathcal{B}_1 \cup \{x\} \subset \mathcal{B}$. Define $\varphi : E \to \mathbb{K}$ by $\varphi(x) = 1$ and $\varphi(t) = 0$ for $t \in \mathcal{B} \backslash \{x\}$. Then φ is a $\mathbb{K}$-linear form on E such that $\varphi_{|M} = 0$, $\varphi(x) = 1$ and φ is continuous ($\varphi_{|U} = 0$), as was to be shown. □

3 Dual systems and orthogonality

Definition 3.1 *Let E be a left vector space over $\mathbb{K}$ and E_1 a right vector space over $\mathbb{K}$. A mapping $B : E \times E_1 \to \mathbb{K}$ is said to be a $\mathbb{K}$-bilinear form if the following conditions hold for all $x, y \in E$, $x_1, y_1 \in E_1$ and $\lambda \in \mathbb{K}$:*
$B(x + y, x_1) = B(x, x_1) + B(y, x_1)$;
$B(\lambda x, x_1) = \lambda B(x, x_1)$;
$B(x, x_1 + y_1) = B(x, x_1) + B(x, y_1)$;
$B(x, x_1\lambda) = B(x, x_1)\lambda$.
(E, E_1) is said to be a dual system if there exists a $\mathbb{K}$-bilinear form $B : E \times E_1 \to \mathbb{K}$ satisfying the following conditions:
(a) for each $x \in E \backslash \{0\}$ there is an $x_1 \in E_1$ such that $B(x, x_1) \neq 0$;
(b) for each $x_1 \in E_1 \backslash \{0\}$ there is an $x \in E$ such that $B(x, x_1) \neq 0$.
We shall also say that (E, E_1) is a dual system with respect to B.

Example 3.2 *If E is a left vector space over $\mathbb{K}$ and E^* is the right vector space over $\mathbb{K}$ of all $\mathbb{K}$-linear forms on E, then (E, E^*) is a dual system with respect to the $\mathbb{K}$-bilinear form*

$$(x, \varphi) \in E \times E^* \longmapsto \varphi(x) \in \mathbb{K},$$

as one readily observes.

Example 3.3 *If (E, τ) is a left linearly topologized space over $\mathbb{K}$ and E' is the right vector space over $\mathbb{K}$ of all continuous $\mathbb{K}$-linear forms on (E, τ), then (E, E') is a dual system with respect to the $\mathbb{K}$-bilinear form*

$$(x, \varphi) \in E \times E' \longmapsto \varphi(x) \in \mathbb{K},$$

condition (a) of Definition 3.1 being a consequence of Proposition 2.6.

Example 3.4 *Let I be an arbitrary index set and, for each $i \in I$, let (E_i, F_i) be a dual system with respect to the $\mathbb{K}$-bilinear form B_i. If E is the left vector space $\prod_{i\in I} E_i$ over $\mathbb{K}$ and E_1 is the right vector space $\oplus_{i\in I} F_i$ over $\mathbb{K}$, then (E, E_1) is a dual system with respect to the $\mathbb{K}$-bilinear form*

$$B : ((x_i)_{i\in I}, (y_i)_{i\in I}) \in E \times E_1 \longmapsto \sum_{i\in I} B_i(x_i, y_i) \in \mathbb{K}.$$

In particular, $(\mathbb{K}^I, \mathbb{K}^{(I)})$ is a dual system with respect to the $\mathbb{K}$-bilinear form

$$((\lambda_i)_{i\in I}, (\mu_i)_{i\in I}) \in \mathbb{K}^I \times \mathbb{K}^{(I)} \longmapsto \sum_{i\in I} \lambda_i \mu_i \in \mathbb{K}.$$

