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Results on Planar Γ – Near – Rings

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Abstract

In this paper we introduce some results for planar Γ- near-rings and planar Γ- near-fields. Through new definition of Γ-near -fields we prove a condition when a planar Γ-near-ring is planar Γ- near field.

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1 . Introduction

Planarity is introduced in algebra by Marshall Hall in his prominent coordinasation of a projective plane by planar ternary rings [4]. At [6] J. L. Zemmer defined planar near-field as a near-ring in which the equation $ax = bx + c$ is a unique solution for every $a \neq b$. Michel Anshel and James R. Clay defined in [1] planar near-rings, which as expected, have geometric interpretations. Here we will give concepts and we will present same auxiliary propositions, which we will use further in the presentation of the main results of the proceeding. Let consider M and Γ as two non empty sets.

Let's consider M and Γ as two non-empty sets. Every map of M x Γ x M in M is called Γ-multiplication in M and is denoted as $(\cdot)_\Gamma$. The result of this multiplication for elements a, b $\in$ M and $\gamma \in \Gamma$ is denoted

$a\gamma b$. According to Satyanarayana [2], Γ- near-ring is a classified ordered triple (M, +, $(\cdot)_\Gamma$) where M and Γ are non empty sets, + is an addition in M, while $(\cdot)_\Gamma$ is Γ - multiplication on M satisfying the following conditions:

(i) *(M, +) is a group.(not necessarily abelian)*
(ii) $\forall (a, b, c, \alpha, \beta) \in M^3 x\Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$
(iii) $\forall (a, b, c, \alpha,) \in M^3 x\Gamma, (a + b)\alpha c = a\alpha c + b\alpha c.$

An e element of Γ-near-ring M is called identity element if for every a $\in$ M and every $\gamma \in \Gamma$ we have

$a\gamma e = e\gamma a = a.$

Let (N, +) be a near – ring. The relation $=_m$ such that a $=_m$ b if only if $ax = bx$ for all $x\in N$ is an equivalence relation and is called the relation of equal multipliers in N [3].

Definition 1.1 [3] *Triple (N, +, ⊔) is planar near - ring when :*

(i) The relation of equal multipliers $=_m$ *has at least three equivalence classes* $|N/=_m| \geq 3$

(ii) $\forall (a, b, c,) \in N^3$ $a \neq_m b$ *the equation* $ax = bx + c$ *has unique solution for* $x \in N$.

Let M be a Γ-near-ring. We define in the M the each relation $=_m$ such that $a =_m b$ then and only then when $a\gamma x = b\gamma x$, for each $x \in M$ and for each $\gamma \in \Gamma$. Obviously, the relation $=_m$ is an equivalence relation.

If $a =_m b$ we will say that a and b are equal multipliers. *The relation* $\boldsymbol{=_m}$ *we going to call it relation of the equal multipliers.*

Definition 1.2[5] *Γ- near-rings (M, +, $(\cdot)_\Gamma$) is called planar if:*

(i) Relation of the equal multipliers $=_m$ has at least three equivalence classes, meaning $|M/=_m| \geq 3$.

(ii) For any three elements a, b, c of M such that $a \neq_m b$ and for each $\gamma \in \Gamma$,

$x\gamma a = x\gamma b + c$ has a unique solution in M .

Example 1.3

Let there be (M_n (P), +) the group of matrices of the order n with elements from a field P, with has at least three elements. For any subset Γ of nonsingular matrices set of order n with elements from the field P we define in M_n (P), Γ- multiplication $(\cdot)_\Gamma$ such that for any two matrices A, B of $M_n(P)$ and for each matrix $\gamma \in \Gamma$, we have

$$A\gamma B = A\,|\gamma|\,|B|,$$

where $|\gamma|$, $|B|$ are respectively the determinants of matrices γ, B. It is easy to be convinced that (M_n (P) +, $(\cdot)_\Gamma$) is Γ-near-ring. In this Γ-near-ring we have $A =_m B$ if and only if for any $X \in M_n(P)$ $\;X\,|\gamma|\,|A| = X\,|\gamma|\,|B|$.

