Special Homomorphisms in R-Vector Spaces

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Abstract

In this paper we introduce the notion of special homomorphism in R-vector spaces and study its properties. We show that the set of all special homomorphisms form an R-vector space with suitable addition and scalar multiplication. Also we prove that $\text{SHom}(V, W)$, where V and W are R-Vector spaces over the same regular ring R , is a normed R-vector space. Further we prove certain results concerning R-Vector spaces.

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Introduction

N. Raja Gopala Rao [4] introduced the concept of vector spaces over regular rings (simply R-vector spaces) as a generalization of Boolean vector space of Subrahmanyam N. V [6]. He introduced the notion of linear endomorphisms and affine transformations in R-vector spaces and studied its properties. Further he made a study on the geometric aspect of these spaces. Later K. Venkateswarlu [2] introduced the notion of direct sums in R-vector spaces and established that every direct sum of R-vector spaces has a basis provided each component has a basis. The set of all linear endomorphisms denoted by $\text{Hom}(V, V)$, where V is an R-vector space in [5], is not closed under usual addition of linear endomorphisms. Keeping this in view, we introduce the notion of special homomorphisms in R-vector spaces and studied its properties. This paper is divided

in 2 sections. The first section is devoted for certain definitions and results concerning regular rings, R-Vector spaces[4]and Boolean Vector spaces of Subrahmanyam [6].In section 2,we introduce the notion of special homomorphisms in R-vector spaces.We give some examples of Special homomorphisms which do not subsume the notion of linear endomorphism of Raja Gopala Rao[4]and B-vector space of Subrahmanyam[6](see example 2.8).We prove that the set of all special homomorphisms in R-vector space forms an R-vector space(see theorem 2.11).We introduce the notion of strong special homomorphism and in theorem 2.14,we prove that the set of all strong special homomorphisms form a subspace.Further we prove that $\text{SHom}(V, W)$ is isomorphic to the finite direct sum of W 's.(see theorem 2.19).

1 Preliminaries

In this section we collect certain definitions and results concerning R-Vector spaces of Raja Gopala Rao[4]and freely make use of the results in [2,4,6].

Definition 1.1. *A ring R is called a regular ring if and only if to each $a \in R$ there is an element $x \in R$ such that $axa = a$. If $a \in R$ and $axa = a$ then we put $a^* = xax$.*

In this B stands for Boolean algebra of idempotents of R-vector spaces, U is the set of units of R-vector spaces and $|a| = aa^*$

Lemma 1.2. *Let R be a regular ring and $a, b \in R$. Then*

1. $|a| = a$ if and only if $a \in B$
2. $a|a| = a = |a|a$
3. If $ab = 0$ then $|a|b = |b|a = |a||b| = 0$
4. $|a|(a + 1 - |a|) = a$

Definition 1.3. *Let $V = (V, +)$ be any group and $R = (R, +, \cdot)$ be a commutative regular ring with unity element 1. Then V is said to be a Vector space over R (or simply R -vector space) if and only if there exists a mapping $: R \times V \rightarrow V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$ the following properties hold:*

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ if $a^2 = a$
3. $1x = x$

4. $(a + b)x = ax + bx$ if $ab = 0$
5. $r(sx) = (rs)x$ if r and s are invertible elements of R
6. $a0 = 0$ imply $a^2 = a$

The zero element of V and as well as that of R will be denoted by the same symbol '0'.

Example 1.4. Let R be any commutative regular ring with 1, which is not a Boolean ring and $V = (V, +)$ be the additive group of R . For any $a \in R$, $x \in V$, define the scalar multiplication of a and x , which will be denoted by $a \otimes x$, by putting $a \otimes x = |a|x$, the ring product of $|a|$ and x . Then it is easy to verify that all the properties 1 - 5 of definition 1.3 are satisfied. But 6 is not true for this system since R is not a Boolean ring.

Remark 1.5. Raja Gopala Rao[4] remarked that every R -vector space can be treated as B -vector space over the Boolean algebra of idempotents of regular ring R with the same scalar multiplication as in the R -vector space.

