International Mathematical Forum, Vol. 9, 2014, no. 22, 1075 - 1088
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.4595

Topologies Induced by (3,j,ρ)-metrics, j∈{1,2}

Sonja Čalamani

Faculty of technical sciences, University "St. Kliment Ohridski"
Bitola, Republic of Macedonia

Dončo Dimovski

"Ss. Cyril and Methodius University", Skopje, Republic of Macedonia

Abstract

For a given (3,j,ρ)-metric d on a set M, j∈{1,2}, we define seven topologies on M induced by d, give some examples, examine the connection among them, and show that some other topologies induced by generalized metrics are special cases of the above ones.

Mathematics Subject Classification: 54A10, 54E35, 54E99

Keywords: metric spaces, (3,j,ρ)-metric spaces, metrizable topological spaces, (3,j,ρ)-metrizable topological spaces

1 Introduction

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered in a lot of papers. We mention some of them: K. Menger ([13]), V. Nemytzki, P.S. Aleksandrov ([1],[15]), Z. Mamuzic ([12]), S. Gähler ([8], [9]), A.V. Arhangelskii, M. Choban, S. Nedev ([2],[4],[16], [17]), R. Kopperman ([11]), J. Usan ([18]), B.C. Dhage, Z. Mustafa, B. Sims ([5],[14]). In [6] we have introduced the notion of an (n,m,ρ)-metric, and in [7] we have examined the connection between some of the topologies induced by a (3,1,ρ)-metric d and the topologies induced by a pseudo-o-metric, o-metric and symmetric as in [17].

For a given (3,j,ρ)-metric d on a set M, we define seven topologies on M induced by d: τ(G,d), τ(H,d), τ(D,d), τ(N,d), τ(W,d), τ(S,d) and τ(K,d). For these topologies we have the following inclusions: τ(W,d)⊆τ(N,d)⊆τ(D,d)⊆τ(G,d), τ(W,d)⊆τ(H,d)⊆τ(G,d), and τ(W,d)⊆τ(S,d)⊆τ(K,d). For a (3,2,ρ)-metric d, these inclusions are, τ(W,d)⊆τ(N,d)=τ(S,d)=τ(K,d)⊆τ(D,d)⊆τ(G,d).

We will illustrate, by examples, that in general, these inclusions are strict.

We show that some of the generalized metrics and their induced topologies considered in, [8], [17], [12], [5] and [14], are special cases of some of the above mentioned metrics and topologies.

Next we prove some properties, such as: a) A topological space (M,τ) is first countable iff there is a (3,1,ρ)-metric d on M such that τ(N,d)=τ(D,d)=τ; b) If τ is the cofinite topology on M, then there is a (3,1,ρ)-metric d on M such that τ(D,d)=τ(G,d)=τ; and c) A space (M,τ) is metrizable iff there is a (3,2)-metric d on M such that τ(D,d)=τ(G,d)=τ.

At the end, we consider several classes of topological spaces "metrizable" via some of the above topologies, show same relations among them and state several questions.

2 (3,j,ρ)-metrics, j∈{1,2}

The notion of an n-metric was considered by K. Menger in [13], and later S. Gähler examined this notion and the induced topologies in series of papers (two of them are [8], [9]). A generalization of this notion, i.e. the notion of an (n,m,ρ)-metric, n>m, was introduced in [6]. Here we will work only with (3,1,ρ)-metrics and (3,2,ρ)-metrics, i.e. with the case n=3, for m=1 and m=2.

Let M be a nonempty set. We denote by $M^{(3)}$ the symmetric third power of M, i.e. $M^{(3)}$ is the factor set M^3/α, where α is the equivalence relation on M^3 defined by:

(x,y,z)α(u,v,w) if and only if (u,v,w) is a permutation of (x,y,z).

Let d: $M^{(3)} \to R_0^+ = [0, \infty)$. We state four conditions for such a map.

(M0) d(x,x,x)=0, for any x∈M;

(M1) d(x,y,z) ≤ d(x,y,a) + d(x,a,z) + d(a,y,z), for any x,y,z,aM;

(M2) d(x,y,z) ≤ d(x,a,b) + d(a,y,b) + d(a,b,z), for any x,y,z,a,b∈M; and

(Ms) d(x,x,y) = d(x,y,y), for any x,y∈M.

For a map d as above we consider a subset ρ_d of $M^{(3)}$ defined by:

$$\rho_d = \{(x,y,z) \mid (x,y,z)\in M^{(3)},\ d(x,y,z)=0\} .$$

If d satisfies (M0) and (M1), then $\rho=\rho_d$ satisfies:

(E0) (x,x,x)∈ρ, for any x∈M; and

(E1) (x,y,a), (x,a,z), (a,y,z)∈ρ implies (x,y,z)∈ρ, for any x,y,z,a∈M.

If d satisfies (M0) and (M2), then $\rho=\rho_d$ satisfies (E0) and

(E2) (x,a,b), (a,y,b), (a,b,z)∈ρ implies (x,y,z)∈ρ, for any x,y,z,a,b∈M.

Definition 2.1. A subset ρ of $M^{(3)}$ satisfying: (E0) and (E1), is called a **(3,1)-equivalence**: satisfying (E0) and (E2) is called a **(3,2)-equivalence**; and satisfying (E0), (E1) and (E2) is called a **3-equivalence** on M.

Examples 2.1. a) The set $\Delta=\{(x,x,x) \mid x\in M\}$ (thin diagonal of $M^{(3)}$) is a 3-equivalence and is a subset of any (3,j)-equivalence, for $j\in\{1,2\}$.

b) Let $\nabla=\{(x,x,y) \mid x,y\in M\}$ (thick diagonal of $M^{(3)}$). The set ∇ is a (3,1)-equivalence, but is not a (3,2)-equivalence, and if ∇ is a subset of a (3,2)-equivalence ρ, then $\rho=M^{(3)}$.

c) Another, very often used example of a (3,1)-equivalence that is not a (3,2)-equivalence is $Coll\subseteq E^{(3)}$, where E is the Euclidean plane, and $(A,B,C)\in Coll$ if and only if the points A,B,C are collinear, i.e. A,B,C are points on a line. Moreover, $\nabla\subseteq Coll$.

Definition 2.2. Let d: $M^{(3)}\to R_0^+$ and $\rho=\rho_d$ be as above.

If d satisfies **(M0)** and **(Mj)**, $j\in\{1,2\}$, we say that d is a **(3,j,ρ)-metric** on M, and if d satisfies **(M0)**, **(M1)** and **(M2)**, we say that d is a **(3,ρ)-metric** on M.

