Representation Type under Excellent Extensions

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Abstract

In this paper, we prove that representation type of finite dimensional algebras over an algebraically closed field is an invariant under excellent extensions.

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1. Introduction

In studying the algebraic structure of group rings, Passman in [9] introduced the notion of the excellent extensions of rings (the name comes from [1]). Such extensions of rings are vital since they include two important classes of extensions of rings: finite matrix rings and skew group rings AG , where the finite group G satisfies the condition $|G|^{-1} \in A$. Many authors have studied the invariant properties of rings under excellent extensions ([1, 5, 6, 9, 10, 11, 12]).

It is well known that determining the representation type of algebras is fundamental and important in representation theory of finite dimensional algebras. Let k be an algebraically closed field. Recall that a finite dimensional k -algebra A is said to be of *finite representation type* or a *representation finite algebra*, if there exist only finitely many isomorphism classes of finitely

generated indecomposable A -modules. Recall that a finite dimensional k -algebra A is said to be of *tame type* or a *tame algebra* if A is not of finite representation type, whereas for any dimension $d > 0$, there are a finite number of $k[T] - A$ -bimodules $\{M_i : i = 1, 2, \dots, n\}$, which are free as left $k[T]$ -modules such that all but a finite number of indecomposable A -modules of dimension d are isomorphic to $k[T]/(T - \lambda) \otimes_{k[T]} M_i$ for $\lambda \in k$. And, a finite dimensional k -algebra A is said to be of *wild type* or a *wild algebra* if there is a finitely generated $k < X, Y > - A$ -bimodule N , which is free as a left $k < X, Y >$ -module such that the functor $- \otimes_{k < X, Y >} N$ from $\text{mod } k < X, Y >$ to $\text{mod } A$ preserves indecomposability and reflects isomorphisms. The famous tame-and-wild theorem of Drozd's in [3, 4] states that *a finite dimensional k -algebra, which is not of finite representation type, is either of tame type or of wild type, and not both.* Therefore, it gives the classification of finite dimensional algebras over an algebraically closed field.

Let $B \geq A$ be an excellent extension of a finite dimensional k -module A . B is of finite representation type if and only if so is A by [6]. On the other hand, AG (G is a finite group such that $|G|^{-1} \in A$, where A is a finite dimensional k -algebra) and A have the same representation type by [8]. And S. Kasjan in [7] showed that A is a tame algebra if and only if so is $A \otimes_k F$, where F is a separable field over k . Based on these facts, it is natural to ask the following question:

Question: Let $B \geq A$ be an excellent extension of a finite dimensional k -algebra, do A and B have the same representation type?

In this paper, we will study the invariance of the representation type of finite dimensional k -algebras under excellent extensions and give a correct answer to this question. This paper is organized as follows.

In Section 2, we give some notations in our terminology and some preliminary results which are often used in this paper. In Section 3, we prove the following

Theorem 1.1 *Let B be an excellent extension of a finite dimensional k -algebra A , then A and B have the same representation type.*

2. Preliminaries

Throughout this paper, All modules are finitely generated right modules unless stated otherwise.

We begin with the definition of excellent extensions of finite dimensional k -algebras.

Definition 2.1. *Let A be a subalgebra of a finite dimensional k -algebra B , such that A and B have the same identity. Then B is called an algebraic extension of A , and denoted by $A \leq B$. An algebraic extension $A \leq B$ is called an excellent extension, if*

(1) $A \leq B$ is right A -projective ([9], p.273), that is, if N_B is a submodule of M_B and if N_A is a direct summand of M_A , denoted by $N_A | M_A$, then $N_B | M_B$.

(2) B is a free normalizing extension of A with a basis that includes 1; that is, there is a finite set $\{b_1, b_2, \dots, b_n\} \in B$ such that $b_1 = 1, B = b_1 A + \dots + b_n A$ and $b_i A = A b_i$, for each i , and B is free with basis $\{b_1, b_2, \dots, b_n\} \in B$ as both a right and left A -module.

Lemma 2.2. (See [[9], p.275, Lemma 2.3].) *Let A be a finite dimensional k -algebra, and let F be a finite separable field extension of k . Then $A \otimes_k F$ is an excellent extension of A .*

Lemma 2.3. (See [[6], Theorem 3.8].) *Let B be an excellent extension of a finite dimensional k -algebra A . If A is of finite representation type if and only if so is B .*

Lemma 2.4. (See [[11], Lemma 1.1].) *Let $A \leq B$ be an algebraic extension such that B is right A -projective. Then we have $M_B | (M \otimes_A B)_B$ for any B -module.*

Let A be a finite dimensional k -algebra. A category is called *generic category*, denoted by $GC(A)$, if its objects are all $k[T]$ - A -bimodules which are free as left $k[T]$ -modules, and its morphisms are all $k[T]$ - A -morphisms. It is closely related with tame algebras and thus with generic modules [2]. The following lemma are taken from [8], which plays an important role in this paper.

