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Asymptotic Formulas
Composite Numbers III

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Abstract

Let $k \geq 1$ and $h \geq 1$ arbitrary but fixed positive integers. Let us consider the numbers such that in their prime factorization there are k primes with exponent h and the remainder of the primes have exponente greater than h. Let $P_{k,h}(x)$ be the number of these numbers not exceeding x. We prove the formula

$$P_{k,h}(x) \sim A_{h+1} \frac{h x^{1/h} (\log \log x)^{k-1}}{(k-1)! \log x},$$

where A_{h+1} is a constant defined in this article.

Let $k \geq 1$, $h \geq 1$ and $t \geq 1$ arbitrary but fixed positive integers. Let us consider the numbers such that in their prime factorization there are k primes with exponent h and the t primes remaining have exponent greater than h. Let $A_{k,h,t}(x)$ be the number of these numbers not exceeding x. We prove the formula

$$A_{k,h,t}(x) \sim A_{t,h+1} \frac{h x^{1/h} (\log \log x)^{k-1}}{(k-1)! \log x},$$

where $A_{t,h+1}$ is a constant defined in this article.

Let $E_{t,h}(x)$ be the number of h-ful numbers with exactly t distinct prime factors in their prime factorization. We prove the asymptotic formula

$$E_{t,h}(x) \sim \frac{h x^{1/h} (\log \log x)^{t-1}}{(t-1)! \log x}.$$

In particular if $h = 1$ then we obtain the following well-known Landau's Theorem

$$E_{t,1}(x) \sim \frac{x(\log\log x)^{t-1}}{(t-1)!\log x},$$

where $E_{t,1}(x)$ is the number of numbers not exceeding x with exactly t distinct prime factors in their prime factorization.

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1 Introduction, Notation and Lemmas

Let n be a number such that its prime factorization if of the form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t},$$

where $a_i \geq h+1$ $(i = 1, 2, \ldots, t)$, $(h \geq 1)$ is fixed and $p_1, p_2, \ldots, p_t$ $(t \geq 1)$ are the different primes in the factorization. Note that the a_i $(i = 1, 2, \ldots, t)$ and t are variable.

These number are well known, they are called $(h+1)$-ful numbers.

There exist various studies on the distribution of these numbers using not elementary methods (see [1]).

Let C_n be the sequence of $(h+1)$-ful numbers and let $C_{h+1}(x)$ be the number of $(h+1)$-ful numbers that do not exceed x. It is well known (see [2] for an elementary proof) that

$$C_n \sim c_{h+1} n^{h+1}, \tag{1}$$

$$C_{h+1}(x) \sim b_{h+1} x^{\frac{1}{h+1}}, \tag{2}$$

where b_{h+1} and c_{h+1} are positive constants. Note that C_n depends of $h+1$. For sake of simplicity we use this notation.

In this article C denotes a $(h+1)$-ful number.

From (1) we can obtain without difficulty the following lemma.

Lemma 1.1 *The following series are convergent. That is, we have*

$$\sum_{n=1}^{\infty} \frac{1}{C_n^{1/h}} = A_{h+1} \qquad \sum_{n=1}^{\infty} \frac{\log C_n}{C_n^{1/h}} = B_{h+1}.$$

Let us consider the sequence P_n of the numbers whose prime factorization is of the form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} p_{t+1}^{h} \cdots p_{t+k}^{h},$$

where $a_i \geq h+1$ $(i = 1, 2, \ldots, t)$ are variable, $(h \geq 1)$ is fixed, $(t \geq 1)$ is variable, $(k \geq 1)$ is fixed and $p_1, p_2, \ldots, p_{t+k}$ are the different primes in the factorization. Note that the sequence P_n depends of k and h. For sake of simplicity we use this notation.

We shall denote these numbers in the compact form $Cp_1^h \cdots p_k^h$ where C denotes the $(h+1)$-ful number $p_1^{a_1}p_2^{a_2}\ldots p_t^{a_t}$ and $p_1^h \cdots p_k^h$ denotes $p_{t+1}^h \cdots p_{t+k}^h$.

The number of these numbers not exceeding x we shall denote $P_{k,h}(x)$

In this article we prove the asymptotic formula

$$P_{k,h}(x) \sim A_{h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}.$$

Let us consider the sequence E_n of the $(h+1)$-ful numbers with t different prime factors, where $t \geq 1$ is a fixed positive integer. Note that the sequence E_n depends of t and $h+1$. For sake of simplicity we use this notation.

We shall denote these numbers in the compact form E.