Definition 3.5 *Let (E, E_1) be a dual system with respect to the $\mathbb{K}$-bilinear form B. For each $x \in E$ (resp. $x_1 \in E_1$) let B_x (resp. B_{x_1}) be the $\mathbb{K}$-linear form on E_1 (resp. E) given by $B_x(w) = B(x, w)$ for $w \in E_1$ (resp. $B_{x_1}(t) = B(t, x_1)$ for $t \in E$). The weak topology $\sigma(E, E_1)$ on E is the initial topology for the family $(\mathbb{K}, B_{x_1})_{x_1\in E_1}$. By Proposition 2.2, $(E, \sigma(E, E_1))$ is a left linearly topologized space over $\mathbb{K}$. Analogously, one defines the weak topology $\sigma(E_1, E)$ on E_1 as the initial topology for the family $(\mathbb{K}, B_x)_{x\in E}$, which makes E_1 a right linearly topologized space over $\mathbb{K}$.*

Example 3.6 *Under the conditions of Example 3.4, $\sigma(E, E_1) = \prod_{i\in I} \sigma(E_i, F_i)$.*

Firstly, let us show that $\sigma(E, E_1)$ is coarser than $\prod_{i\in I} \sigma(E_i, F_i)$. In fact, let $x_1 = (y_i^1)_{i\in I}, \ldots, x_n = (y_i^n)_{i\in I} \in E_1$ be arbitrary and put $U = \bigcap_{j=1}^{n} Ker(B_{x_j})$. Let I_0 be a finite subset of I such that $y_i^j = 0$ for $j = 1, \ldots, n$ and $i \in I \backslash I_0$, and define $V_i = \bigcap_{j=1}^{n} Ker((B_i)_{y_i^j})$ for $i \in I_0$ and $V_i = E_i$ for $i \in I \backslash I_0$. Then $V = \prod_{i\in I} V_i$ is a $\prod_{i\in I} \sigma(E_i, F_i)$- neighborhood of 0 in E such that $V \subset U$, and $\sigma(E, E_1)$ is coarser than $\prod_{i\in I} \sigma(E_i, F_i)$.

Conversely, let us show that $\prod_{i\in I} \sigma(E_i, F_i)$ is coarser than $\sigma(E, E_1)$. In fact, let I_0 be a finite subset of I and $V = \prod_{i\in I} V_i$, where V_i is a $\sigma(E_i, F_i)$-neighborhood of 0 in E_i for $i \in I_0$ and $V_i = E_i$ for $i \in I \backslash I_0$. For each $i \in I_0$ we can find $y_i^1, \ldots, y_i^{n_i} \in F_i$ so that $\bigcap_{j=1}^{n_i} Ker((B_i)_{y_i^j}) \subset V_i$. Put $n = \max\{n_i \,|\, i \in I_0\}$ and, for $i \in I_0$ and $n_i < j \leq n$, put $y_i^j = 0$. Then $\bigcap_{j=1}^{n} Ker((B_i)_{y_i^j}) \subset V_i$ for all $i \in I_0$. Finally, for $j \in I_0$ and $1 \leq k \leq n$, define $z^{j,k} = (z_i^{j,k})_{i\in I} \in E_1$ by

$z_i^{j,k} = y_i^k$ if $i = j$ and $z_i^{j,k} = 0$ if $i \neq j$. Therefore $U = \bigcap_{j \in I_0, 1 \leq k \leq n} Ker(B_{z^{j,k}})$ is a $\sigma(E, E_1)$-neighborhood of 0 in E such that $U \subset V$, and $\prod_{i \in I} \sigma(E_i, F_i)$ is coarser than $\sigma(E, E_1)$.

The next proposition was stated in [4]:

Proposition 3.7 *Under the conditions of Definition 3.5, let $\varphi \in E^*$. In order that $\varphi \in (E, \sigma(E, E_1))'$, it is necessary and sufficient that $\varphi = B_{x_1}$ for a unique $x_1 \in E_1$. Consequently, the $\mathbb{K}$-linear mapping*

$$x_1 \in E_1 \longmapsto B_{x_1} \in (E, \sigma(E, E_1))'$$

is an isomorphism.