Definition 1.6. An R -vector space V is said to be normed if and only if there exists a mapping norm $\| \| : V \rightarrow B$ satisfying the following properties.

- (1). $\| x \| = 0 \Leftrightarrow x = 0$ and (2). $\| ax \| = a \| x \|$ for all $x \in V, a \in B$

Definition 1.7. A finite subset of non zero elements $x_1, \dots, x_n$ of an R -vector space V is called (linearly) independent over R if and only if $a_1x_1 + \dots + a_nx_n = 0$ and $a_1 \dots a_n \neq 0$ imply $x_1 + \dots + x_n = 0$ and a subset S (of non zero elements) of V is said to be an independent subset of V if and only if every finite subset of S is linearly independent over R .

Definition 1.8. Let V be a vector space over a regular ring R . An independent subset of V which spans V is called basis for V .

Remark 1.9. If G is a basis group for V , then $a_{x,g}$ represents the uniquely determined coefficient of any $g \in G$ in the representation of x in terms of G and $x = \sum_{g \in G} a_{x,g}g$.

Definition 1.10. Let V, W be R -vector spaces over a regular ring R . A mapping $T : V \rightarrow W$ is a homomorphism of V to W provided

$$(1.10) \quad T(ax + by) = aTx + bTy \text{ if } ab = 0, \forall x, y \in V \text{ and } a, b \in R.$$

The set of homomorphisms from V to W will be denoted by $\text{Hom}(V, W)$.

2 Special Homomorphisms

Even though the condition 6 of definition 1.3 is independent of the other conditions 1 through 5, which was substantiated by an example in [4] given by Raja Gopala Rao, it can be relaxed without affecting the validity of the results obtained in [4, 5]. We restate the definition of R -vector space below by relaxing the condition 6 and adopt the same nomenclature given by Raja Gopala Rao [4].

Definition 2.1. Let $V = (V, +)$ be any group and $R = (R, +, \cdot)$ be a commutative regular ring with unity element 1. Then V is said to be a Vector space over R (or simply R -vector space) if and only if there exists a mapping $\cdot : R \times V \rightarrow V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$, all the following properties hold:

1. $a^2(x + y) = ax + ay$
2. $a(bx) = (ab)x$ if $a^2 = a$
3. $1x = x$
4. $(a + b)x = ax + bx$ if $ab = 0$
5. $r(sx) = (rs)x$ if r and s are invertible elements of R

Remark 2.2. The set of all linear endomorphisms defined by Raja Gopala Rao [4] in definition 1.10 is not closed under the binary operation. Consider the following

Example 2.3. Let $V = (Z_6, +)$ be the group of addition modulo 6 and $R = (Z_3, +, \cdot)$ be the ring of residues modulo 3, which is a commutative regular ring with 1. Define scalar multiplication as $\cdot : R \times V \rightarrow V$, by $0x = 0, 1x = x, 2x = x + 3, \forall x \in V$. Define $T, S : V \rightarrow V$ by $T(0) = 0, T(1) = 5, T(2) = 1, T(3) = 3, T(4) = 2, T(5) = 4$ and $S(0) = 0, S(1) = 4, S(2) = 1, S(3) = 3, S(4) = 2, S(5) = 5$. Letting $a = 0, b = 2$ and $y = 1$ in (1.10), $(T + S)(0x + 2y) = 4 \neq 0 = 0(T + S)y$. Thus $(T + S) \notin \text{Hom}(V, V)$.

Remark 2.4. In view of the above example, we introduce the notion of special homomorphism in R -vector spaces which in turn gives the set of all special homomorphisms closed under the binary operation '+ '.

Definition 2.5. Let V, W be R -vector spaces over a regular ring R . A mapping $T : V \rightarrow W$ is a special homomorphism of V to W provided
(2.5) $T(ax + by) = a \mid Tx + b \mid Ty$ if $ab = 0, \forall x, y \in V$ and $a, b \in R$.

The set of special homomorphisms from V to W will be denoted by $\text{SHom}(V, W)$.