The number of these numbers not exceeding x we shall denote $E_{t,h+1}(x)$.

Let us consider the sequence A_n of the numbers whose prime factorization is of the form

$$p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}p_{t+1}^h \cdots p_{t+k}^h,$$

where $a_i \geq h+1$ $(i = 1, 2, \ldots, t)$ are variable, $(h \geq 1)$ is fixed, $(t \geq 1)$ is fixed, $(k \geq 1)$ is fixed and $p_1, p_2, \ldots, p_{t+k}$ are the different primes in the factorization. Note that the sequence A_n depend of k, h and t. For sake of simplicity we use this notation.

We shall denote these numbers in the compact form $Ep_1^h \cdots p_k^h$ where E denotes the $(h+1)$-ful numbers with t different prime factors $p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$ and $p_1^h \cdots p_k^h$ denotes $p_{t+1}^h \cdots p_{t+k}^h$.

The number of these numbers not exceeding x we shall denote $A_{k,h,t}(x)$.

Since in this case the E numbers are $(h+1)$-ful numbers, Lemma 1.1 imply that the following series are convergent, that is

$$\sum_{n=1}^{\infty}\frac{1}{E_n^{1/h}} = A_{t,h+1} \qquad \sum_{n=1}^{\infty}\frac{\log E_n}{E_n^{1/h}} = B_{t,h+1}. \tag{3}$$

In this article we prove the asymptotic formula

$$A_{k,h,t}(x) \sim A_{t,h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}.$$

On the other hand (2) imply that from a certain value of x we have

$$E_{t,h+1}(x) \leq (1+\epsilon)b_{h+1}x^{\frac{1}{h+1}} \qquad (\epsilon > 0). \tag{4}$$

Let $\pi(x)$ be the number of primes not exceeding x. We shall need the prime number Theorem which we shall use as a lemma.

Lemma 1.2 *The following formula holds*

$$\pi(x) = \frac{x}{\log x} + f(x)\frac{x}{\log x},$$

where $|f(x)| \leq M$ *if* $x \geq 2$ *and* $f(x) \to 0$.

Let us consider the numbers whose prime factorization is of the form

$$p_1 p_2 \dots p_k,$$

where $k \geq 2$ is fixed and $p_1, p_2, \dots, p_k$ are different primes.

Let $B_k(x)$ be the number of these numbers not exceeding x. We have the following theorem (Landau's Theorem) which we shall use as a lemma (see [1]).

Lemma 1.3 *The following asymptotic formula holds*

$$B_k(x) = \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} + f(x)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x},$$

where $|f(x)| \leq M$ *if* $x \geq 3$ *and* $f(x) \to 0$. *Note that* $f(x)$ *and* M *depend of* k.

We shall also need the following two lemmas whose proofs are simple.

Lemma 1.4 *The nonnegative function* $(x \geq e)$ $(k \geq 2)$

$$f(x) = \frac{(\log\log x)^{k-1}}{\log x}$$

is bounded. That is, there exist $H > 0$ *such that* $f(x) \leq H$. *Note that* $f(x)$ *and* H *depend of* k.

Lemma 1.5 *The function* $(c > 1)$

$$f(x) = \frac{\log\log\left(\frac{x}{c}\right)}{\log\log x}$$

is increasing from a certain value of x. *Note that* $f(x)$ *depends of* c.

In this article we also prove the asymptotic formula (see (4))

$$E_{t,h}(x) \sim \frac{hx^{1/h}(\log\log x)^{t-1}}{(t-1)!\log x}.$$

In particular if $h = 1$ then we obtain the following well-known Landau's Theorem

$$E_{t,1}(x) \sim \frac{x(\log\log x)^{t-1}}{(t-1)!\log x},$$

where $E_{t,1}(x)$ is the number of numbers not exceeding x with exactly t distinct prime factors in their prime factorization.

2 Main Lemmas

The method of proof in the following Lemma 2.1 is similar to the method used in [4]. For sake of completeness we give the proof. Note that the meaning of E is different here.