Proof: Since the sufficiency is obvious, let us prove the necessity. In fact, since $\varphi \in (E, \sigma(E, E_1))'$, there are $z_1, \ldots, z_n \in E_1$ so that the relations $x \in E$, $B_{z_1}(x) = \ldots = B_{z_n}(x) = 0$ imply $\varphi(x) = 0$, which means that $\bigcap_{i=1}^{n} Ker(B_{z_i}) \subset Ker(\varphi)$. By a known result of Linear Algebra, there are $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $\varphi = \sum_{i=1}^{n} B_{z_i} \lambda_i$. Therefore, if we put $x_1 = \sum_{i=1}^{n} z_i \lambda_i \in E_1$, it follows that $\varphi = B_{x_1}$. Finally, as the uniqueness is clear, the proof is concluded. □

Definition 3.8 *Let (E, E_1) be a dual system with respect to the $\mathbb{K}$-bilinear form B and $X \subset E$. The orthogonal of X is the subspace*

$$X^{\perp} = \{x_1 \in E_1 \mid B_x(x_1) = 0 \text{ for all } x \in X\}$$

of E_1. In the same way, one defines the orthogonal of a subset of E_1. Moreover,the biorthogonal of X, denoted by $X^{\perp\perp}$, is defined as the orthogonal of $X^{\perp}$; obviously, $X \subset X^{\perp\perp}$.

Proposition 3.9 *If (E, E_1) is a dual system with respect to the $\mathbb{K}$-bilinear form B, the following assertions hold:*
(a) if $X_1 \subset X_2 \subset E$, then $X_2^{\perp} \subset X_1^{\perp}$;
(b) if $X \subset E$, then $X^{\perp}$ is $\sigma(E_1, E)$-closed.

Proof: Since (a) is evident, let us prove (b). Indeed,

$$X^{\perp} = \bigcap_{x \in X} \{x_1 \in E_1 \mid B_x(x_1) = 0\} = \bigcap_{x \in X} Ker(B_x).$$

Thus, since $Ker(B_x)$ is clearly $\sigma(E_1, E)$-closed for all $x \in E$, (b) is valid. □

The next result will be important for our purposes.

Proposition 3.10 (theorem of the biorthogonals) *If (E, E_1) is a dual system with respect to the $\mathbb{K}$-bilinear form B and M is a subspace of E, then $\overline{M^{\sigma(E,E_1)}} = M^{\perp\perp}$.*

Proof: By Proposition 3.9(b), $\overline{M^{\sigma(E,E_1)}} \subset M^{\perp\perp}$. Now, let $x \in E\backslash\overline{M^{\sigma(E,E_1)}}$. Since $\overline{M^{\sigma(E,E_1)}}$ is a $\sigma(E, E_1)$-closed subspace of E, Proposition 2.6 guarantees the existence of a $\varphi \in (E, \sigma(E, E_1))'$ such that $\varphi_{|\overline{M^{\sigma(E,E_1)}}} = 0$ (hence, $\varphi_{|M} = 0$) and $\varphi(x) = 1$. On the other hand, by Proposition 3.7 there is a unique $x_1 \in E_1$ so that $\varphi = B_{x_1}$. Consequently, $x \notin M^{\perp\perp}$, and the equality $\overline{M^{\sigma(E,E_1)}} = M^{\perp\perp}$ is established. □

Proposition 3.11 *Let (E, E_1) be a dual system with respect to the $\mathbb{K}$-bilinear form B and (F, F_1) a dual system with respect to the $\mathbb{K}$-bilinear form C. In order that a $\mathbb{K}$-linear mapping $u : E \to F$ be $\sigma(E, E_1)$-$\sigma(F, F_1)$-continuous, it is necessary and sufficient that the $\mathbb{K}$-linear form*

$$x \in (E, \sigma(E, E_1)) \longmapsto C(u(x), y_1) \in \mathbb{K}$$

be continuous for all $y_1 \in F_1$. Analogously, in order that a $\mathbb{K}$-linear mapping $v : F_1 \to E_1$ be $\sigma(F_1, F)$-$\sigma(E_1, E)$-continuous, it is necessary and sufficient that the $\mathbb{K}$-linear form

$$y_1 \in (F_1, \sigma(F_1, F)) \longmapsto B(x, v(y_1)) \in \mathbb{K}$$

be continuous for all $x \in E$.

