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Some Properties of $\mathscr{D}$-sets of a Group[1]

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Abstract

A subset D of a group G is called a $\mathscr{D}$-set if every element of G which is not in D has its inverse in D. In this paper, we gave some of the properties of a $\mathscr{D}$-set.

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1 Introduction

Some investigations in group theory are concerned with finding a subset H of a group G which is also a group under the same operation (in this case, H is called a subgroup of G). Some even endeavored to find all subgroups of a given

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group and how each of these subsets are related. In contrast to such idea, we introduced in this research special subsets of a group which are normally not subgroups. A subset D of a group G is called a $\mathscr{D}$-set if every element of G which is not in D has its inverse in D. It will be shown that a proper subset of a group, which is a $\mathscr{D}$-set, is not a subgroup.

The following concepts are taken from [1], [2] and [3].

A *binary operation* on a nonempty set G is a function $G \times G \to G$. A *semigroup* is a nonempty set G together with a binary operation on G which is associative. A *group* is a semigroup which contains an element e with the property that $ge = g = eg$ for all $g \in G$ (in this case we call e the identity element); and for each $g \in G$, there exists $h \in G$ such that $gh = e = hg$.

Let G and H be semigroups. A function $f : G \to H$ is a *homomorphism* if for all $a, b \in G$, $f(ab) = f(a)f(b)$. If f is surjective, then we call f an *epimorphism*. On the other hand, a bijective homomorphism is called an *isomorphism.*

2 Results

We now give some properties of a $\mathscr{D}$-set.

Theorem 2.1 *Let G be a group. Then*

i. The set T of all $\mathscr{D}$-sets of G is a semigroup under the set operation union.

ii. The set T^C of complements of all $\mathscr{D}$-sets of G is a semigroup under the set operation intersection.

Proof: Let $D_1, D_2 \in T$ and consider $G\backslash(D_1 \cup D_2)$. Let $x \in G\backslash(D_1 \cup D_2)$. Then $x \notin D_1 \cup D_2$, that is, $x \notin D_1$ and $x \notin D_2$. Since D_1 and D_2 are $\mathscr{D}$-sets, $x^{-1} \in D_1$ and $x^{-1} \in D_2$. Thus, $x^{-1} \in D_1 \cup D_2$. This shows that $D_1 \cup D_2$ is a $\mathscr{D}$-sets of G. Since associativity holds for the operation union of sets, T is a semigroup.

Let $E_1, E_2 \in T^C$. Thus, $G\backslash E_1$ and $G\backslash E_2$ are $\mathscr{D}$-sets of G. By the first part, $(G\backslash E_1) \cup (G\backslash E_2)$ is also a $\mathscr{D}$-set. Since $(G\backslash E_1) \cup (G\backslash E_2) = G\backslash(E_1 \cap E_2)$, we get $E_1 \cap E_2 \in T^C$. Finally, since associativity holds for intersection, T^C is a semigroup. ■

Theorem 2.2 *Let D be a $\mathscr{D}$-set and $x \in G$ such that $x^2 = e$. Then $x \in D$.*

Proof: Suppose $x \notin D$. Since D is a $\mathscr{D}$-set, there exists $y \in D$ such that $xy = e$. Note that y is necessarily x^{-1}. Since $x^2 = e$, $x = x^{-1} = y \in D$. This is a contradiction. Hence, $x \in D$. ■

The elements of order 2 of a group are what we called *involutions*. Theorem 2.2 suggests that involutions together with the identity element belong to any $\mathscr{D}$-set of G. The next theorem compares the cardinality of a $\mathscr{D}$-set and its complement. We borrow a convention from [4] which states that for sets X and Y, $|X| \preceq |Y|$ only if there exists an injection $f : X \to Y$.

Theorem 2.3 *Let D be a $\mathscr{D}$-set of a group G. Then $|G \backslash D| \preceq |D|$.*

Proof: Define $f : G\backslash D \to D$ by $f(x) = x^{-1}$. Let $x, y \in G\backslash D$. Then by the uniqueness of the inverse, we have, $x = y \Leftrightarrow x^{-1} = y^{-1} \Leftrightarrow f(x) = f(y)$. Hence, f is a well-defined injection. Accordingly, $|G\backslash D| \preceq |D|$. ■

For a finite group G, it is evident that a $\mathscr{D}$-set has more elements than its complement.

Theorem 2.4 *Let G be a group and $x \in G$. Then $T(x) = \{D \in T : x \in D\}$ is a sub-semigroup of T.*

Proof: All we need is to show the closure property. Let G be a group containing an element x and T be the set of all $\mathscr{D}$-sets of G. Let $T(x) = \{D \in T : x \in D\}$ and suppose D_i and D_j be any two elements of $T(x)$. By Theorem 2.1, $D_i \cup D_j$ is also in T. But since x is clearly in $D_i \cup D_j$, then $D_i \cup D_j$ must also be in $T(x)$. ■

Theorem 2.5 *Let x be a non-identity element of a group G. Then x is an involution if and only if $T(x) = T$.*

Proof: Suppose x is an involution of G. Then by Theorem 2.2, x is in every $\mathscr{D}$-set of G. Hence, $T(x) \supseteq T$. Since $T(x) \subseteq T$, we have $T(x) = T$

Conversely, assume that $T(x) = T$ and x is not an involution. If x is non-involution and x is not the identity, then $x^{-1} \neq x$. Note that if D is a $\mathscr{D}$-set, then $(D\backslash\{x\}) \cup \{x^{-1}\}$ is a $\mathscr{D}$-set that do not contain x. This is a contradiction. Hence, x must be an involution. ■

Theorem 2.6 *Let G be a group with an element x whose order is 2. Then any $\mathscr{D}$-set of G contains a nontrivial subgroup of G.*

Proof: Let D be a $\mathscr{D}$-set of G. If $x \in G$ is of order 2 then $\langle x \rangle = \{x, e\}$ with $x^2 = e$. By Theorem 2.2, $x \in D$. Hence, $\langle x \rangle \subseteq D$. ■

The concept of $\mathscr{D}$-set can be used to characterize groups and subgroups. The following theorems reveal how $\mathscr{D}$-set show that a subgroup is improper and when can the group be trivial.