Lemma 2.1 *Let $\epsilon > 0$. There exists x_ϵ such that if $x \geq x_\epsilon$ then we have the following inequality*

$$A_{1,h,t}(x) \leq (A_{t,h+1} + \epsilon)\frac{hx^{1/h}}{\log x}. \tag{5}$$

Proof. We have

$$Ep^h \leq x,$$
$$p^h \leq \frac{x}{E},$$
$$E \leq \frac{x}{p^h} \leq \frac{x}{2^h},$$
$$\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \geq 2.$$

Therefore (lemma 1.2)

$$\begin{aligned}
A_{1,h,t}(x) &= \sum_{E \leq \frac{x}{2^h}} \sum_{p^h \leq \frac{x}{E}} 1 - F_1(x) = \sum_{E \leq \frac{x}{2^h}} \sum_{p \leq \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}} 1 - F_1(x) \\
&= \sum_{E \leq \frac{x}{2^h}} \pi\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) - F_1(x) = \sum_{E \leq \frac{x}{2^h}} \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\
&+ \sum_{E \leq \frac{x}{2^h}} f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} - F_1(x) \\
&= \frac{hx^{\frac{1}{h}}}{\log x} \sum_{E \leq \frac{x}{2^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} + G_1(x) - F_1(x).
\end{aligned} \tag{6}$$

Substituting $x = 2^h E_n$ into

$$\sum_{E \leq \frac{x}{2^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

we obtain the sequence

$$\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}}. \tag{7}$$

Note that if $E_i \leq E_n$ then

$$\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq \frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_i^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}}\frac{\log 2+\log E_i^{\frac{1}{h}}}{\log 2} = \frac{1}{E_i^{\frac{1}{h}}} + \frac{1}{h\log 2}\frac{\log E_i}{E_i^{\frac{1}{h}}} \tag{8}$$

and if E_i is fixed then

$$\lim_{n\to\infty}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}}. \tag{9}$$

We have (see (7))

$$\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} = \sum_{i=1}^{k}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} + \sum_{i=k+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}}, \tag{10}$$

where (see (8))

$$\sum_{i=k+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq \sum_{i=k+1}^{n}\frac{1}{E_i^{\frac{1}{h}}} + \frac{1}{h\log 2}\sum_{i=k+1}^{n}\frac{\log E_i}{E_i^{\frac{1}{h}}}. \tag{11}$$

There exists k such that ($\epsilon > 0$) (see (3))

$$A_{t,h+1} - \epsilon < \sum_{i=1}^{k}\frac{1}{E_i^{\frac{1}{h}}} < A_{t,h+1}, \tag{12}$$

$$\frac{1}{h\log 2}\sum_{i=k+1}^{\infty}\frac{\log E_i}{E_i^{\frac{1}{h}}} < \epsilon. \tag{13}$$

If $n \geq k+1$, (11), (12) and (13) give

$$0 \leq \sum_{i=k+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq 2\epsilon. \tag{14}$$

On the other hand (see(9))

$$\lim_{n\to\infty}\sum_{i=1}^{k}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} = \sum_{i=1}^{k}\frac{1}{E_i^{\frac{1}{h}}}.$$

Consequently there exists $n' > k+1$ such that for all $n \geq n'$ we have

$$\sum_{i=1}^{k} \frac{1}{E_i^{\frac{1}{h}}} - \epsilon \leq \sum_{i=1}^{k} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq \sum_{i=1}^{k} \frac{1}{E_i^{\frac{1}{h}}} + \epsilon. \tag{15}$$

(12) and (15) give

$$A_{t,h+1} - 2\epsilon \leq \sum_{i=1}^{k} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq A_{t,h+1} + \epsilon. \tag{16}$$

Therefore for all $n \geq n'$ we have (see (10), (14) and (16))

$$A_{t,h+1} - 3\epsilon \leq \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} \leq A_{t,h+1} + 3\epsilon. \tag{17}$$

Consequently

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} = A_{t,h+1}. \tag{18}$$

Now, we have

$$\begin{aligned} & \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n+1} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_{n+1}^{\frac{1}{h}}}} \right) \\ = & \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_{n+1}^{\frac{1}{h}}}} - \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log 2E_{n+1}^{\frac{1}{h}}}{\log 2} \right) \\ = & \ 0 \end{aligned} \tag{19}$$

and

$$\lim_{n\to\infty} \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log 2E_{n+1}^{\frac{1}{h}}}{\log 2} = 0. \tag{20}$$

(19) and (20) give

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log 2E_{n+1}^{\frac{1}{h}}}} \right) = 0. \tag{21}$$

Therefore (see (18))

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}}=A_{t,h+1}, \tag{22}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_{n+1}^{\frac{1}{h}}}}=A_{t,h+1}. \tag{23}$$