Proof: We shall prove the first claim. Indeed, by the definition of $\sigma(F, F_1)$, u is $\sigma(E, E_1)$-$\sigma(F, F_1)$-continuous if and only if $C_{y_1} \circ u : (E, \sigma(E, E_1)) \to \mathbb{K}$ is continuous for all $y_1 \in F_1$. Consequently, since

$$(C_{y_1} \circ u)(x) = C_{y_1}(u(x)) = C(u(x), y_1) \text{ for } x \in E,$$

the proof of the first claim is concluded. □

Definition 3.12 *Under the conditions of Proposition 3.11, suppose that u is $\sigma(E, E_1)$-$\sigma(F, F_1)$-continuous and let $y_1 \in F_1$ be arbitrary. Then, by Propositions 3.11 and 3.7, there is a unique $x_1 \in E_1$ such that*

$$C(u(x), y_1) = B(x, x_1)$$

for all $x \in E$. Therefore the mapping

$$u^t : y_1 \in F_1 \longmapsto x_1 \in E_1$$

is well defined and the fundamental property

$$C(u(x), y_1) = B(x, u^t(y_1)) \ (x \in E, \ y_1 \in F_1)$$

holds; u^t is a $\mathbb{K}$-linear mapping, called the transpose of u.

Remark 3.13 *u^t is the unique $\mathbb{K}$-linear mapping $v : F_1 \to E_1$ such that $C(u(x), y_1) = B(x, v(y_1))$ for $x \in E$ and $y_1 \in F_1$.*

Proposition 3.14 *Under the conditions of Proposition 3.11, assume that u is $\sigma(E, E_1)$-$\sigma(F, F_1)$-continuous. Then u^t is $\sigma(F_1, F)$-$\sigma(E_1, E)$-continuous.*

Proof: By Proposition 3.11, u^t is $\sigma(F_1, F)$-$\sigma(E_1, E)$-continuous if and only if the $\mathbb{K}$-linear form

$$y_1 \in (F_1, \sigma(F_1, F)) \longmapsto B(x, u^t(y_1)) \in \mathbb{K}$$

is continuous for all $x \in E$. But, since $B(x, u^t(y_1)) = C(u(x), y_1)$ and since the $\mathbb{K}$-linear form

$$y_1 \in (F_1, \sigma(F_1, F)) \longmapsto C(u(x), y_1) \in \mathbb{K}$$

is continuous, the result follows. □

Remark 3.15 *Under the conditions of Proposition 3.14, $(u^t)^t = u$.*
In fact, put $w = (u^t)^t$. Then, for $x \in E$ and $y_1 \in F_1$,

$$B(x, u^t(y_1)) = C(w(x), y_1).$$

Consequently, for each $x \in E$, we have that $C(u(x), y_1) = C(w(x), y_1)$ for all $y_1 \in F_1$, and hence $u(x) = w(x)$. Therefore $u = (u^t)^t$.

Proposition 3.16 *Under the conditions of Proposition 3.14, the following properties hold:*
(a) $(Im(u))^\perp = Ker(u^t)$;
(b) $(Ker(u^t))^\perp = \overline{Im(u)}^{\sigma(F,F_1)}$.