If d is a **(3,j,ρ)-metric** and satisfies (Ms), we say that d is a **(3,j,ρ)-symmetric** on M, and if d is a **(3,ρ)-metric** and satisfies (Ms), we say that d is a **(3,ρ)-symmetric** on M.

If d is a (3,j,Δ)-metric on M, we say that d is a **(3,j)-metric** on M, and if d is a (3,Δ)-metric on M, we say that d is a **3-metric** on M.

If d is a (3,j,Δ)-symmetric on M, we say that d is a **(3,j)-symmetric** on M, and if d is a (3,Δ)-symmetric on M, we say that d is a **3-symmetric** on M.

Remark 2.1. The notion of (n,m,ρ)-metrics, for the case n=2 allows only the possibility m=1. A (2,1,ρ)-metric d on M, for an equivalence ρ on M, is the well known notion of a pseudometric on M i.e. a map d: $M^2\to R_0^+$, satisfying:
a) $d(x,y)=0$ if and only if $(x,y)\in\rho$; **b)** $d(x,y)=d(y,x)$; and **c)** $d(x,y)\le d(x,u)+d(u,y)$.

If ρ is the diagonal $\Delta=\{(x,x) \mid x\in M\}$, then the notion of (2,1,Δ)-metric on M is the notion of a metric on M.

According to this, we could say: **3-pseudometric; 3-pseudosymmetric; (3,j)-pseudometric;** and **(3,j)-pseudosymmetric,** for: (3,ρ)-metric; (3,ρ)-symmetric; (3,j,ρ)-metric; and (3,j,ρ)-symmetric, respectively.

Examples 2.2. In a) and b) let M be a nonempty set.

a) The map d: $M^{(3)}\to R_0^+$ defined by $d(x,x,x)=0$ and $d(x,y,z)=1$ otherwise, is a 3-symmetric, called **discrete 3-metric**.

b) Let D: $M^2\to R_0^+$ be a pseudometric on M, let $\alpha=\{(x,y) \mid D(x,y)=0\}$, let k be a positive real number, and let $\rho=\{(x,y,z) \mid$ x,y,z are in the same equivalence class of the equivalence $\alpha\}$.

b.1) The map d_k: $M^{(3)}\to R_0^+$ defined by:

$$d_k(x,y,z) = k\cdot(D(x,y)+D(x,z)+D(y,z))$$

is a (3,ρ)-symmetric, and $d_k(x,x,y)=D(x,y)$. When D is a metric, d_k is a 3-symmetric. For k=1/2, we denote d_k by d_D.

b.2) The map d_{kmax}: $M^{(3)}\rightarrow R_0^+$ defined by:

$$d_{kmax}(x,y,z) = k\cdot\max\{D(x,y),D(x,z),D(y,z)\}$$

is a (3,ρ)-symmetric, and $d_{kmax}(x,x,y)=D(x,y)$. When D is a metric, d_k is a 3-symmetric.

c) Let E be the Euclidean plane, and let d_P: $E^{(3)}\rightarrow R_0^+$ be defined by:

$$d_P(A,B,C) = \text{Perimeter of the "triangle" } ABC.$$

Since Perimeter of the triangle ABC is the sum of the lengths of the segments AB, AC and BC, b.1) implies that d_P is a 3-symmetric on E.

d) Let E be the Euclidean plane, and let d_A: $E^{(3)}\rightarrow R_0^+$ be defined by:

$$d_A(A,B,C) = \text{Area of the "triangle" } ABC.$$

It is easy to check that d_A is a (3,1,Coll)-symmetric, but is not a (3,2,Coll)-symmetric on E (see Example 2.1. c) for Coll).

e) Let $\rho\subseteq M^{(3)}$ be a (3,j)-equivalence on M, $j\in\{1,2\}$. Then, it is easy to check that $d:M^{(3)}\rightarrow R_0^+$ defined by $d(x,y,z)=0$ if $(x,y,z)\in\rho$ and $d(x,y,z)=1$ if $(x,y,z)\notin\rho$ is a (3,j,ρ)-metric on M.

Proposition 2.1. Let d be (3,j,ρ)-metric on M, $j\in\{1,2\}$. Let $d':M^{(3)}\rightarrow R_0^+$ be defined by:

$$d'(x,y,z) = d(x,y,z)/(1+d(x,y,z)).$$

Then, d' is also a (3,j,ρ)-metric on M, and $d'(x,y,z)<1$ for any $x,y,z\in M$. Moreover, if d is a (3,j,ρ)-symmetric, then d' is also a (3,j,ρ)-symmetric.

Proof. Extending the well known fact that $c/(1+c) \leq a/(1+a) + b/(1+b)$ for any $a,b,c\in R_0^+$ with $c\leq a+b$, to the fact that $c/(1+c) \leq a/(1+a) + b/(1+b) + e/(1+e)$ for any $a,b,e,c\in R_0^+$ with $c\leq a+b+e$, we obtain that d' satisfies the axiom (Mj). It is obvious that $d'(x,y,z)=0$ iff $d(x,y,z)=0$. This shows that d' is a (3,j,ρ)-metric on M. The last part follows directly from the definition of d'. □

Remark 2.2. Any (3,j,ρ)-metric d on M induces a map $D_d:M^2\rightarrow R_0^+$ defined by: $D_d(x,y)=d(x,x,y)$.

It is easy to check the following facts.

a) For any (3,j,ρ)-metric d, $D_d(x,x)=0$. (D_d is called a **distance** in [12] and a **pseudo o-metric** in [17].)

b) For any (3,j)-metric d, $D_d(x,y)=0$ if and only if x=y. (D_d is called an **o-metric** in [17].)

c) For any (3,j)-symmetric d, $D_d(x,y)=D_d(y,x)$. (D_d is called a **symmetric** in [17].)

d) For a (3,2,ρ)-metric d, $D_d(x,y)\leq 2D_d(z,x)+D_d(z,y)$ and $D_d(x,y)\leq 2D_d(y,x)$.

e) For any (3,2)-symmetric d, $D_d(x,y)=D_d(y,x)\leq 3(D_d(x,z)+D_d(z,y))/2$. (In the literature D_d is called a **quasimetric**, a **nearmetric** or an **inframetric**.)

