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Convergence Theorems for a Finite Family of Relatively Quasi-Nonexpansive Mappings and System of Equilibrium Problems

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Abstract

In this paper, we employ an iteration scheme introduced in [13], to prove strong convergence to a common element of the set of fixed points of a family $\{T_i\}_{i=1}^N$ of L_i-Lipschitzian mappings and the set of common solutions to a system of equilibrium problems, in a uniformly convex Banach space which is also uniformly smooth. We do not require that the family $\{T_i\}_{i=1}^N$ be uniformly continuous, as in [13].

Introduction

Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called Lipschitzian, if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||,$$

for all $x, y \in C$. If $L < 1$, then T is called a contraction. If $L = 1$, then T is called nonexpansive.
A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is defined as $F(T) := \{x \in C : Tx = x\}$. If $F(T) \neq \emptyset$ and T satisfies

$$||Tx - p|| \leq ||x - p||,$$

for all $x \in C$, $p \in F(T)$, then T is called Quasi-Nonexpansive. Observe that every nonepansive mapping with a nonempty fixed point set is a quasi-nonexpansive mapping.

Let E^{E^*}, be the dual space of E, defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^*\}.$$

If E is smooth, then J is single-valued.
Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) := ||x||^2 - 2\langle x, j(y)\rangle + ||y||^2,$$

for all $x, y \in C$.

A point $y \in C$ is said to be an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to y and $\lim ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\overline{F}$. We say that a mapping T is *relatively nonexpansive* (see for example [3-5] and the references therein) if the following conditions are satisfied:
(R_1) $F(T) \neq \emptyset$
(R_2) $\phi(p, Tx) \leq \phi(p, x), \ \forall x \in C, \ p \in F(T)$
(R_3) $F(T) = \overline{F}(T)$
If T satisfies R_1 and R_2, then T is said to be *relatively quasi-nonexpansive.* It is easily seen that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive maps. Several authors (see for example [9], [10] and the references therein) have studied the approximations of fixed points of relatively quasi-nonexpansive mappings. It is easy to see that in a Hilbert space,H, the classes of relatively quasi-nonexpansive and relatively nonexpansive mappings coincide. This is because $\phi(x, y) = ||x-y||^2, \forall x, y \in H$ and this implies $\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p|| \ \forall x \in C, \ p \in F(T)$. Examples of relatively quasi-nonexpansive mappings are given in [10].

Let F be a bi-function of $C \times C \to \Re$. The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of solutions to (1.1) is denoted by $EP(F) := \{x^* \in C : F(x^*, y) \geq 0 \ \forall y \in C\}$. Many problems in Engineering, Economics and optimization are reduced to finding a solution of (1.1). Several methods have been proposed for solving (1.1) (see for example [2], [6] and the references therein).

Takahashi and Zembayashi [12] introduced a hybrid iterative scheme for the approximation of fixed points of relatively nonexpansive mappings which are also solutions to equilibrium problems, in a uniformly smooth real Banach space which is also uniformly convex.

Recently, Yekini [13] proved strong convergence theorems for a finite family of relatively quasi-nonexpansive mappings and a system of equilibrium problems in a real uniformly convex Banach space which is also uniformly smooth. More precisely, he proved the following theorem:

Theorem 1 [13]: Let E be a real uniformly convex Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k = 1, 2, ..., m$ let F_k be a function from $C \times C \to \Re$, satisfying $(A_1) - (A_4)$ and let $\{T_i\}_{i=1}^N$ be a finite family of closed relatively quasi-nonexpansive mappings of C into itself such that $F := \cap_{i=1}^N F(T_i) \cap (\cap_{k=1}^m EP(F_k)) \neq \emptyset$. Assume that T_i is uniformly continuous for each $i = 1, 2, 3, ..., N$. Let $\{x_n\}$ be iteratively generated by $x_0 \in C,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n),\ n \geq 1 \\ u_n = T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}} ... T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} y_n \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n),\ n \geq 1 \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 1 \end{cases}$$

where J is the duality mapping on E and $T_n := T_{n(modN)}$. Suppose $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\liminf \alpha_n(1 - \alpha_n) > 0$ and $\{r_{k,n}\}, (k = 1, 2, ..., m)$ satisfying $\liminf r_{k,n} > 0, (k = 1, 2, ..., m)$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Motivated by the above theorem, we prove strong convergence theorems for a finite family $\{T_i\}_{i=1}^N$ of L_i-Lipschitzian relatively quasi-nonexpansive mappings and a system of equilibrium problems in a uniformly convex Banach space which is also uniformly smooth. Our finite family $\{T_i\}_{i=1}^N$ is different from that in [13], as ours is a family of L_i-Lipschitzian mappings as opposed to the uniform continuity assumed for the family $\{T_i\}_{i=1}^N$ in [13].

Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \{\frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq \tau\}.$$

E is uniformly smooth if and only if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$.
The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\{1 - \|\frac{x + y}{2}\| : ||x|| = ||y|| = 1; \epsilon = ||x - y||.$$

E is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| \leq 1, |||y|| \leq 1$ and $||x - y|| \geq \epsilon$, then $||\frac{1}{2}(x + y)|| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. A normed space is called strictly convex if for all $x, y \in E, x \neq y, ||x|| = ||y|| = 1$, we have $||\lambda x + (1 - \lambda)y|| < 1,\ \forall \lambda \in (0, 1)$.