Proof: (a) For $y_1 \in F_1$, we have that $y_1 \in (Im(u))^\perp$ if and only if $0 = C(u(x), y_1) = B(x, u^t(y_1))$ for all $x \in E$, which is equivalent to $y_1 \in Ker(u^t)$.
(b) By Propositions 3.10 and 3.16(a),

$$\overline{Im(u)}^{\sigma(F,F_1)} = ((Im(u))^\perp)^\perp = (Ker(u^t))^\perp. \square$$

Corollary 3.17 *Under the conditions of Proposition 3.14, u is injective if and only if $Im(u^t)$ is $\sigma(E_1, E)$-dense in E_1.*

Proof: By applying Proposition 3.16(b), with u^t in place of u, we get

$$\overline{Im(u^t)}^{\sigma(E_1,E)} = (Ker((u^t)^t))^\perp = (Ker(u))^\perp$$

(recall Remark 3.15). On the other hand, since $(Ker(u))^\perp = E_1$ if and only if $Ker(u) = \{0\}$, the proof is concluded. □

Now we may state a version of Theorem 13 of [5] in our context:

Theorem 3.18 *Let (E, E_1) and (F, F_1) be two dual systems. Let u be a $\sigma(E, E_1)$-$\sigma(F, F_1)$-continuous linear mapping such that $Im(u)$ is $\sigma(F, F_1)$-closed in F and let $y_0 \in F$. In order that the equation $u(x) = y_0$ admits at least one solution $x \in E$, it is necessary and sufficient that $y_0 \in (Ker(u^t))^\perp$.*

Proof: The necessity holds without the assumption that $Im(u)$ is $\sigma(F, F_1)$-closed, since $Im(u) \subset ((Im(u))^\perp)^\perp = (Ker(u^t))^\perp$, the equality being a consequence of Proposition 3.16(a). And the sufficiency follows from the equalities

$$Im(u) = \overline{Im(u)}^{\sigma(F,F_1)} = ((Im(u))^\perp)^\perp = (Ker(u^t))^\perp,$$

the second (resp. third) being a consequence of Proposition 3.10 (resp. Proposition 3.16(a)). □

Remark 3.19 *The condition that $Im(u)$ is $\sigma(F, F_1)$-closed is essential for the validity of the sufficiency of Theorem 3.18. In fact, if $Im(u)$ is not $\sigma(F, F_1)$-closed, there is a $y_0 \in \overline{Im(u)}^{\sigma(F,F_1)} \setminus Im(u)$. But, as we have seen in the proof of Theorem 3.18, $y_0 \in (Ker(u^t))^\perp$.*

Example 3.20 *Let E, F be two left vector spaces over $\mathbb{K}$, $u : E \to F$ a $\mathbb{K}$-linear mapping and $y_0 \in F$. If $u^t : \psi \in F^* \longmapsto \psi \circ u \in E^*$ is the transpose of u, it is known ([2], p. 304, Proposition 12) that the the equation $u(x) = y_0$ admits a solution $x \in E$ if and only if $y_0 \in (Ker(u^t))^\perp$. This fact can be easily obtained from Theorem 3.18, as follows. Consider the dual systems (E, E^*) and (F, F^*) (Example 3.2). Then u is $\sigma(E, E^*)$-$\sigma(F, F^*)$-continuous and $Im(u)$ is $\sigma(F, F^*)$-closed (it is easily seen that every subspace of F is $\sigma(F, F^*)$-closed). Therefore Theorem 3.18 ensures the validity of the above-mentioned fact.*

4 Weak linear topologies versus dual systems

Throughout we shall denote by $Lts_{\mathbb{K}}$ the category whose objects are the left linearly topologized spaces over $\mathbb{K}$ where, for $(E, \tau), (F, \theta) \in Ob(Lts_{\mathbb{K}})$, $Mor_{Lts_{\mathbb{K}}}(E, F)$ is the additive group of all continuous $\mathbb{K}$-linear mappings from (E, τ) into (F, θ).