The function of x (E_i fixed, $E_i \leq E_n$)

$$\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \tag{24}$$

is decreasing in the interval $\left[2^hE_n, 2^hE_{n+1}\right)$. Therefore if $x\in\left[2^hE_n, 2^hE_{n+1}\right)$ we have

$$\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_{n+1}^{\frac{1}{h}}}}\leq\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}\leq\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log 2E_n^{\frac{1}{h}}}}. \tag{25}$$

Consequently (22), (23) and (25) give

$$\lim_{x\to\infty}\sum_{E\leq\frac{x}{2^h}}\frac{1}{E^{\frac{1}{h}}}\frac{1}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}=A_{t,h+1}. \tag{26}$$

There exists x_0 such that (lemma 1.2)

$$\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|<\epsilon \quad if \quad \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\geq x_0, \quad that\ is\ if \quad E\leq\frac{x}{x_0^h},$$

$$\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|\leq M \quad if \quad 2\leq\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}<x_0, \quad that\ is\ if \quad \frac{x}{x_0^h}<E\leq\frac{x}{2^h}.$$

Therefore (see(6))

$$\begin{aligned}|G_1(x)| &= \left|\sum_{E\leq\frac{x}{2^h}} f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}\right| \\ &\leq \sum_{E\leq\frac{x}{2^h}}\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}\end{aligned}$$

$$\leq \epsilon \sum_{E \leq \frac{x}{x_0^h}} \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} + Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{2^h}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}$$

$$= \frac{\epsilon h x^{\frac{1}{h}}}{\log x} \sum_{E \leq \frac{x}{x_0^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} + Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{2^h}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}. \quad (27)$$

Now (see(4))

$$Mx_0 \sum_{\frac{x}{x_0^h} < E \leq \frac{x}{2^h}} \frac{1}{\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \leq Mx_0 \sum_{E \leq x} \frac{1}{\log 2} = \frac{Mx_0}{\log 2} E_{t,h+1}(x) \leq \frac{Mx_0}{\log 2}(1+\epsilon) b_{h+1} x^{\frac{1}{h+1}} \quad (28)$$

and (see (26))

$$\frac{\epsilon h x^{\frac{1}{h}}}{\log x} \sum_{E \leq \frac{x}{x_0^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq \frac{\epsilon h x^{\frac{1}{h}}}{\log x} \sum_{E \leq \frac{x}{2^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq \frac{\epsilon h x^{\frac{1}{h}}}{\log x}(A_{t,h+1} + \epsilon). \quad (29)$$

Consequently (27), (28) and (29) give

$$G_1(x) = o\left(\frac{x^{\frac{1}{h}}}{\log x}\right). \quad (30)$$

Equations (6), (26) and (30) give

$$A_{1,h,t}(x) = A_{t,h+1} \frac{h x^{1/h}}{\log x} + o\left(\frac{x^{1/h}}{\log x}\right) - F_1(x). \quad (31)$$

Now, $F_1(x) \geq 0$, therefore (31) gives

$$A_{1,h,t}(x) \leq A_{t,h+1} \frac{h x^{1/h}}{\log x} + o\left(\frac{x^{1/h}}{\log x}\right) \leq (A_{t,h+1} + \epsilon) \frac{h x^{1/h}}{\log x}.$$

That is, equation (5). The lemma is proved.

The method of proof in the following Lemma 2.2 is similar to the method used in [5]. For sake of completeness we give the proof. Note that the meaning of E is different here.

Lemma 2.2 *Let $\epsilon > 0$. There exists x_ϵ such that if $x \geq x_\epsilon$ then we have the following inequality*

$$A_{k,h,t}(x) \leq (A_{t,h+1} + \epsilon) \frac{h x^{1/h} (\log\log x)^{k-1}}{(k-1)! \log x} \qquad (k \geq 2). \quad (32)$$

Proof. Let P_k be the product of the first k primes, that is, $P_2 = 2.3 = 6, P_3 = 2.3.5 = 30$, etc. We have

$$Ep_1^h p_2^h \dots p_k^h \leq x,$$

$$p_1^h p_2^h \dots p_k^h \leq \frac{x}{E},$$

$$E \leq \frac{x}{p_1^h p_2^h \dots p_k^h} \leq \frac{x}{P_k^h},$$

$$\frac{x}{E} \geq P_k^h,$$

$$\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \geq P_k \geq 6.$$

Therefore (lemma 1.3)

$$\begin{aligned} A_{k,h,t}(x) &= \sum_{E \leq \frac{x}{P_k^h}} \sum_{p_1^h \dots p_k^h \leq \frac{x}{E}} 1 - F_k(x) = \sum_{E \leq \frac{x}{P_k^h}} \sum_{p_1 \dots p_k \leq \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}} 1 - F_k(x) \\ &= \sum_{E \leq \frac{x}{P_k^h}} B_k\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) - F_k(x) = \sum_{E \leq \frac{x}{P_k^h}} \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\ &+ \sum_{E \leq \frac{x}{P_k^h}} f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right) \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}} \frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} - F_k(x) \\ &= \frac{hx^{\frac{1}{h}}\left(\log\log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)!\log x} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \\ &+ G_k(x) - F_k(x). \end{aligned} \tag{33}$$