Example 2.6. Let V be an R -vector space as in example 2.3 above and $T : V \rightarrow V$ be a map defined by $T(0) = 0 = T(3)$, $T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$. Then by easy verification of (2.5) T is a special homomorphism.

Remark 2.7. It seems to be that special homomorphism defined above coincides with the linear homomorphism in B -Vector space of $N.V.$ Subrahmanyam [6]. But this is not the case. For instance in any B -vector space the property $(a + b)x = ax + bx - abx$ holds. However this is not true in the case of special homomorphism of R -vector spaces. It is justified in the following example given under

Example 2.8. Let V be an R -vector space as in example 2.3 and $T : V \rightarrow V$ be a map defined by $T(0) = 0 = T(3)$, $T(1) = 2 = T(4)$, $T(2) = 3 = T(5)$. Let $a = 1, b = 2$ in Z_3 and $x = 1$ in Z_6 . Clearly $(1 + 2)Tx = 3T1 = 0T1 = 0 \neq 2 = 1T1 + 2T1 - (1.2)T1 = 1Tx + 2Tx - (1)(2)Tx$.

We have the following

Lemma 2.9. Let V and W be R -vector spaces over R and T be a mapping from V to W . $T \in SHom(V, W) \Leftrightarrow T(ax) = |a|Tx, \forall x \in V$ and $a \in R$.

Proof. $(\Rightarrow)$ Suppose $T \in SHom(V, W)$. Letting $b = 0$ in (2.5), $T(ax) = |a|Tx$. Conversely, let $T(ax) = |a|Tx$ for all $a \in R$ and $ab = 0$. Then $T(ax + by) = (|a| + 1 - |a|)T(ax + by) = |a|T(ax + by) + (1 - |a|)T(ax + by) = T(|a|(ax + by)) + T((1 - |a|)(ax + by)) = T[(|a|a)x + (|a|b)y] + T[(1 - |a|)ax + (1 - |a|)by] = T(ax) + T(by)$ (by lemma 1.2(2), theorem 1.2(3)) $= |a|Tx + |b|Ty$. $\square$

Theorem 2.10. Let V and W be R -Vector spaces. Define $+$ on $SHom(V, W)$ by $(T + S)(x) = Tx + Sx$ for all $x \in V$ and $T, S \in SHom(V, W)$. Then $(SHom(V, W), +)$ is an abelian group.

Proof. Let $T, S \in SHom(V, W)$. Then $(T + S)(ax + by) = T(ax + by) + S(ax + by) = (|a|Tx + |b|Ty) + (|a|Sx + |b|Sy) = |a|(Tx + Sx) + |b|(Ty + Sy) = |a|(T + S)x + |b|(T + S)y$. Hence $T + S \in SHom(V, W)$. It is routine to verify the remaining axioms of the abelian group. $\square$

Theorem 2.11. Let V and W be R -Vector spaces. Then $SHom(V, W)$ is an R -vector space if the scalar multiplication is defined by $(aT)(x) = |a|Tx, \forall x \in V, T \in SHom(V, W)$ and $a \in R$.

Proof. Routine verification of axioms of definition 2.1. $\square$

Definition 2.12. Let V, W be R -vector spaces over a regular ring R . A mapping $T : V \rightarrow W$ is a strong special homomorphism of V to W provided (2.12) $T(ax + by) = |a|Tx + |b|Ty, \forall x, y \in V$ and $a, b \in R$.

The set of strong special homomorphism from V to W will be denoted by $SSHom(V, W)$.

We characterize $SSHom(V, W)$ in the following

Theorem 2.13. Let V, W be R -vector spaces and $T \in SHom(V, W)$.

$T \in SSHom(V, W) \Leftrightarrow T(x + y) = Tx + Ty, \forall x, y \in V$.

Proof. Letting $a = b = 1$ in (2.12), then we have $T(x + y) = Tx + Ty$. Conversely, let $T(x + y) = Tx + Ty, \forall x, y \in V$. Then $T(ax + by) = T(ax) + T(by) = |a|Tx + |b|Ty$ (by lemma 2.9). $\square$

Theorem 2.14. Let V and W be R -vector spaces. Then $SSHom(V, W)$ is a subspace of $SHom(V, W)$.