Definition 4.1 *[9] For $(E, \tau) \in Ob(Lts_{\mathbb{K}})$, τ is said to be a weak topology if $\tau = \sigma(E, E')$.*

Remark 4.2 *By Proposition 3.7, $\sigma(E, E_1)$ is a weak topology for every dual system (E, E_1).*

Example 4.3 *For every index set I, the product topology τ on $\mathbb{K}^I$ is weak. In fact, since $\sigma(\mathbb{K}, \mathbb{K})$ is the discrete topology ξ, it follows from Example 3.6 that $\tau = \xi^I = (\sigma(\mathbb{K}, \mathbb{K}))^I = \sigma(\mathbb{K}^I, \mathbb{K}^{(I)})$. Hence, in view of Remark 4.2, τ is weak.*

Proposition 4.4 *[9] If τ_1, τ_2 are weak linear topologies on the left vector space E over $\mathbb{K}$, we have that τ_1 is coarser than τ_2 if and only if $(E,\tau_1)' \subset (E,\tau_2)'$. Consequently, $\tau_1 = \tau_2$ if and only if $(E,\tau_1)' = (E,\tau_2)'$.*

Proof: It is clear that $(E,\tau_1)' \subset (E,\tau_2)'$ if τ_1 is coarser than τ_2. Conversely, assume that $G_1 = (E,\tau_1)' \subset (E,\tau_2)' = G_2$. Then $\sigma(E,G_1)$ is coarser than $\sigma(E,G_2)$, which is the same as saying that τ_1 is coarser than τ_2, because τ_1 and τ_2 are weak topologies. □

We shall denote by $Ltsw_{\mathbb{K}}$ the subcategory of $Lts_{\mathbb{K}}$ whose objects are the left linearly topologized spaces over $\mathbb{K}$ whose topologies are weak.

Now, given a dual system (E,E_1) with respect to B and a dual system (F,F_1) with respect to C, we shall denote by $\mathcal{L}_a(E,F)$ (resp. $\mathcal{L}_a(F_1,E_1)$) the additive group of all $\mathbb{K}$-linear mappings from E into F (resp. F_1 into E_1), and we shall denote by $H = \mathcal{L}_a(E,F_1,\mathbb{K})$ the additive group of all $\mathbb{K}$-bilinear forms from $E \times F_1$ into $\mathbb{K}$. Let us define the group homomorphism $\theta_1 : \mathcal{L}_a(E,F) \to H$ (resp. $\theta_2 : \mathcal{L}_a(F_1,E_1) \to H$) given by $\theta_1(u)(x,y_1) = C(u(x),y_1)$ for $u \in \mathcal{L}_a(E,F)$ and $(x,y_1) \in E \times F_1$ (resp. $\theta_2(v)(x,y_1) = B(x,v(y_1))$ for $v \in \mathcal{L}_a(F_1,E_1)$ and $(x,y_1) \in E \times F_1$) and put

$$\mathcal{L}_a(E,F) \times_H \mathcal{L}_a(F_1,E_1) = \{(u,v) \in \mathcal{L}_a(E,F) \times \mathcal{L}_a(F_1,E_1) \,|\, \theta_1(u) = \theta_2(v)\}$$

($\mathcal{L}_a(E,F) \times_H \mathcal{L}_a(F_1,E_1)$ is a subgroup of the product group $\mathcal{L}_a(E,F) \times \mathcal{L}_a(F_1,E_1)$).

As in [10] we shall consider the category $Dual_{\mathbb{K}}$ whose objects are the dual systems where, for $(E,E_1),(F,F_1) \in Ob(Dual_{\mathbb{K}})$,

$$Mor_{Dual_{\mathbb{K}}}((E,E_1),(F,F_1)) = \mathcal{L}_a(E,F) \times_H \mathcal{L}_a(F_1,E_1).$$

Following an idea exploited in Theorem 10 of [1], but in a completely different context, we shall show that there is an equivalence between the categories $Ltsw_{\mathbb{K}}$ and $Dual_{\mathbb{K}}$.