Substituting $x = P_k^h E_n$ into

$$\sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

we obtain the sequence

$$\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}}. \tag{34}$$

Note that if $E_i \le E_n$ then

$$\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \le \frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_i^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}}\frac{\log P_k + \log E_i^{\frac{1}{h}}}{\log P_k} = \frac{1}{E_i^{\frac{1}{h}}}+\frac{1}{h\log P_k}\frac{\log E_i}{E_i^{\frac{1}{h}}} \tag{35}$$

and if E_i is fixed then

$$\lim_{n\to\infty}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \frac{1}{E_i^{\frac{1}{h}}}. \tag{36}$$

We have (see (34))

$$\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \sum_{i=1}^{j}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} + \sum_{i=j+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}}, \tag{37}$$

where (see (35))

$$\sum_{i=j+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \le \sum_{i=j+1}^{n}\frac{1}{E_i^{\frac{1}{h}}} + \frac{1}{h\log P_k}\sum_{i=j+1}^{n}\frac{\log E_i}{E_i^{\frac{1}{h}}}. \tag{38}$$

There exists j such that $(\epsilon > 0)$ (see (3))

$$A_{t,h+1} - \epsilon < \sum_{i=1}^{j}\frac{1}{E_i^{\frac{1}{h}}} < A_{t,h+1}, \tag{39}$$

$$\frac{1}{h\log P_k}\sum_{i=j+1}^{\infty}\frac{\log E_i}{E_i^{\frac{1}{h}}} < \epsilon. \tag{40}$$

If $n \ge j+1$, (38), (39) and (40) give

$$0 \le \sum_{i=j+1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \le 2\epsilon. \tag{41}$$

On the other hand (see (36))

$$\lim_{n\to\infty}\sum_{i=1}^{j}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = \sum_{i=1}^{j}\frac{1}{E_i^{\frac{1}{h}}}.$$

Consequently there exists $n' > j+1$ such that for all $n \geq n'$ we have

$$\sum_{i=1}^{j} \frac{1}{E_i^{\frac{1}{h}}} - \epsilon \leq \sum_{i=1}^{j} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq \sum_{i=1}^{j} \frac{1}{E_i^{\frac{1}{h}}} + \epsilon. \tag{42}$$

(39) and (42) give

$$A_{t,h+1} - 2\epsilon \leq \sum_{i=1}^{j} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq A_{t,h+1} + \epsilon. \tag{43}$$

Therefore for all $n \geq n'$ we have (see (37), (41) and (43))

$$A_{t,h+1} - 3\epsilon \leq \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} \leq A_{t,h+1} + 3\epsilon. \tag{44}$$

Consequently

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} = A_{t,h+1}. \tag{45}$$

Now, we have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n+1} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} \right) \\
= &\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} - \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log P_k E_{n+1}^{\frac{1}{h}}}{\log P_k} \right) \\
= &\ 0
\end{aligned} \tag{46}$$

and

$$\lim_{n\to\infty} \frac{1}{E_{n+1}^{\frac{1}{h}}} \frac{\log P_k E_{n+1}^{\frac{1}{h}}}{\log P_k} = 0. \tag{47}$$

(46) and (47) give

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}} - \sum_{i=1}^{n} \frac{1}{E_i^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}} \right) = 0. \tag{48}$$

Therefore (see (45))

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}}=A_{t,h+1}, \tag{49}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}}=A_{t,h+1}. \tag{50}$$

The function of x (E_i fixed, $E_i \leq E_n$)

$$\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \tag{51}$$

is decreasing in the interval $\left[P_k^h E_n, P_k^h E_{n+1}\right)$. Therefore if $x \in \left[P_k^h E_n, P_k^h E_{n+1}\right)$ we have

$$\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_{n+1}^{\frac{1}{h}}}}\leq\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}\leq\sum_{i=1}^{n}\frac{1}{E_i^{\frac{1}{h}}}\frac{1}{1-\frac{\log E_i^{\frac{1}{h}}}{\log P_k E_n^{\frac{1}{h}}}}. \tag{52}$$