Taking a nonsingular matrix X, we have $|A| = |B|$. Since P has at least three elements we have at least three matrices that does not have the same determinants, therefore $|M_n(P)/=_m| \geq 3$. If $A \neq_m B$, meaning $|A| \neq |B|$, matrix equation $X|\gamma|\,|A| = X|\gamma|\,|B| + C$

has the unique solution the matrix $X = \dfrac{1}{|\gamma|\left(|A|-|B|\right)}C$.

Therefore, Γ-near-ring $(M_n(P), +, (\cdot)_\Gamma)$ is planar.

For a planar Γ-near-ring $(M, +, (\cdot)_\Gamma)$ note $A = \{a \in M \mid a =_m 0\}$

To make it easy write the set $M \backslash A = M^\circ$. For each $a \in M^\circ$ and for each $\gamma \in \Gamma$, the equation

$x\gamma a = x\gamma 0 + a = a$ there is a unique solution which is noted 1_a^γ. It can happen that $1_a^\gamma = 1_b^\gamma$ although $a \neq b$. For a $a \in M^\circ$ and for each $\gamma \in \Gamma$ note

$$B_a^\gamma = \{b \in M^\circ \mid 1_a^\gamma \gamma b = b\}.$$

Theorem 1.4[5] *For any planar Γ-near-ring $(M, +, (\cdot)_\Gamma)$ the following statements are true:*

(i) $M = A \cup \left[\bigcup_{(a,\gamma)\in M^\circ\times\Gamma} B_a^\gamma\right]$

(ii) $B_a^\gamma \gamma M^\circ = B_a^\gamma$ *for each* $a \in M^\circ$ *and for each* $\gamma \in M^\circ$.

Definition 1.5[4] *A Γ-near-ring M is called Γ-near -field if for every $\alpha \in \Gamma$, the near-ring*

$M_\alpha = (M, +, (\alpha))$ *is near-field.*

Let there be $(M, +, (\cdot)_\Gamma)$ a planar Γ-near-ring. If for every $\gamma \in \Gamma$, near-ring $(M, +, \gamma)$ is unitary, then $(M, +, (\cdot)_\Gamma)$ would be called *unitary planar Γ-near-ring*. If for each $\gamma \in \Gamma$, near-ring $(M, +, \gamma)$ is near-field, then $(M, +, (\cdot)_\Gamma)$ will be called *planar Γ-near-field*.[5]

Proposition 1.6 [5] *If (M, +, $(\cdot)_\Gamma$) is a unitary planar Γ-near-ring, then it is a planar Γ-near-field.*

2. Results on planar Γ– near –rings

Theorem 2.1 *Let be (M, +, $(\cdot)_\Gamma$) a planar Γ-near-ring:*

For each $\gamma \in \Gamma$ and for each $a \in M^\circ$, B_a^γ is closed with respect to binary operation $\circ_\gamma$ of M for which $x \circ_\gamma y = x\gamma y$ and is a group in relation with the operation induced in it.

Proof .For $a \in M^\circ$ we have $1_a^\gamma \gamma a = a = 1_a^\gamma (1_a^\gamma \gamma a) = (1_a^\gamma \gamma 1_a^\gamma)\gamma a$. The equation $x\gamma a = a$ there is a unique solution so we have $1_a^\gamma \gamma 1_a^\gamma = 1_a^\gamma$. Thus, $1_a^\gamma \in B_a^\gamma$ and consequently 1_a^γ in the left identity of subgroup

$(B_a^\gamma, \circ_\gamma)$.

Assume that a' is the unique solution to the equation $x\gamma a = 1_a^\gamma$, meaning $a'\gamma a = 1_a^\gamma$. Then

$(1_a^\gamma \gamma a')\gamma a = 1_a^\gamma \gamma(a'\gamma a) = 1_a^\gamma \gamma 1_a^\gamma = 1_a^\gamma$.