Proof. Since $\tilde{0}(ax + by) = 0 = |a| \tilde{0}x + |b| \tilde{0}y = |a|0 + |b|0$, it is clear that $\tilde{0} \in SSHom(V, W)$. Let $T, S \in SSHom(V, W)$ and $x, y \in V$. $(T - S)(x + y) = (T + (-S))(x + y) = T(x + y) + (-S)(x + y) = Tx + Ty + (-S)x + (-S)y = (T - S)x + (T - S)y$. Thus $T - S \in SSHom(V, W)$. Finally, $(aT)(x + y) = |a|T(x + y) = |a|(Tx + Ty) = |a|Tx + |a|Ty = (aT)x + (aT)y$. Thus $(aT) \in SSHom(V, W)$. Hence, $SSHom(V, W)$ is a subspace of $SHom(V, W)$. $\square$

Remark 2.15. Observe that $T(ax) = |a|Tx$ and $(aT)(x) = |a|Tx$. Hence $T(ax) = (aT)(x)$.

Theorem 2.16. Let V and W be R -vector spaces with V having basis G^* . Let $L : G \rightarrow W$ be a map such that $L0 = 0$, where G is a group. Then L can be uniquely extended to an element of $SHom(V, W)$.

Proof. If $x \in V$, then $x = \sum_{g \in G^*} a_{x,g}g, a_{x,g} \in R$. Define $T : V \rightarrow W$ by

$$T(x) = \sum_{g \in G^*} |a_{x,g}| Lg, x \in V. \text{ Now let } c \in R, \text{ then } T(cx) = \sum_{g \in G^*} |a_{cx,g}| Lg =$$

$$\sum_{g \in G^*} |ca_{x,g}| Lg = \sum_{g \in G^*} |c| |a_{x,g}| Lg = |c| \sum_{g \in G^*} |a_{x,g}| Lg = |c| T(x). \text{ Thus,}$$

$T \in SHom(V, W)$. We claim that $Tg = Lg$ for all $g \in G^*$.

$$\text{For } x = 0, T0 = \sum_{g \in G^*} |a_{0,g}| Lg = |a_{0,0}| L0 + \sum_{gi \neq 0} |a_{0,gi}| Lgi = L0.$$

$$\text{For } x = g, Tg = \sum_{g \in G^*} |a_{x,g}| Lg = |a_{g,g}| Lg = 1.Lg = Lg. \text{ Thus, } Tg = Lg, \forall g \in$$

G . Therefore, T is an extension of L to an element of $SHom(V, W)$. Suppose S is an extension of L such that $S \in SHom(V, W)$. Since S is an extension of L $Sg = Lg, \forall g \in G$. Let $x \in V$, then $x = \sum_{g \in G^*} a_{x,g}g$. $S(x) = Sx = S(\sum_{g \in G^*} a_{x,g}g) =$

$$\sum_{g \in G^*} |a_{x,g}| Sg = \sum_{g \in G^*} |a_{x,g}| Lg = Tx. \text{ Thus } Sx = Tx, \forall x \in V. \text{ Hence the theorem.}$$

$\square$

Theorem 2.17. Let V, W be normed R -vector spaces and V has basis G^* . If B is complete up to cardinality of G^* , then $\|T\| = \sum_{g \in G^*} |Tg|$ defines a norm on $SHom(V, W)$ and hence $SHom(V, W)$ is a normed R -vector space.

Proof. Let $\tilde{0} \in SHom(V, W)$. Then $\tilde{0}(x) = 0, \forall x \in V$ and $\|\tilde{0}\| = \sum_{g \in G^*} |\tilde{0}g| =$

$\sum_{g \in G^*} |0| = 0$. Let $T \in SHom(V, W)$ and $\|T\| = 0$. Then $\sum_{g \in G^*} |Tg| = 0$. Hence

$|Tg| = 0, \forall g \in G^*$ and $T0 = 0$. Thus $Tg = 0 = \tilde{0}g, \forall g \in G$. Hence $Tx = 0 = \tilde{0}x, \forall x \in V$. Thus $T = \tilde{0}$. Let $a \in B$ and $T \in SHom(V, W)$.