Consequently (49), (50) and (52) give

$$\lim_{x\to\infty}\sum_{E\leq\frac{x}{P_k^h}}\frac{1}{E^{\frac{1}{h}}}\frac{1}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}=A_{t,h+1}. \tag{53}$$

The function of x (E fixed)

$$0<\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}<1$$

is increasing from a certain value of x (see lemma 1.5) and

$$\lim_{x\to\infty}\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}=1.$$

Let us consider the function

$$\sum_{E\leq\frac{x}{P_k^h}}\frac{1}{E^{\frac{1}{h}}}\frac{1}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}\left(1-\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}\right).$$

There exists x_0 such that

$$A_{t,h+1} - \epsilon \leq \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \leq A_{t,h+1} + \epsilon. \tag{54}$$

Now

$$\lim_{x\to\infty} \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} = \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}}. \tag{55}$$

Therefore there exist $x_1 > x_0$ such that if $x \geq x_1$ we have (see (54) and (55))

$$A_{t,h+1} - 2\epsilon \leq \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq A_{t,h+1} + 2\epsilon \tag{56}$$

and if $E \leq \frac{x_0}{P_k^h}$ we have

$$0 < 1 - \left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}} \right)^{k-1} < \epsilon. \tag{57}$$

(56) and (57) give

$$0 < \sum_{E \leq \frac{x_0}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}} \right)^{k-1} \right) \leq (A_{t,h+1} + 2\epsilon)\epsilon. \tag{58}$$

On the other hand (see (56) and (53))

$$\begin{aligned} 0 \quad &< \sum_{\frac{x_0}{P_k^h} < E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}} \right)^{k-1} \right) \\ &\leq \sum_{\frac{x_0}{P_k^h} < E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \leq 4\epsilon. \end{aligned} \tag{59}$$

(58) and (59) give

$$\lim_{x\to\infty} \sum_{E \leq \frac{x}{P_k^h}} \frac{1}{E^{\frac{1}{h}}} \frac{1}{1 - \frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \left(1 - \left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}} \right)^{k-1} \right) = 0. \tag{60}$$

(53) and (60) give

$$\lim_{x\to\infty}\sum_{E\leq\frac{x}{P_k^h}}\frac{1}{E^{\frac{1}{h}}}\frac{\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}=A_{t,h+1}. \tag{61}$$

There exists x_0 such that (lemma 1.3)

$$\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|<\epsilon \quad if \quad \frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\geq x_0,\quad that\ is\ if \quad E\leq\frac{x}{x_0^h},$$

$$\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|\leq M \quad if \quad P_k\leq\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}<x_0,\quad that\ is\ if \quad \frac{x}{x_0^h}<E\leq\frac{x}{P_k^h}.$$

Therefore

$$\begin{aligned}
|G_k(x)| &= \left|\sum_{E\leq\frac{x}{P_k^h}}f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}\right| \\
&\leq \sum_{E\leq\frac{x}{P_k^h}}\left|f\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right|\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\
&\leq \epsilon\sum_{E\leq\frac{x}{x_0^h}}\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}+Mx_0\sum_{\frac{x}{x_0^h}<E\leq\frac{x}{P_k^h}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)} \\
&= \frac{\epsilon h x^{\frac{1}{h}}\left(\log\log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)!\log x}\sum_{E\leq\frac{x}{x_0^h}}\frac{1}{E^{\frac{1}{h}}}\frac{\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}} \\
&+ Mx_0\sum_{\frac{x}{x_0^h}<E\leq\frac{x}{P_k^h}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}.
\end{aligned} \tag{62}$$

Now (see lemma 1.4 and (4))

$$Mx_0\sum_{\frac{x}{x_0^h}<E\leq\frac{x}{P_k^h}}\frac{\left(\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)\right)^{k-1}}{(k-1)!\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}\leq\frac{Mx_0H}{(k-1)!}\sum_{E\leq x}1$$

$$\leq \frac{Mx_0H}{(k-1)!}E_{t,h+1}(x) \leq \frac{Mx_0H}{(k-1)!}(1+\epsilon)b_{h+1}x^{\frac{1}{h+1}} \tag{63}$$

and (see (61))

$$\frac{\epsilon h x^{\frac{1}{h}}\left(\log\log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)!\log x}\sum_{E\leq\frac{x}{x_0^h}}\frac{1}{E^{\frac{1}{h}}}\frac{\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