Theorem 2.7 *G has a trivial $\mathscr{D}$-set if and only if G is trivial.*

Proof: Suppose G is a trivial group. Then $G = \{e\}$ is a $\mathscr{D}$-set of G.

Conversely, suppose that G has a trivial $\mathscr{D}$-set and G is a nontrivial group. Then there exists a non-identity element x in G. Note that x is necessarily in $G\backslash\{e\}$. Since $\{e\}$ is a $\mathscr{D}$-set, there exists $y \in \{e\}$ (i.e. $y = e$) such that $xy = e$. This implies that $x = e$. This is a contradiction. Hence G must be trivial. ■

Theorem 2.8 *Let T be the set of all $\mathscr{D}$-set of G and $S = \{x \in G : x^2 = e\}$. Then $|T| = 1$ if and only if $G = S$.*

Proof: Assume that $|T| = 1$ and $S \neq G$. Then there exists $x \in G\backslash S$ such that $x^2 \neq e$ (i.e. $x \neq x^{-1}$). Let $D \in T$ and consider the following cases:

Case 1: $x \notin D$

If $x \notin D$, then $D \cup \{x\}$ is another $\mathscr{D}$-set of G, contradicting the fact that D is the only $\mathscr{D}$-set of G.

Case 2: $x \in D$

If $x \in D$, then $(D\backslash\{x\}) \cup \{x^{-1}\}$ is another $\mathscr{D}$-set of G, contradicting the assumption that D is the only $\mathscr{D}$-set of G.

Hence, S must be equal to G.

Conversely, suppose $G = S$. Note that for any D in T, $S \subseteq D$. Hence, $S \subseteq D \subseteq G$, which implies that $D = G$. Accordingly, $|T| = 1$. ■

In the next theorem, the usual subgroup notation will be used; that is, $H \leq G$ denotes that H is a subgroup of G.

Theorem 2.9 *Let $H \leq G$. Then H contains a $\mathscr{D}$-set of G if and only if $H = G$.*

Proof: Suppose H contains a $\mathscr{D}$-set D of G and $H \neq G$. Then there exists $x \in G\backslash H$. Since $x \notin H$, x must also be not in D. This implies that x^{-1} is in D. But this means that $x = (x^{-1})^{-1}$ is in H. This is a contradiction. Thus, $H = G$.

If $H = G$, then G is a $\mathscr{D}$-set of G contained in H. ■

Remark 2.10 *If each nontrivial subgroup contains a $\mathscr{D}$-set of a nontrivial group G then G is simple.*

To see this, suppose that G is a nontrivial group which is not simple. This means that G has a proper nontrivial normal subgroup H. Assume that each nontrivial subgroup contains a $\mathscr{D}$-set of G. Accordingly, H must contain a $\mathscr{D}$-set of G. By Theorem 2.9, $H = G$ which is a contradiction.

Theorem 2.11 *Let G_1 and G_2 be groups, and $\phi : G_1 \to G_2$ be an epimorphism of groups.*

i. If D_1 is a $\mathscr{D}$-set of G_1, then $\phi(D_1)$ is a $\mathscr{D}$-set of G_2.

ii. If D_2 is a $\mathscr{D}$-set of G_2, then $\phi^{-1}(D_2)$ is a $\mathscr{D}$-set of G_1.

Proof: Let D_1 be a $\mathscr{D}$-set of G_1 and $y \in G_2\backslash\phi(D_1)$, then there exists $x \in G_1$ such that $y = \phi(x)$. Note that $x \in G_1\backslash D_1$, otherwise, $y = \phi(x) \in \phi(D_1)$. Since D_1 is a $\mathscr{D}$-set, $x^{-1} \in D_1$. Necessarily, $\phi(x^{-1}) \in \phi(D_1)$. Hence, there exists $z = \phi(x^{-1}) \in \phi(D_1)$ such that $yz = \phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(e_1) = e_2$. This shows that $\phi(D_1)$ is a $\mathscr{D}$-set of G_2.

Let D_2 be a $\mathscr{D}$-set of G_2 and $x \in G_1\backslash\phi^{-1}(D_2)$. Note that $\phi(x) \notin D_2$, otherwise $x \in \phi^{-1}(D_2)$. Since D_2 is a $\mathscr{D}$-set, $\phi(x)^{-1} = \phi(x^{-1}) \in D_2$. Hence $x^{-1} \in \phi^{-1}(D_2)$. This shows that $\phi^{-1}(D_2)$ is a $\mathscr{D}$-set of G_1. ■

This section culminates with a theorem which shows that the families T of $\mathscr{D}$-sets of a group and T^C of their corresponding complements are structurally alike.

Theorem 2.12 *T is isomorphic to T^C.*

Proof: We form $\phi : T \to T^C$ defined by $\phi(D) = G\backslash D$, where $D \in T$ and thus $G\backslash D$ must be in T^C. Then ϕ is obviously a bijection. Now, let D_1 and D_2 be in T. This means that $G\backslash D_1$ and $G\backslash D_2$ are in T^C. Further $\phi(D_1 \cup D_2) = G\backslash(D_1 \cup D_2) = (G\backslash D_1) \cap (G\backslash D_2) = \phi(D_1) \cap \phi(D_2)$. Hence, $\phi : T \to T^C$ is an isomorphism. ■

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