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the generalized projection Π_C from

E onto C is defined by

$$\Pi_C(x) := argmin_{y\in C}\phi(x,y),\ \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x,y)$ and the strict monotonicity of the mapping J (see for example [1]). If E is a Hilbert space, then Π_C is the metric projection from H onto C. It is well known (see for example [7]), that in uniformly convex and uniformly smooth Banach spaces

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2,\ \forall x,y \in E.$$

We now state some well known lemmas, which will be useful in the sequel.
Lemma 1 [1]: Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, E. Then

$$\phi(x,\Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x,y),\ \forall x \in C,\ \forall y \in E$$

Lemma 2[1]: Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, E. Let $x \in E$ and $z \in C$. Then

$$z = \Pi_C x \Leftrightarrow \langle y - z, J(x) - J(z)\rangle \leq 0,\ \forall y \in C$$

Lemma 3[10]: Let T be a closed relatively quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.
Lemma 4 [7]: Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, E. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n,y_n) = 0$, then $\lim_{n\to\infty}||x_n - y_n|| = 0$.
Lemma 5 [14]: Let E be a real uniformly convex Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0,1]$. Then there is a continuous strictly increasing convex function $g : [0,2r] \to \Re,\ g(0) = 0$ such that for every $x,y \in B_r(0)$, the following inequality holds:

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||^2)$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \Re$, we assume F satisfies the following conditions:
(A_1) $F(x,x) = 0 \forall x \in C$
(A_2) $F is monotone, i.e F(x,y) + F(y,x) \leq 0 \forall x,y \in C$
(A_3) $for each, x,y \in C, lim_{t\to 0}F(tz + (1-t)x,y) \leq F(x,y)$
(A_4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
In [2], the authors proved the following lemma:
Lemma 6 [2]: Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, E, and let F be a bifunction of $C \times C$ into

$\Re$ satisfying $(A_1)-(A_4)$. Let $r>0$ and $x\in E$. Then there exists $z\in C$ such that

$$F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0,$$

for all $y\in C$.

Takahashi and Zembayashi [11], proved the following lemma:

Lemma 7 [11]: Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space, E. Assume that $F: C\times C\to \Re$ satisfies $(A_1)-(A_4)$. For $r>0$ and $x\in E$, define a mapping $T_r^F: E\to C$ as follows:

$$T_r^F(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0,\ \forall y\in C\},$$

for all $z\in E$. Then, the following hold:
1. T_r^F is single - valued
2. T_r^F is a firmly nonexpansive-type mapping, i.e. for all $x,y\in E$,

$$\langle T_r^F x-T_r^F y, JT_r^F x-JT_r^F y\rangle \leq \langle T_r^F x-T_r^F y, Jx-Jy\rangle$$

3. $F(T_r^F)=EP(F)$
4. $EP(F)$ is closed and convex

Main Results

Theorem 2: Let E be a real uniformly convex Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k=1,2,...,m$ let F_k be a function from $C\times C\to\Re$, satisfying $(A_1)-(A_4)$ and let $\{T_i\}_{i=1}^N$ be a finite family of closed relatively quasi-nonexpansive mappings of C into itself such that $F:=\cap_{i=1}^N F(T_i)\cap(\cap_{k=1}^m EP(F_k))\neq\emptyset$. Assume that T_i is L_i-Lipschitzian for each $i=1,2,3,...,N$. Let $\{x_n\}$ be iteratively generated by $x_0\in C,\ C_1=C,\ x_1=\Pi_{C_1}x_0$,

$$\begin{cases} y_n=J^{-1}(\alpha_n Jx_n+(1-\alpha_n)JT_nx_n),\ n\geq 1\\ u_n=T_{r_{m,n}}^{F_m}T_{r_{m-1,n}}^{F_{m-1}}...T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}y_n\\ C_{n+1}=\{w\in C_n:\phi(w,u_n)\leq\phi(w,x_n),\ n\geq 1\\ x_{n+1}=\Pi_{C_{n+1}}x_0,\ n\geq 1\end{cases}$$

where J is the duality mapping on E and $T_n:=T_{n(modN)}$. Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\liminf\alpha_n(1-\alpha_n)>0$ and $\{r_{k,n}\},(k=1,2,...,m)$ satisfying $\liminf r_{k,n}>0,(k=1,2,...,m)$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof: The facts that C_n is closed and convex, $\{x_n\}$ is a Cauchy sequence in C, $\lim_{n\to\infty}||x_{n+l}-x_n||=0\ \forall\ l=1,2,...,N; \lim||x_n-T_nx_n||=0$ all follow from the proof of theorem 3.1 of [13].

Now since T_i is L_i-Lipschitzian for each $i = 1, 2, ..., N$ set $L = max\{L_i\}$. Then $||T_i x - T_i y|| \leq L||x - y||$. Now

$$\begin{aligned} ||x_n - T_{n+l}x_n|| &\leq ||x_n - x_{n+l}|| + ||x_{n+l} - T_{n+l}x_{n+l}|| + ||T_{n+l}x_{n+l} - T_{n+l}x_n|| \\ &\leq ||x_n - x_{n+l}|| + ||x_{n+l} - T_{n+l}x_{n+l}|| + L||x_n - x_{n+l}|| \\ &= (1 + L)||x_n - x_{n+l}|| + ||x_{n+l} - T_{n+l}x_{n+l}|| \to 0 \text{ as } n \to \infty \end{aligned}$$

The rest of the proof now follows as from (3.7) of [13].

Remark: The classes of lipschitzian and uniformly continuous maps are independent of each other. Hence our results complement the results in [13].

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