$$\leq \frac{\epsilon h x^{\frac{1}{h}}\left(\log\log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)!\log x}\sum_{E\leq\frac{x}{P_k^h}}\frac{1}{E^{\frac{1}{h}}}\frac{\left(\frac{\log\log\left(\frac{x^{\frac{1}{h}}}{E^{\frac{1}{h}}}\right)}{\log\log x^{\frac{1}{h}}}\right)^{k-1}}{1-\frac{\log E^{\frac{1}{h}}}{\log x^{\frac{1}{h}}}}$$

$$\leq \frac{\epsilon h x^{\frac{1}{h}}\left(\log\log x^{\frac{1}{h}}\right)^{k-1}}{(k-1)!\log x}(A_{t,h+1}+\epsilon). \tag{64}$$

Consequently (62), (63) and (64) give

$$G_k(x) = o\left(\frac{x^{\frac{1}{h}}(\log\log x)^{k-1}}{\log x}\right). \tag{65}$$

Equations (33), (61) and (65) give

$$A_{k,h,t}(x) = A_{t,h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x} + o\left(\frac{x^{1/h}(\log\log x)^{k-1}}{\log x}\right) - F_k(x). \tag{66}$$

Now, $F_k(x) \geq 0$, therefore (66) gives

$$A_{k,h,t}(x) \leq A_{t,h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x} + o\left(\frac{x^{1/h}(\log\log x)^{k-1}}{\log x}\right)$$

$$\leq (A_{t,h+1}+\epsilon)\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}.$$

That is, equation (32). The lemma is proved.

Lemma 2.1 and Lemma 2.2 can be united in the following lemma.

Lemma 2.3 *Let $\epsilon > 0$. There exists x_ϵ such that if $x \geq x_\epsilon$ then we have the following inequality*

$$A_{k,h,t}(x) \leq (A_{t,h+1}+\epsilon)\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x} \qquad (k \geq 1). \tag{67}$$

Lemma 2.4 *Let $\epsilon > 0$. There exists x_ϵ such that if $x \geq x_\epsilon$ then we have the following inequality*

$$P_{k,h}(x) \leq (A_{h+1} + \epsilon)\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x} \qquad (k \geq 1). \tag{68}$$

Proof. The proof is the same as Lemma 2.1 and Lemma 2.2. In the proofs of Lemma 2.1 and Lemma 2.2 we replace $A_{k,h,t}(x)$ by $P_{k,h}(x)$, E by C, E_i by C_i, E_n by C_n, $A_{t,h+1}$ by A_{h+1}, E_{n+1} by C_{n+1} and $E_{t,h+1}(x)$ by $C_{h+1}(x)$. The lemma is proved.

3 Main Results

Theorem 3.1 *We have the following asymptotic formula*

$$A_{k,h,t}(x) \sim A_{t,h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x} \qquad (k \geq 1). \tag{69}$$

Proof. In the sums (see (6) and (33))

$$\sum_{E \leq \frac{x}{2^h}} \sum_{p^h \leq \frac{x}{E}} 1,$$

$$\sum_{E \leq \frac{x}{P_k^h}} \sum_{p_1^h \ldots p_k^h \leq \frac{x}{E}} 1,$$

are generated undesirable numbers. The number of these undesirable numbers not exceeding x is $F_k(x)$ $(k \geq 1)$. Let us consider the number (we take $h = 1$)

$$p_1 p_2 \cdots p_k E$$

This number is undesirable when some primes p_i appear in the prime factorization of the $2-ful$ number with t different prime factors E. For example the number

$$p_1 p_2 p_3 p_4 p_5^2 p_6^2 p_3^5 p_7^5 p_8^7 p_4^9 p_9^9 p_{10}^{14}, \tag{70}$$

where $k = 4$ and $t = 8$ is undesirable. This number is

$$p_1 p_2 p_5^2 p_6^2 p_3^6 p_7^5 p_8^7 p_4^{10} p_9^9 p_{10}^{14}. \tag{71}$$

This number can be generated in various ways. For example equation (70) is one way, we obtain this way if withdraw one prime p_3 and one prime p_4 of (71). We obtain other way if we withdraw one prime p_8 and one prime p_{10} of (71). This new way is

$$p_1 p_2 p_8 p_{10} p_5^2 p_6^2 p_3^6 p_7^5 p_8^6 p_4^{10} p_9^9 p_{10}^{13}. \tag{72}$$