Now, $\|aT\| = \sum_{g \in G^*} |(aT)g| = \sum_{g \in G^*} \|a\| |Tg| = \|a\| \sum_{g \in G^*} |Tg| = \|a\| \|T\|$. $\square$

Theorem 2.18. Let V, W be R -vector spaces and G^* be basis of V . If H is a subgroup of G and $S \in SHom(\langle H \rangle, W)$, then S can be extended to an element of $SHom(V, W)$.

Proof. Define $R : G \rightarrow W$ by $Rg = 0$ for $g \notin H$ and $Rg = Sg$ for $g \in H$. Since $S \in SHom(\langle H \rangle, W)$, $S0 = 0$. Hence $R0 = S0 = 0$. Thus $R : G \rightarrow W$ such that $R0 = 0$. By theorem 2.16, R can be extended to $T \in SHom(V, W)$. Let H^* is a basis of $\langle H \rangle$ and $x \in \langle H \rangle$. Then $x = a_1g_1 + \dots + a_ng_n$. Now $Tx = T(\sum_{g_i \in H^*} a_i g_i) = \sum_{g_i \in H^*} |a_i| Tg_i = \sum_{g_i \in H^*} |a_i| Rg_i = \sum_{g_i \in H^*} |a_i| Sg_i = S(\sum_{g_i \in H^*} a_i g_i) = Sx$. Where $1 \leq i \leq n$. Hence, T is an extension of S to an element of $SHom(V, W)$. $\square$

We conclude this paper by the following

Theorem 2.19. Let V and W be R -vector spaces over R . If V has a finite basis $G^* = \{x_1, \dots, x_n\}$, then $SHom(V, W)$ is isomorphic to $\sum_{i=1}^n W_i, W_i = W, \forall 1 \leq i \leq n$.

Proof. Let $y \in \sum_{i=1}^n W_i$. Then $y = (y_1, \dots, y_n), y_i \in W, \forall 1 \leq i \leq n$. Define

a mapping $\theta : SHom(V, W) \rightarrow \sum_{i=1}^n W_i$ by $\theta(T) = (Tx_1, \dots, Tx_n)$ for each

$T \in SHom(V, W)$. Let $T, S \in SHom(V, W)$ and $a \in R$. Now $\theta(S + T) = ((S + T)x_1, \dots, (S + T)x_n) = (Sx_1 + Tx_1, \dots, Sx_n + Tx_n) = (Sx_1, \dots, Sx_n) + (Tx_1, \dots, Tx_n) = \theta(S) + \theta(T)$. Similarly, $\theta(aT) = ((aT)x_1, \dots, (aT)x_n) = (|a|Tx_1, \dots, |a|Tx_n) = |a|(\theta(T))$. Hence, θ is a homomorphism.

Let $T, S \in SHom(V, W)$ such that $\theta(T) = \theta(S) \Leftrightarrow (Tx_1, \dots, Tx_n) = (Sx_1, \dots, Sx_n) \Leftrightarrow Tx_i = Sx_i, \forall 1 \leq i \leq n$. Also we have $S0 = 0 = T0$. So $Tx = Sx, \forall x$. Thus $T = S$.

Let $y \in \sum_{i=1}^n W_i$. Then $y = (y_1, \dots, y_n)$ for some $y_i \in W$. Let $\alpha : G \rightarrow W$ be the mapping defined by $\alpha 0 = 0$ and $\alpha x_i = y_i, i = 1, 2, 3, \dots, n$. Then by theorem 2.16, there is a unique extension $T : SHom(V, W) \rightarrow \sum_{i=1}^n W_i$ of α .

Consequently, $\theta(T) = (Tx_1, \dots, Tx_n) = (\alpha x_1, \dots, \alpha x_n) = (y_1, \dots, y_n)$. Hence, θ is an isomorphism of $SHom(V, W)$ on to $\sum_{i=1}^n W_i$. $\square$

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