From the iniquity of the solution of the equation $x\gamma a = 1_a^\gamma$ we have $1_a^\gamma \gamma a' = a'$ so a′ $\in B_a^\gamma$ from *Theorem1.4*

Let there be b an element of the subgroup $(B_a^\gamma, \circ_\gamma)$. For an element b there is an element

$b' \in B_b^\gamma$ such that $b'\gamma b = 1_b^\gamma$.

Now we will prove that $B_a^\gamma = B_b^\gamma$. Let c be an element of $B_a^\gamma \cap B_b^\gamma$, which is not empty because

$b \in B_a^\gamma \cap B_b^\gamma$. Then $1_a^\gamma \gamma c = c = 1_b^\gamma \gamma c$. Thus, 1_a^γ and 1_b^γ are the solutions of the equation $x\gamma c = c$ and hence from the iniquity of the solution of this equation we have $1_a^\gamma = 1_b^\gamma$. From this equalities and definitions of sets B_a^γ, B_b^γ we obtain the equality $B_a^\gamma = B_b^\gamma$. So, for every $b \in B_a^\gamma$ we have $b'\gamma b = 1_a^\gamma = 1_b^\gamma$, which shows that each element b of the subgroup (B_a^γ, $\circ_\gamma$) has a left inverse with respect to his left identity 1_a^γ. Thus, subgroup (B_a^γ, $\circ_\gamma$) is a group.

Proposition 2.2 *For each two elements a, c of* M° *the map* $\varphi : B_a^\gamma \to B_c^\gamma$ *such that* $\varphi(x) = 1_c^\gamma \gamma x$ *is an isomorphism.*

Proof. By Theorem 1.4 (ii) the definition of the map $\varphi : \underset{x \mapsto 1_c^\gamma \gamma x}{B_a^\gamma \to B_c^\gamma}$ is correct because $1_c^\gamma \gamma x$ is the element of the group B_c^γ. For any two elements x, y of B_a^γ the equalities are true:

$$\varphi(x \circ_\gamma y) = 1_c^\gamma \gamma(x \circ_\gamma y) = 1_c^\gamma \gamma(x\gamma y) = (1_c^\gamma \gamma x)\gamma y = [(1_c^\gamma \gamma x)\gamma\ 1_c^\gamma]\gamma y = (1_c^\gamma \gamma x)\gamma(1_c^\gamma \gamma y) = \varphi(x) \circ_\gamma \varphi(y),$$

which implies that the map φ is a homomorphism of the group (B_a^γ, $\circ_\gamma$) in group (B_c^γ, $\circ_\gamma$).

For every $b \in B_c^\gamma$ we have $1_c^\gamma \gamma b = b$. because of this equality we have:

$(1_a^\gamma \gamma 1_c^\gamma) = 1_a^\gamma \gamma(1_c^\gamma \gamma b) = 1_a^\gamma \gamma b$, obtaining that the elements $1_a^\gamma \gamma 1_c^\gamma$, 1_a^γ are solutions of equation $x\gamma b = 1_a^\gamma \gamma(1_c^\gamma \gamma b)$, which has a unique solution. Thus, we have $1_a^\gamma \gamma 1_c^\gamma = 1_a^\gamma$. In the same way we can prove even the equality $1_c^\gamma \gamma 1_a^\gamma = 1_c^\gamma$.