Remark 2.3. a) The notion of a 2-metric considered in [13] is the same as the notion of a (3,1,ρ)-metric with the additional requirement: $\nabla\subseteq\rho$.

b) The notion of a 2-metric considered in [8] is the same as the notion of a (3,1,ρ)-metric with the additional requirements: $\nabla\subseteq\rho$ and for any $x\neq y$ from M, there is a $z\in M$ such that $d(x,y,z)\neq 0$.

c) The notion of a D-metric considered in [5] is the same as the notion of a (3,1)-metric.

d) The notion of a G-metric considered in [14] is the same as the notion of a 3-metric d with the additional requirements:

$d(x,x,y) \leq d(x,y,z)$ for any $x,y,z \in M$ with $x \neq y$; and

$d(x,y,z) \leq d(x,a,a)+d(a,y,z)$ for any $x,y,z,a \in M$.

Moreover, for a G-metric d, the map $D:M^2 \to R_0^+$ defined by $D(x,y)=d(x,x,y)+d(x,y,y)$ is a metric on M. For a G-metric d, that is also a 3-symmetric, the map $D_d:M^2 \to R_0^+$ defined by $D_d(x,y)=d(x,x,y)$ is a metric on M.

3 Topologies induced by (3,j,ρ)-metrics, j∈{1,2}

Definition 3.1. Let d be a (3,j,ρ)-metric on M, $j \in \{1,2\}$, let $x,y \in M$ and let $\varepsilon>0$. We define the following ε-balls, as subsets of M:

a) $B(x,y,\varepsilon)=\{z \mid z \in M, d(x,y,z)<\varepsilon\}$–ε-ball with center at (x,y) and radius ε;

b) $L(x,\varepsilon)=\{z \mid z \in M, d(x,z,z)<\varepsilon\}$–"litle" ε-ball with center at x and radius ε; and

c) $B(x,\varepsilon)=\{z \mid z \in M$, there is a $v \in M$ such that $d(x,z,v)<\varepsilon\}$– ε-ball with center at x and radius ε.

Remark 3.1. a) For x=y, $B(x,x,\varepsilon)=B(x,y,\varepsilon)=\{z \mid z \in M, d(x,x,z)<\varepsilon\}$. For any $a \in M$, $a \in B(a,\varepsilon)$, $a \in L(a,\varepsilon)$ and $a \in B(a,a,\varepsilon)$, but, it is possible for some $x \neq a$ to have $a \notin B(a,x,\varepsilon)$.

b) For a pseudo o-metric $D:M^2 \to R_0^+$, there is only one possibility for defining ε-balls, i.e. $B(x,\varepsilon)=\{z \mid z \in M, D(x,z)<\varepsilon\}$.

c) For a (3,j,ρ)-metric d there is one more possibility to define an ε-ball, as a subset of $M^{(2)}$, i.e. $K(x,\varepsilon)=\{(u,v) \mid (u,v) \in M^{(2)}, d(x,u,v)<\varepsilon\}$, but here we are not going to discuss these ε-balls. For general (n,m,ρ)-metrics, the number of possible ε-balls is much bigger and depends on n.

Proposition 3.1. For any (3,j,ρ)-metric d on M, $j \in \{1,2\}$, and any $a \in M$:

a) $L(a,\varepsilon) \subseteq B(a,\varepsilon)$, $B(a,a,\varepsilon) \subseteq B(a,\varepsilon)$ and $B(a,x,\varepsilon) \subseteq B(a,\varepsilon)$ for any $x \in M$;

b) $B(a,\varepsilon)=\cup\{B(a,x,\varepsilon) \mid x \in B(a,\varepsilon)\}$.

Proof. The inclusions in **a)** follow directly from the definitions. For **b)**, the inclusion $\cup\{B(a,x,\varepsilon) \mid x \in B(a,\varepsilon)\} \subseteq B(a,\varepsilon)$ follows from **a)**. If $y \in B(a,\varepsilon)$, there is $x \in M$ such that $d(a,y,x)<\varepsilon$. This implies that $x \in B(a,\varepsilon)$ and $y \in B(a,x,\varepsilon)$. So $y \in \cup\{B(a,x,\varepsilon) \mid x \in B(a,\varepsilon)\}$ and $B(a,\varepsilon) \subseteq \{B(a,x,\varepsilon) \mid x \in B(a,\varepsilon)\}$. □

Definition 3.2. Let d be a (3,j,ρ)-metric on M, $j \in \{1,2\}$. We define seven topologies on M induced by d as follows:

1) τ(G,d) – the topology generated by all the ε-balls $B(x,y,\varepsilon)$, i.e. the topology whose base is the set of the finite intersections of all the ε-balls $B(x,y,\varepsilon)$;

2) τ(H,d) – the topology generated by all the ε-balls $B(x,\varepsilon)$;

3) τ(D,d) – the topology generated by all the ε-balls $B(x,x,\varepsilon)$;

4) $\tau(K,d)$ – the topology generated by all the ε-balls $L(x,\varepsilon)$;

5) $\tau(N,d)$ – the topology defined by: $U\in\tau(N,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$ such that $B(x,x,\varepsilon)\subseteq U$;

6) $\tau(S,d)$ – the topology defined by: $U\in\tau(S,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$ such that $L(x,\varepsilon)\subseteq U$;

7) $\tau(W,d)$ – the topology defined by: $U\in\tau(W,d)$ iff $\forall x\in U$, $\exists\varepsilon>0$ such that $B(x,\varepsilon)\subseteq U$.

Proposition 3.2. For any $(3,j,\rho)$-metric d on M, $j\in\{1,2\}$:

a) $\tau(W,d) \subseteq \tau(N,d) \subseteq \tau(D,d) \subseteq \tau(G,d)$;

b) $\tau(W,d) \subseteq \tau(H,d) \subseteq \tau(G,d)$; and

c) $\tau(W,d) \subseteq \tau(S,d) \subseteq \tau(K,d)$.

Proof. If $V\in\tau(N,d)$ then $V=\cup\{B(x,x,\varepsilon_x) \mid x\in V\}$, the ε_x are provided by the definition. This implies $\tau(N,d)\subseteq\tau(D,d)$. Similarly $\tau(S,d)\subseteq\tau(K,d)$ and $\tau(W,d)\subseteq\tau(H,d)$. Since each $B(x,x,\varepsilon)$ is also a $B(x,y,\varepsilon)$ for $x=y$, it follows that $\tau(D,d)\subseteq\tau(G,d)$. The definition and Poposition 3.1.a) imply that $\tau(W,d)\subseteq\tau(N,d)$ and $\tau(W,d)\subseteq\tau(S,d)$. The definition and Proposition 3.1.b) imply that $\tau(H,d)\subseteq\tau(G,d)$. □

Proposition 3.3. If d is a $(3,j,\rho)$-symmetric, $j\in\{1,2\}$, then $\tau(N,d)=\tau(S,d)$ and $\tau(D,d)=\tau(K,d)$.