Clearly we can withdraw of (71) two primes with exponent greater than 2. In contrary case we do not obtain a $2 - ful$ number. The number of possible ways is then bounded by $\binom{8}{2}$. Therefore if $k \leq t$ we have

$$F_k(x) \leq \binom{t}{1} A_{k-1,h,t}(x) + \binom{t}{2} A_{k-2,h,t}(x) + \cdots + \binom{t}{k-1} A_{1,h,t}(x) + \binom{t}{k} E_{t,h+1}(x). \tag{73}$$

In particular

$$F_1(x) \leq \binom{t}{1} E_{t,h+1}(x). \tag{74}$$

If $k > t$ we have

$$F_k(x) \leq \binom{t}{1} A_{k-1,h,t}(x) + \binom{t}{2} A_{k-2,h,t}(x) + \cdots + \binom{t}{t} A_{k-t,h,t}(x). \tag{75}$$

Equations (73), (74), (75), (67) and (4) give

$$\lim_{x \to \infty} \frac{F_k(x)}{\frac{x(\log \log x)^{k-1}}{\log x}} = 0.$$

That is

$$F_k(x) = o\left(\frac{x^{1/h} (\log \log x)^{k-1}}{\log x} \right) \qquad (k \geq 1). \tag{76}$$

Finally equations (31), (66) and (76) give equation (69). The theorem is proved.

Theorem 3.2 *We have the following asymptotic formula*

$$P_{k,h}(x) \sim A_{h+1} \frac{h x^{1/h} (\log \log x)^{k-1}}{(k-1)! \log x}. \tag{77}$$

Proof. We have

$$\sum_{t=1}^{\infty} A_{t,h+1} = A_{h+1}.$$

Let $\epsilon > 0$. There exists n such that

$$\sum_{t=1}^{n} A_{t,h+1} > A_{h+1} - \frac{\epsilon}{2}.$$

Now, we have (see (69))

$$P_{k,h}(x) \geq \sum_{t=1}^{n} A_{k,h,t}(x)$$

$$
\begin{aligned}
&= \sum_{t=1}^{n}\left(A_{t,h+1}\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}+o\left(\frac{x^{1/h}(\log\log x)^{k-1}}{\log x}\right)\right)\\
&= \left(\sum_{t=1}^{n}A_{t,h+1}\right)\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}+o\left(\frac{x^{1/h}(\log\log x)^{k-1}}{\log x}\right)\\
&\geq \left(A_{h+1}-\frac{\epsilon}{2}\right)\frac{hx^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}+o\left(\frac{x^{1/h}(\log\log x)^{k-1}}{\log x}\right)\\
&\geq (A_{h+1}-\epsilon)\frac{x^{1/h}(\log\log x)^{k-1}}{(k-1)!\log x}. \qquad (78)
\end{aligned}
$$

Equations (68) and (78) give (77), since ϵ is arbitrarily small. The theorem is proved.

Let us consider the h-ful numbers with exactly t distinct primes in their prime factorization. If $h = 1$ we obtain the numbers with exactly t distinct primes in their prime factorization. The number of these numbers not exceeding x is (see the introduction) $E_{t,h}(x)$.

Theorem 3.3 *The following asymptotic formula holds*

$$E_{t,h}(x) \sim \frac{hx^{1/h}(\log\log x)^{t-1}}{(t-1)!\log x}. \qquad (79)$$

Proof. Let us consider the numbers whose prime factorization is of the form

$$p_1^h p_2^h \dots p_t^h,$$

where $t \geq 1$ and $h \geq 1$ are fixed and $p_1, p_2, \dots, p_t$ are different primes.

Let $B_{t,h}(x)$ be the number of these numbers not exceeding x. We have the following asymptotic formula

$$B_{t,h}(x) \sim \frac{hx^{1/h}(\log\log x)^{t-1}}{(t-1)!\log x}. \qquad (80)$$

The proof of this formula is an immediate consequence of Lemma 1.2 and Lemma 1.3.

We have (see (80), (67) and (4))

$$
\begin{aligned}
E_{t,h}(x) &= B_{t,h}(x) + A_{t-1,h,1}(x) + A_{t-2,h,2}(x) + \cdots + A_{1,h,t-1}(x) + E_{t,h+1}(x)\\
&\sim B_{t,h}(x) \sim \frac{hx^{1/h}(\log\log x)^{t-1}}{(t-1)!\log x}.
\end{aligned}
$$

The theorem is proved.

If $h = 1$ then we obtain as corollary of Theorem 3.3 the following well-known Landau's result.

Corollary 3.4 *The following asymptotic formula holds*

$$E_{t,1}(x) \sim \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}.$$

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