If $1_c^\gamma \gamma x = 1_c^\gamma\ \gamma y$ for every two elements x, y of B_a^γ , then the equalities are true:

$$1_a^\gamma\ \gamma(1_c^\gamma \gamma x) = (1_a^\gamma \gamma 1_c^\gamma)\gamma x = 1_a^\gamma \gamma x = x = 1_a^\gamma\ \gamma(1_c^\gamma \gamma x) = 1_a^\gamma\ \gamma(1_c^\gamma \gamma y) = (1_a^\gamma \gamma 1_c^\gamma)\gamma y = 1_a^\gamma \gamma y = y,$$

which proved that the homomorphism φ is monomorphism. For each element $y \in B_c^\gamma$ element $1_a^\gamma \gamma y \in B_a^\gamma$ Since the equalities are true:

$$\varphi(1_a^\gamma \gamma y) = 1_c^\gamma \gamma(1_a^\gamma \gamma y) = (1_c^\gamma \gamma 1_a^\gamma)\gamma y = 1_a^\gamma \gamma y = y,$$

monomorphism φ is epimorphism and consequently φ is isomorphism of group $(B_a^\gamma, \circ_\gamma)$ in group

$(B_c^\gamma, \circ_\gamma)$.

Proposition 2.3 *For each $a \in M^\circ$ and for each element $\gamma \in \Gamma$, 1_a^γ is a γ- right*

identity element, ($\forall d \in M$, $d\gamma 1_a^\gamma = d$).

Proof. For each element $d \in M$ we take the equation $x\gamma 1_a^\gamma = d\gamma 1_a^\gamma$.

This equation has solution an element d and in the same way the element $d\gamma 1_a^\gamma$ is its solution because the equalities are true: $(d\gamma 1_a^\gamma)\gamma\ 1_a^\gamma = d\gamma(\ 1_a^\gamma \gamma 1_c^\gamma) = d\gamma\ 1_a^\gamma$.

From the iniquity of the solution of the equation, obtain that for every $d \in M$ we have

$d\gamma\ 1_a^\gamma = d$, meaning the element $d \in M$ is γ-right identity.

Proposition 2.4 *If $(M, +, (\cdot)_\Gamma)$ is a planar Γ-near-ring such that*

$$\forall (a, b) \in M^2,\ a =_m b \Leftrightarrow a = b \text{ and } \forall (\gamma, a, b) \in \Gamma \times (M^\circ)^2,\ 1_a^\gamma = 1_b^\gamma,$$

$(M, +, (\cdot)_{\Gamma})$ is planar Γ-near-field.

Proof. For three elements a, b, c of M, where $a \neq b$ and for each $\gamma \in \Gamma$ equation

$a\gamma x = b\gamma x + c$ has only one solution. If $a \neq 0$, then $a \neq_m 0$ and consequently $A = \{0\}$. Since for each $\gamma \in \Gamma$ $(a, b) \in (M^{\circ})^2$, $1_a^{\gamma} = 1_b^{\gamma}$, for any two elements a, b different from zero of M, we have $B_a^{\gamma} = B_b^{\gamma} = M^{\circ}$. Thus, $M^{\circ} = M \setminus \{0\} = M^* = B_a^{\gamma}$ and therefore M° therefore is closed with respect binary operation $\circ_{\gamma}$ of M for which $x \circ_{\gamma} y = x\gamma y$ and forms the group in relation with the operation inducted in it. Therefore, for each $\gamma \in \Gamma$ near-ring $(M, +, \circ_{\gamma})$ is planar by definition 1.5 meaning $(M, +, (\cdot)_{\Gamma})$ is planar Γ-near-field by Proposition 1.6

If Γ-near-ring $(M, +, (\cdot)_{\Gamma})$ has no zero divisor, then it would call that *Γ-near-ring is integral domain*. If these Γ-near-ring $(M, +, (\cdot)_{\Gamma})$ is planar, then we would call it that *planar Γ-near-ring is integral domain.*

By the Proposition 2.4 we obtain:

Corollary 2.8 *If $(M, +, (\cdot)_{\Gamma})$ planar Γ-near-ring is integral domain such as*

$$\forall (a, b) \in M^2,\ a =_m b \Leftrightarrow a = b, \text{ then it is a planar } \Gamma\text{-near-field.}$$

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