Proof. Follows directly from the definitions. □

Proposition 3.4. For any $(3,2,\rho)$-metric d on M:

$$\tau(W,d) \subseteq \tau(N,d) = \tau(S,d) = \tau(K,d) \subseteq \tau(D,d) \subseteq \tau(G,d).$$

Proof. Using Proposition 3.2., it is enough to show that $\tau(N,d)=\tau(S,d)$ and $\tau(K,d)\subseteq\tau(S,d)$.

For a $(3,2,\rho)$-metric d on M, for any $x,y\in M$ and any $\varepsilon>0$, $d(x,x,y)\le 2d(x,y,y)$ and $d(x,y,y)\le 2d(x,x,y)$. This implies that $B(x,x,\varepsilon)\subseteq L(x,2\varepsilon)$ and $L(x,\varepsilon)\subseteq B(x,x,2\varepsilon)$. These inclusions, together with the definitions imply that $\tau(N,d)=\tau(S,d)$.

Next, let $x\in M$, $\varepsilon>0$ and let $y\in L(x,\varepsilon)$. Let $2\delta=\varepsilon-d(x,y,y)$. For any $z\in B(y,y,\delta)$, we have that $d(x,z,z)\le d(x,y,y)+2d(z,y,y)<d(x,y,y)+2\delta=\varepsilon$. So, $z\in B(y,y,\delta)\subseteq L(x,\varepsilon)$. Hence, $L(x,\varepsilon)\in\tau(N,d)$, and since $\tau(N,d)=\tau(S,d)$, the definition of $\tau(K,d)$ implies that $\tau(K,d)\subseteq\tau(S,d)$. □

Corollary 2.1. For any $(3,2,\rho)$-symmetric d on M:

$$\tau(W,d) \subseteq \tau(N,d) = \tau(S,d) = \tau(K,d) = \tau(D,d) \subseteq \tau(G,d). \ \square$$

The following example shows that in general, the inclusions in Proposition 3.2. are strict.

Example 3.1. Let $M=\{a,b,c\}$ and let $\rho=\Delta\cup\{(a,a,b),(b,b,c)\}\subseteq M^{(3)}$. It can be checked that ρ is a (3,1)-equivalence on M, so by Examples 2.2.e), $d:M^{(3)}\to R_0^+$ defined by: $d(x,y,z)=0$ if $(x,y,z)\in\rho$ and $d(x,y,z)=1$ if $(x,y,z)\notin\rho$ is a $(3,1,\rho)$-metric on M. Note that d is not a $(3,2,\rho)$-metric on M. Directly from the definition of d, it follows that for $\varepsilon>1$, all the ε-balls are equal to M, and for any $0<\varepsilon\leq 1$, any ε-ball is equal to the ball with radius 1, i.e. the 1-ball. Using the facts that $d(x,y,z)=0$ only for (a,a,a), (b,b,b), (c,c,c), (a,a,b) and (b,b,c) the 1-balls are the following:

a) $B(a,1)=\{a,b\}$, $B(b,1)=\{a,b,c\}$ and $B(c,1)=\{b,c\}$;
b) $B(a,a,1)=\{a,b\}$, $B(b,b,1)=\{b,c\}$ and $B(c,c,1)=\{c\}$;
c) $B(a,b,1)=\{a\}$, $B(a,c,1)=\varnothing$ and $B(b,c,1)=\{b\}$; and
d) $L(a,1)=\{a\}$, $L(b,1)=\{a,b\}$ and $L(c,1)=\{b,c\}$.

The definition of the topologies induced by d implies that:

1) $\tau(G,d) = \mathfrak{D}$ – the discrete topology on M;
2) $\tau(H,d) = \{\varnothing, \{b\}, \{a,b\}, \{b,c\}, M\}$;
3) $\tau(D,d) = \{\varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}, M\}$;
4) $\tau(K,d) = \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{b,c\}, M\}$;
5) $\tau(N,d) = \{\varnothing, \{c\}, \{b,c\}, M\}$;
6) $\tau(S,d) = \{\varnothing, \{a\}, \{a,b\}, M\}$; and
7) $\tau(W,d) = \{\varnothing, M\} = \mathfrak{I}$ – the indiscrete topology on M

Example 3.2. Let d be the discrete 3-metric on M, (see Example 2.2.a)). Then all the seven topologies on M induced by d are equal to the discrete topology $\mathfrak{D}$, i.e. for any $X\in\{W,N,S,D,K,H,G\}$, $\tau(X,d) = \mathfrak{D}$.

Since we will often use similar expression as in Example 3.2., we use the following notation:

$$\mathfrak{M} = \{W,N,S,D,K,H,G\}.$$

Proposition 3.5. Let D be a pseudometric on M, let d_D be the (3,ρ)-symmetric, defined in Example 2.2.b.1), and let τ_D be the topology on M induced by the pseudometric D. Then for each $X\in\mathfrak{M}$, $\tau(X,d_D) = \tau_D$.

Proof. Denote by $T(x,\varepsilon)=\{y \mid D(x,y)<\varepsilon\}$ the open ball in τ_D with center x and radius ε. The definitions directly imply that $B(x,x,\varepsilon)=L(x,\varepsilon)=T(x,\varepsilon)$. So, $\tau(X,d_D)=\tau_D$ for any $X\in\mathfrak{M}\setminus\{W,G\}$.

Let $y\in B(x,\varepsilon)$ and $d_D(x,y,z)<\varepsilon$. Then, $2D(x,y)\leq D(x,y)+D(x,z)+D(y,z)= 2d_D(x,y,z)<2\varepsilon$, implies that $y\in T(x,\varepsilon)$. So, $B(x,\varepsilon)\subseteq T(x,\varepsilon)$. Conversely, let $y\in T(x,\varepsilon)$. Then, $d_D(x,x,y)=D(x,y)<\varepsilon$, implies that $y\in B(x,\varepsilon)$. So, $T(x,\varepsilon)=B(x,\varepsilon)$, and this implies that $\tau(W,d_D)=\tau_D$.