For this purpose, for each $(E,\tau) \in Ob(Ltsw_{\mathbb{K}})$ let us define $\mathcal{F}((E,\tau)) = (E,E') \in Ob(Dual_{\mathbb{K}})$ (Example 3.3), where $E' = (E,\tau)'$. And for each $u \in Mor_{Ltsw_{\mathbb{K}}}((E,\tau),(F,\theta))$ let us define $\mathcal{F}(u) = (u,u^t)$. As above, let $H = \mathcal{L}_a(E,F',\mathbb{K})$ and $\theta_1 : \mathcal{L}_a(E,F) \to H$ (resp. $\theta_2 : \mathcal{L}_a(F',E') \to H$) be given by $\theta_1(u)(x,\psi) = \psi(u(x))$ for $u \in \mathcal{L}_a(E,F)$, $(x,\psi) \in E \times F'$ (resp. $\theta_2(v)(x,\psi) = (v(\psi))(x)$ for $v \in \mathcal{L}_a(F',E')$, $(x,\psi) \in E \times F'$). It is easily seen that

$$\mathcal{F}(f) \in \mathcal{L}_a(E,F) \times_H \mathcal{L}_a(F',E') = \{(u,v) \in \mathcal{L}_a(E,F) \times \mathcal{L}_a(F',E') \,|\, \theta_1(u) = \theta_2(v)\}.$$

Therefore $\mathcal{F} : Ltsw_{\mathbb{K}} \longrightarrow Dual_{\mathbb{K}}$ is a covariant functor, and we may state the following

Theorem 4.5 *$\mathcal{F}$ establishes an equivalence between $Ltsw_{\mathbb{K}}$ and $Dual_{\mathbb{K}}$.*

Proof: Let (E, E_1) be a dual system with respect to the $\mathbb{K}$-bilinear form B. Then, by Remark 4.2, $(E, \sigma(E, E_1)) \in Ob(Ltsw_{\mathbb{K}})$. And, by definition, $\mathcal{F}((E, \sigma(E, E_1))) = (E, (E, \sigma(E, E_1))')$ (dual system with respect to the $\mathbb{K}$-bilinear form $C(x, \varphi) = \varphi(x)$). But, by Proposition 3.7, $\varphi \in (E, \sigma(E, E_1))'$ if and only if $\varphi = B_{x_1}$ for a unique $x_1 \in E_1$; consequently, for $(x, \varphi) \in E \times (E, \sigma(E, E_1))'$,

$$C(x, \varphi) = C(x, B_{x_1}) = B_{x_1}(x) = B(x, x_1).$$

Therefore $\mathcal{F}((E, \sigma(E, E_1))) = (E, E_1)$, proving the surjectivity of $\mathcal{F}$.

Now, let $(E, \tau), (F, \theta) \in Ob(Ltsw_{\mathbb{K}})$. We claim that the mapping

$$u \in Mor_{Ltsw_{\mathbb{K}}}(E, F) \longmapsto \mathcal{F}(u) \in Mor_{Dual_{\mathbb{K}}}(\mathcal{F}(E), \mathcal{F}(F))$$

is bijective. In fact, since its injectivity is clear, let us show its surjectivity. For this purpose, let

$$(u, v) \in Mor_{Dual_{\mathbb{K}}}(\mathcal{F}(E), \mathcal{F}(F)) = \mathcal{L}_a(E, F) \times_H \mathcal{L}_a(F', E')$$

be arbitrary, where $H = \mathcal{L}_a(E, F', \mathbb{K})$; then $(v(\psi))(x) = \psi(u(x))$ for all $(x, \psi) \in E \times F'$. By Remark 4.2, $(E, \sigma(E, E')), (F, \sigma(F, F')) \in Ob(Ltsw_{\mathbb{K}})$. Moreover, u is $\sigma(E, E')$-$\sigma(F, F')$-continuous in view of Proposition 3.11, because $\psi \circ u = v(\psi) \in E'$ for all $\psi \in F'$. Finally, $v = u^t$, since

$$(v(\psi))(x) = (\psi \circ u)(x) = (u^t(\psi))(x)$$

for all $(x, \psi) \in E \times F'$. Consequently, $\mathcal{F}(u) = (u, v)$, proving the surjectivity of the above-mentioned mapping. Therefore, by Theorem 1, Chapter IV, §4 of [8], $\mathcal{F}$ establishes an equivalence between $Ltsw_{\mathbb{K}}$ and $Dual_{\mathbb{K}}$, as was to be shown. □

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