Next let $x,y\in M$, $x\neq y$ and let $\varepsilon>0$. For $\varepsilon\leq D(x,y)$, $B(x,y,\varepsilon)=\varnothing$. Let $D(x,y)<\varepsilon$. For $z\in B(x,y,\varepsilon)$, let $\delta_z = \varepsilon-d_D(x,y,z)$. Then, for $u\in T(z,\delta_z)$, we have:

$d_D(x,y,u)=(D(x,y)+D(x,u)+D(y,u))/2\leq(D(x,y)+D(x,z)+D(z,u)+D(y,z)+D(z,u))/2= d_D(x,y,z)+D(z,u)<d_D(x,y,z)+\delta_z = \varepsilon$.

We have shown that $u\in B(z,z,\varepsilon_z)=T(z,\varepsilon_z)\subseteq B(x,y,\varepsilon_z)$. This implies that $\tau(G,d_D)=\tau(D,d_D)=\tau_D$. □

Proposition 3.6. Let d be a (3,j,ρ)-metric on M, $j\in\{1,2\}$ and let d' be the (3,j,ρ)-metric on M defined in Proposition 2.1. Then, for any $X\in\mathfrak{M}$, $\tau(X,d)=\tau(X,d')$.

Proof. Denote the ε-balls for d' by B'(x,y,ε), B'(x,ε) and L'(x,ε). It is easy to check that:

a) for any ε>0, B(x,y,ε)⊆B'(x,y,δ), B(x,ε)⊆B'(x,δ) and L(x,ε)⊆B'(x,δ) for δ=ε/(1+ε); and

b) for any 1>ε>0, B'(x,y,ε)⊆B(x,y,δ), B'(x,ε)⊆B(x,δ) and L'(x,ε)⊆B(x,δ) for δ=ε/(1–ε).

The conclusion follows from the definitions of the induced topologies. □

Example 3.3. Let M=A∪B, for A and B disjoint equivalent subsets of M. Let f:A→B be a bijection, and let g:M→M be the bijection defined by: g(x)=f(x) for x∈A, and g(x)=f^{-1}(x) for x∈B. Let d:$M^{(3)} \to R_0^+$ be defined by:

a) d(x,x,g(x))=1=d(x,g(x),g(x)), for any x∈M;

b) d(x,g(x),y)=1/2, for any x∈M and any y∈M, y≠x; and

c) d(x,y,z)=0 for all (x,y,z) not defined by a) and b).

It is easy to check that d satisfies **(M0)**, **(M1)** and **(Ms)**, and so, it is a (3,1,ρ)-symmetric

As in Example 3.1, we consider only the ε-balls for ε=1 and ε=1/2. We have: B(x,1)=M; B(x,1/2)=M\{g(x)}; B(x,x,1)=M\{g(x)}=B(x,x,1/2)=L(x,1)= L(x,1/2); B(x,g(x),1)=M\{x,g(x)}; B(x,g(x),1/2)=∅; B(x,y,1)=M for y≠x,g(x); and B(x,y,1/2)=M\{g(x),g(y)} for y≠x,g(x). All this implies that: τ(W,d)=τ(N,d)= τ(S,d)=𝒮 and τ(D,d)=τ(K,d)=τ(K,d)=τ(G,d)=τ_{cof} is the cofinite topology on M.

Example 3.4. Let M=R and let d:$M^{(3)} \to R_0^+$ be defined by: d(x,x,x)=0; d(x,x,y)=1, for y<x; d(x,x,y)=y–x, for x<y; and d(x,y,z)=max{x,y,z}–min{x,y,z} + 3/2, for x≠y≠z≠x.

It can be checked that this d is a (3,1)-metric, that is not a (3,1)-symmetric.

The ε-balls are as follows: B(x,ε)=M for ε>3/2; B(x,ε)=(x–ε,x+ε)–the open interval in R for ε≤3/2; B(x,x,ε)=[x,x+ε), L(x,ε)=(x–ε,x]–half open intervals in R; and for x<y<x+ε, B(x,y,ε)={x}.

This implies that: τ(G,d)=𝒟; τ(W,d)=τ(H,d)=τ_D – the topology on R induced by the metric D(x,y) = |x – y|; τ(N,d)=τ(D,d)=τ_L – the topology on R generated by the half open intervals [a,b); τ(S,d)=τ(K,d)=τ_R – the topology on R generated by the half open intervals (a,b].

In the literature, the space (R,τ_L) (i.e. (R,τ_R)) is called Sorgenfrey line.

4. (3,j,ℬ)-metrizable spaces, j∈{1,2}, ℬ⊆𝔐={W,N,D,G,S,K,H}

Definition 4.1. Let ℬ⊆𝔐 and j∈{1,2}. We say that a topological space (M,τ) is:

1) **(3,j,ρ)-ℬ-metrizable;**	2) **(3,j,ρ)-ℬ-symmetrizable;**
3) **(3,j)-ℬ-metrizable;**	4) **(3,j)-ℬ-symmetrizable;**
5) **(3,ρ)-ℬ-metrizable;**	6) **(3,ρ)-ℬ-symmetrizable;**

7) **3-$\mathcal{B}$-metrizable;** 8) **3-$\mathcal{B}$-symmetrizable;**

if there is a: 1) $(3,j,\rho)$-metric; 2) $(3,j,\rho)$-symmetric; 3) $(3,j)$-metric; 4) $(3,j)$-symmetric; 5) $(3,\rho)$-metric; 6) $(3,\rho)$-symmetric; 7) 3-metric; and 8) 3-symmetric d; such that $\tau = \tau(X,d)$, for each $X\in\mathcal{B}$.

When $\mathcal{B}=\{X\}$ we write X instead of $\mathcal{B}$, and when $\mathcal{B}=\{X,Y\}$, we write X-Y instead of $\mathcal{B}$.

Examples 4.1. a) Proposition 3.5., implies that any metrizable space is 3-$\mathcal{M}$-symmetrizable and any pseudometrizable space is $(3,\rho)$-$\mathcal{M}$-symmetrzable.

b) Example 3.2., implies that any discrete space is 3-$\mathcal{M}$-symmetrizable.

c) In [7] the foolowing connections with notions from [17} are shown:

c.1) A space is $(3,1,\rho)$-N-metrizable iff it is pseudo-o-metrizable;

c.2) A space is $(3,1)$-N-metrizable iff it is o-metrizable; and

c.3) A space is $(3,1)$-N-symmetrizable iff it is simmetrizable.

d) A space is 2-metrizable as in [8] iff it is $(3,1,\rho)$-G-metrizable, with ρ satisfying the additional requirements as in Remark 2.3.b).

e) A space is D-metrizable as in [5] iff it is $(3,1)$-S-metrizable.

f) A space is G-metrizable as in [14] iff it is 3-S-symmetrizable.

Proposition 4.1. Any space (M,τ_{cof}) with the cofinite topology is $(3,1,\rho)$-$\mathcal{B}$-symmetrizable for $\mathcal{B}=\{D,K,H,G\}$.

Proof. If M is a finite set, then the space is discrete, so it is $(3,1,\rho)$-$\mathcal{B}$-symmetrizable, by Example 4.1.b). If M is infinite set, then it is possible to choose disjoint subsets A and B of M and a bijection $f:A\to B$, such that $M=A\cup B$. This can be done by well ordering M and choosing A, B and f using the order. The conclusion follows from Example 3.3. □

Proposition 4.2. If a space (M,τ) is $(3,j)$-D-metrizable, $j\in\{1,2\}$, then it is T_1. Moreover, if (M,τ) is $(3,2)$-N-D-metrizable, then (M,τ) is regular.

Proof. Let d be a $(3,j)$-metric, such that $\tau=\tau(D,d)$, let $x,y\in M$ and $x\neq y$. Since d is $(3,j)$-metric it follows that $d(x,x,y)\neq 0$ and $d(x,y,y)\neq 0$. For $\varepsilon=\min\{d(x,x,y), d(x,y,y)\}$, $x\notin B(y,y,\varepsilon)$ and $y\notin(B,x,x,\varepsilon)$.

Next, let d be a $(3,2)$-metric and $\tau=\tau(N,d)=\tau(D,d)$. Let F be a closed set and let $x\notin F$. Since $M\backslash F$ is open, there is an $\varepsilon>0$, such that $B(x,x,\varepsilon)\subseteq B(x,x,2\varepsilon)\subseteq M\backslash F$. For each $y\in F$, $d(x,x,y)>\varepsilon$. Let $\delta=\varepsilon/6$. Using the fact that d is a $(3,2)$-metric, (see Remark 2.2.d)) we have the following inequalities: $d(x,x,y)\leq 2d(x,z,z)+d(y,z,z)\leq 4d(x,x,z)+2d(y,z,z)<6\delta=\varepsilon$. If for some $y\in F$, $B(x,x,\delta)\cap B(y,y,\delta)\neq\varnothing$, we will obtain $d(x,x,y)<\varepsilon$. Hence, for each $y\in F$, $B(x,x,\delta)\cap B(y,y,\delta)=\varnothing$. For $V=\cup\{B(y,y,\delta) \mid y\in F\}$, we have that: V and $B(x,x,\delta)$ are open, $F\subseteq V$, $x\in B(x,x,\delta)$ and $V\cap B(x,x,\delta)=\varnothing$. □

Proposition 4.3. A space is $(3,1,\rho)$-N-D-metrizable iff it satisfies the I^{st} axiom of countability. Moreover, a space is $(3,1)$-N-D-metrizable iff it satisfies the I^{st} axiom of countability and is T_1.

Proof. Proposition 4.2. implies that each (3,1)-N-D-metrizable space is T_1, and directly from the definitions it follows that each (3,1,ρ)-N-D-metrizable space is first countable.

Conversely, let (M,τ) be a first countable topological space. For each $x\in M$ we choose a countable local base $\{U_n(x)\}$, such that $U_{n+1}(x)\subseteq U_n(x)$, for each n. We define $d:M^{(3)}\to R_0^+$ by:

a) $d(x,x,x)=0$;

b) $d(x,x,y)=0$ for $x\neq y$ and $y\in U_n(x)$, for each n;

c) $d(x,x,y)=1/n$ for $x\neq y$, and $y\in U_n(x)$ but $y\notin U_{n+1}(x)$; and

d) $d(x,y,z)=1$ for $x\neq y\neq z\neq x$.

1) It can be easily checked that d satisfies (M0) and (M1), and so it is a (3,1,ρ)-metric on M.

2) The definition of d implies that $B(x,x,1/n)=U_{n+1}(x)$, for any n and $x\in M$.

3) The definitions of τ(N,d) and τ(D,d), and the fact that $\{U_n(x)\}$, is a local base at x, together with 2) imply that τ=τ(N,d)=τ(D,d), and so (M,τ) is (3,1,ρ)-N-D-metrizable.

For the moreover case, the fact that (M,τ) is T_1, together with the definition of d, implies that ρ=Δ. So, d is a (3,1)-metric on M and (M,τ) is (3,1)-N-D-metrizable. □

Corollary 4.1. Any topological space (M,τ), with M a finite set, is (3,1,ρ)-N-D-metrizable. □

The above Proposition 4.2., together with the Urison Metrization Theorem (see e.g. [10]), implies that any (3,2)-N-D-metrizable topological space having a countable base is metrizable. But, this is a consequence of the following stronger property that any (3,2)-N-D-metrizable topological space is metrizable.

Proposition 4.4. Let (M,τ) be a (3,2,ρ)-N-D-metrizable topological space. Then:

a) Any open cover of M has an open σ-discrete refinement;

b) (M,τ) has a σ-discrete base; and

c) If ρ=Δ, then (M,τ) is metrizable.

Proof. Let d be a (3,2,ρ)-metric, such that τ=τ(N,d)=τ(D,d), and let $D=D_d$, where D_d is defined in Remark 2.2. The proof is almost the same as the proof that any pseudometric space (X,D) has a σ-discrete base (for example as in [10]), but we have to be careful how to use D. We use the standard notations:

$D(x,A)=\inf\{D(x,a)\mid a\in A\}$ and $D(A,B)=\inf\{D(a,b)\mid a\in A,\ b\in B\}$.

Note that in general $D(A,B)\neq D(B,A)$, but it is always $D(A,B)\leq 2\cdot D(B,A)$.

a) Let $\mathcal{U}$ be an open cover of M. For $U\in\mathcal{U}$ and $n\in N$, let $U_n=\{x\mid D(x,M\setminus U)\geq 5^{-n}\}$. The definition implies that $U_n\subseteq U_{n+1}\subseteq U$, for any n.

a.1) Let $x\in U_n$, $y\in M\setminus U_{n+1}$. Then, there is $z\in M\setminus U$, such that $D(y,z)<5^{-n-1}$ and $D(x,z)\geq 5^{-n}$. The inequalities

$$5^{-n}\leq D(x,z)\leq 2D(y,x)+D(y,z)\leq 4D(x,y)+D(y,z)<4D(x,y)+5^{-n-1}$$

imply that $D(x,y)>5^{-n-1}$. This shows that $D(U_n,M\setminus U_{n+1})\geq 5^{-n-1}$.

Next, choose a well ordering < on $\mathcal{U}$ and let $U_n^*=U_n\setminus\cup\{V_{n+1} \mid V<U\}$. It is obvious that if V<U, then $U_n^*\subseteq M\setminus V_{n+1}$.

a.2) Let V<U, let $x\in V_n^*\subseteq V_n$ and let $y\in U_n^*$. Since V<U, $U_n^*\subseteq M\setminus V_{n+1}$ implies that $y\in M\setminus V_{n+1}$. Now, a.1) implies that $D(x,y)\geq 5^{-n-1}$. All this implies that for any $U,V\in\mathcal{U}$, $2\cdot D(U_n^*,V_n^*)\geq 5^{-n-1}$.

Next, let $U_n^{\sim}=\{x \mid D(x,U_n^*)\geq 5^{-n-3}\}$.

a.3) Let $z\in M\setminus U$. Then, for any $y\in U_n^*$, $D(y,z)\geq 5^{-n}$, and so $2\cdot D(z,y)\geq 5^{-n}$, i.e. $D(z,y)\geq 5^{-n-1}$. All this implies that $U_n^{\sim}\subseteq U$.

a.4) Let $x\in U_n^{\sim}$. There is a $v\in U_n^*$ with $D(x,v)<5^{-n-3}$. Let $2\cdot\delta=5^{-n-3}-D(x,v)$. Then, for $y\in B(x,x,\delta)$, we have:

$$D(y,v)\leq 2\cdot D(x,y)+D(x,v)=2\cdot d(x,x,y)+D(x,v)<2\cdot\delta+D(x,v)=5^{-n-3}.$$

With this, we have shown that $U_n^{\sim}$ is open.

a.5) We will show that $D(U_n^{\sim},V_n^{\sim})\geq 5^{-n-2}$, for any $U,V\in\mathcal{U}$. Let $x\in U_n^{\sim}$ and $y\in V_n^{\sim}$. Then, there are $a\in U_n^*$ and $b\in V_n^*$ such that $D(x,a)<5^{-n-3}$ and $D(y,b)<5^{-n-3}$. A simple calculation, show that:

$$D(a,b)\leq 2\cdot D(y,a)+D(y,b)\leq 2\cdot(2\cdot D(x,y)+D(x,a))+D(y,b)<4\cdot D(x,y)+3\cdot 5^{-n-3};\ \text{and}$$
$$D(b,a)\leq 2\cdot D(y,b)+D(y,a)\leq 2\cdot D(y,b)+2\cdot D(x,y)+D(x,a)<4\cdot D(x,y)+3\cdot 5^{-n-3}.$$

Now, a.2) implies that $5^{-n-1}<4\cdot D(x,y)+3\cdot 5^{-n-3}$. All this implies that $D(U_n^{\sim},V_n^{\sim})\geq 5^{-n-2}$.

Next, let $\mathcal{V}_n=\{U_n^{\sim} \mid U\in\mathcal{U}\}$.

a.6) Let $x\in M$. If $y,z\in B(x,x,5^{-n-3})$, then $D(y,z)\leq 2\cdot D(x,y)+D(x,z)< 3\cdot 5^{-n-3}<5^{-n-2}$. This implies that $\mathcal{V}_n$ is a discrete family of sets, i.e. any point x has a neighborhood intersecting at most one member of $\mathcal{V}_n$.

a.7) Let $x\in M$. Using the well ordering on $\mathcal{U}$, let U be the smallest open set in $\mathcal{U}$ such that $x\in U$. Then, there is an $n\in N$, such that $B(x,x,5^{-n})\subseteq U$. This implies that $x\in U_n$ and since U is the smallest, it follows that $x\in U_n^*$. So, $D(x,U_n^*)=0$, i.e. $x\in U_n^{\sim}$.

Now, let $\mathcal{V}_{\mathcal{U}}=\cup\{\mathcal{V}_n \mid n\in N\}$. Then, a.3), a.4), a.6) and a.7) imply that $\mathcal{V}_{\mathcal{U}}$ is an open σ-discrete refinement of $\mathcal{U}$.

b) For $n\in N$, let $\mathcal{U}_n$ be the open cover $\{B(x,x,1/n) \mid x\in M\}$, and let $\mathcal{V}_n$ be its σ-discrete refinement provided by **a)**. The definitions of τ(N,d) and τ(D,d) imply that $\mathcal{U}=\cup\{\mathcal{U}_n \mid n\in N\}$ is a base for τ=τ(N,d)=τ(D,d). Since each $\mathcal{V}_n$ is a σ-discrete refinement of $\mathcal{U}_n$, it follows that $\mathcal{V}=\cup\{\mathcal{V}_n \mid n\in N\}$ is a σ-discrete refinement of $\mathcal{U}$. Hence, $\mathcal{V}$ is a σ-discrete base for τ.

c) For ρ=Δ, Proposition 4.3., implies that (M,τ) is T_1 and regular. The Bing Metrization Theorem implies that (M,τ) is metrizable. □

Remark 4.1. In its original form, the Bing Metrization Theorem is Theorem 3. in [3], where a topological space is assumed to be T_1.

Theorem 3. [3] A necessary and sufficient condition that a regular topological space be metrizable is that it be perfectly screenable.

We use the Bing Metrization Theorem as stated in [10].

A necessary and sufficient condition that a T_1 and regular topological space be metrizable is that it has a σ-discrete base.

Corollary 4.2. a) A topological space is metrizable iff it is (3,2)-N-D-metrizable.

b) A topological space is (3,2)-N-D-metrizable iff it is 3-$\mathcal{M}$-metrizable. □

Proposition 4.5. a) A space is (3,1,ρ)-N-D-symmetrizable iff it is (3,1,ρ)-$\mathcal{M}$-symmetrizable.

b) A space is (3,1)-N-D-symmetrizable iff it is (3,1)-$\mathcal{M}$-symmetrizable.

Proof. a) Let (M,τ) be (3,1,ρ)-N-D-symmetrizable space, and d be a (3,j,ρ)-symmetric such that τ=τ(N,d)=τ(D,d). Let $d':M^{(3)}\to R_0^+$ be defined by: d'(x,y,z)=max{d(x,x,y),d(x,x,z),d(y,y,z)}. Directly from the definition follows that d' satisfies (M0) and that d'(x,x,y)=d(x,x,y). The proof that d' satisfies (M1) is as follows:

$$d'(x,y,u)+d'(x,u,z)+d'(u,y,z)\geq d(x,x,y)+d(x,x,z)+d(y,y,z)$$
$$\geq \max\{d(x,x,y),d(x,x,z), d(y,y,z)\}=d'(x,y,z).$$

We will denote the ε-balls for d' by B'(x,ε), B'(x,y,ε) and L'(x,e). The definition of d' implies that: B'(x,ε)=B(x,x,ε)=L(x,ε)=B'(x,x,ε)=L'(x,ε), and B'(x,y,ε)=B(x,x,ε)∩B(y,y,ε), for B'(x,y,ε)≠∅. All this implies that for each X∈$\mathcal{M}$, τ=τ(N,d)= τ(D,d)=τ(X,d').

b) The definition of d' in a) implies that if ρ=Δ, then d' is a (3,1)-symmetric. □

For any $\mathcal{B}\subseteq\mathcal{M}$ and j∈{1,2}, we consider the following eight classes of topological spaces:

$\mathcal{C}$(o,$\mathcal{B}$,j) – the class of all the (3,j,ρ)-$\mathcal{B}$-metrizable topological space;

$\mathcal{C}$(o,s,$\mathcal{B}$,j) – the class of all the (3,j,ρ)-$\mathcal{B}$-symmetrizable topological space;

$\mathcal{C}$($\mathcal{B}$,j) – the class of all the (3,j)-$\mathcal{B}$-metrizable topological space;

$\mathcal{C}$(s,$\mathcal{B}$,j) – the class of all the (3,j)-$\mathcal{B}$-symmetrizable topological space;

$\mathcal{C}$(o,$\mathcal{B}$) – the class of all the (3,ρ)-$\mathcal{B}$-metrizable topological space;

$\mathcal{C}$(o,s,$\mathcal{B}$) – the class of all the (3,ρ)-$\mathcal{B}$-symmetrizable topological space;

$\mathcal{C}$($\mathcal{B}$) – the class of all the 3-$\mathcal{B}$-metrizable topological space; and

$\mathcal{C}$(s,$\mathcal{B}$) – the class of all the 3-$\mathcal{B}$-symmetrizable topological space.

There are some obvious relations among these classes, and some of them are characterized by some of the above propositions. For example: a) $\mathcal{C}$(o,N-D,1) is the class of all the first countable spaces; b) $\mathcal{C}$(N-D,1) is the class of all the first countable and T_1 spaces; c) $\mathcal{C}$(o,N,1) is the class of pseudo-o-metrizable space as in [17]; d) $\mathcal{C}$(N-D,2)=$\mathcal{C}_m$=$\mathcal{C}$($\mathcal{M}$,2)=$\mathcal{C}$($\mathcal{M}$)=$\mathcal{C}$(s,$\mathcal{M}$), where $\mathcal{C}_m$ is the class of all the metrizable spaces; and d) $\mathcal{C}$(o,s,N-D,1)=$\mathcal{C}$(o,s,$\mathcal{M}$,1). In general, we do not have good characterizations of these classes. The class $\mathcal{C}_m$ is a subclass of each of the above classes. We state several questions:

Question 4.1. Is $\mathcal{C}$(o,s,N-D,1) the class of all the first countable spaces having a σ-discrete base? Is $\mathcal{C}$(s,N-D,1) the class of all the first countable and T_1 spaces having a σ-discrete base?

Let d be a (3,j,ρ)-metric on M, and for a,b∈M, let F_{ab}:M→R be defined by F_{ab}(x)=d(a,b,x).
In general, F_{ab} is not continuous for any of the induced topologies on M by d.

Question 4.2. What conditions on d would imply the continuity of F_{ab}?

For a∈M, the multivalued map F_a defined by F_a(x)={d(a,x,y) | y∈M} is l.s.c. (lower semi continuous) for the induced topologies τ(G,d) and τ(H,d).

Question 4.3. What additional requirements on d would imply the existence of a continuous selection for F_a?

References

[1] П.С. Александров, В.В. Немыцкий, Условия метризуемости топологических пространств и аксиома симетрии, *Мат. сб.* 3:3 (1938), 663-672

[2] А.В. Архангельский, О поведении метризуемости при факторных тображениях, *ДАН* 164, № 2 (1965), 247-250

[3] R.H. Bing, Metrization of topological spaces, *Canadian J.Math.*3 (1951), 175-186

[4] М. Чобан, О симетрисуемыих пространствах, *Вестн.Моск. Ун-та, сер.Матем., мех*, № 3 (1959), 44-50

[5] B.C. Dhage, Generalized metric spaces and topological structure I, *An. stiint. Univ. Al.I. Cuza Iasi. Mat(N.S),* 46 (2000), 3-24

[6] D. Dimovski, Generalized metrics - (n,m,r)-metrics, *Mat. Bilten*, 16, Skopje (1992), 73-76

[7] D. Dimovski, (3,1,ρ)-metrizable topological spaces, *Math. Macedonica*, 3 (2005), 59-64

[8] S. Gähler, 2-metrische räume und ihre topologische Struktur, *Math. Nachr*. 26 (1963), 115-148

[9] S. Gähler, Untersuchungen über verallgemenerte m-metrische Räume. I and II, *Math. Nach*. 40 (1969), 165-189 and 229-264

[10] J.L. Kelley, *General topology*, D. Van Nostrand Company, New York, 1959

[11] R. Kopperman, All topologies come from generalized metrics, *A. Math. Month*. V 95 i2, 89-97

[12] Z. Mamuzič, Abstract distance and neighborhood spaces, *Proc. Prague Symp*. (1962), 261-266

[13] K. Menger, Untersuchungen über allgemeine Metrik, *Math. Ann*. 100 (1928), 75-163

[14] Z. Mustafa, B. Sims, (2006) A new approach to generalized metric spaces, *Jurnal of Nonlinear and Convex Analysis*, Vol. 7, Number 2 (2006), 289-297

[15] V. Nemytzki, On the "third axiom of metric spaces", *Tr. Amer. Math. Soc*. 29 (1927) , 507-513

[16] С. Недев, Об обобщено метризуемых пространствах, *Докл. Болг. Акад. Наук*, 20, № 6 (1967), 513-516

[17] С. Недев, о-Метризуемыие пространства, *Тр. Моск. Мат. Общ*. Том 24 (1971), 201-236

[18] J. Usan, <Nm,E>-seti s (n+1)-rastojaniem, *Review of Research, PMF*, Novi Sad, Ser. Mat. 17 (1989), 2, 65-87

Received: May 9, 2014