Lemma 2.5. *Let $X \in GC(A)$. Then X is indecomposable in $GC(A)$ if and only if $k[T]/(T-\lambda) \otimes_{k[T]} X$ is indecomposable as a $k[T]/(T-\lambda)$ - A -bimodule for $\lambda \in k$.*

3. Representation type

By the definition of excellent extensions, it is easy to prove the following

Lemma 3.1. *Let $B \geq A$ be an excellent extension. If A is a finite dimensional k -algebra, then so is B .*

Proposition 3.2 *Let B be an excellent extension of a finite dimension*

k -algebra A , then B is of tame type if and only if so is A .

Proof. Our proof is based on the definition of a tame algebra.

We first show the sufficiency of the condition. Let $B \geq A$ be an excellent extension and let A be a tame algebra. By Lemma 2.3, B is not of finite representation type. Let d be a positive integer and X an indecomposable right B -module with $\dim_k X = d$. By Lemma 2.4, there is an indecomposable direct summand Y_A of X_A such that $X_B|(Y \otimes_A B)_B$. Hence $\dim_k Y \leq d$.

Since that A is a tame algebra, there are a finite number of $k[T]$ - A -bimodules $\{M_j : j = 1, \dots, m\}$, which are free as left $k[T]$ -modules such that all but a finite number of indecomposable A -modules of dimension not bigger than d are isomorphic to $k[T]/(T-\lambda) \otimes_{k[T]} M_j$ for some $\lambda \in k$. No loss of generality, we may assume that $Y_A \cong k[T]/(T-\lambda) \otimes_{k[T]} M_t$ for some t and some $\lambda \in k$. It is easy to see that $M_t \otimes_A B$ is free as a left $k[T]$ -module from the same property of $(M_t)_A$, because B is a finite free left A -module. Hence $M_t \otimes_A B \in GC(B)$. Thus we have $M_t \otimes_A B = \bigoplus_{s=1}^{m_t} M_t^s$ for some positive integer m_t , where each M_t^s is indecomposable in $GC(B)$. By Lemma 2.5, $k[T]/(T-\lambda) \otimes_{k[T]} M_t^s$ is indecomposable as an $k[T]/(T-\lambda)$ - B -bimodule. That is, $k[T]/(T-\lambda) \otimes_{k[T]} M_t^s$ is an indecomposable right B -module because $k[T]/(T-\lambda) \cong k$ as algebras.

Notice that $Y \otimes_A B \cong (k[T]/(T-\lambda) \otimes_{k[T]} M_t) \otimes_A B \cong \bigoplus_{s=1}^{m_t} ((k[T]/(T-\lambda) \otimes_{k[T]} M_t^s)$. We have $X_B \cong k[T]/(T-\lambda) \otimes_{k[T]} M_t^i$ for some i . Because the set $\{M_j^i : j = 1, \dots, m; i = 1, \dots, m_j\}$ is finite, B is a tame algebra.

Conversely, assume that B is a tame algebra. Then A is not of finite representation type by Lemma 2.3 again. Let d be a positive integer and X' an indecomposable right A -module with $\dim_k X' = d$, there is an indecomposable direct summand Y'_B of $X' \otimes_A B$ such that $X'_A|Y'_A$.

Notice that B is a tame algebra, there are a finite number of $k[T]$ - B -bimodules $\{N_j : j = 1, 2, \dots, n\}$, which are free as left $k[T]$ -modules such that all but a finite number of indecomposable B -modules of dimension not bigger than d are isomorphic to $k[T]/(T-\lambda) \otimes_{k[T]} N_j$ for some j and some $\lambda \in k$. To prove that A is a tame algebra, it is no harm to assuming that $Y'_B \cong k[T]/(T-\lambda) \otimes_{k[T]} (N_t)_B$ for some t and some $\lambda \in k$. It is easy to see that $(N_t)_A$ is free as a left $k[T]$ -module from the same property of $(N_t)_B$. Hence, $(N_t)_A \in GC(A)$. We have $(N_t)_A = \bigoplus_{s=1}^{m_t} N_t^s$ for some positive integer m_t , where each N_t^s are indecomposable in $GC(A)$. By Lemma 2.5, $X'_A \cong k[T]/(T-\lambda) \otimes_{k[T]} N_t^s$ is indecomposable as an $k[T]/(T-\lambda)$ - A -bimodule. It is equivalent to saying that $X'_A \cong k[T]/(T-\lambda) \otimes_{k[T]} N_t^s$ is an indecomposable A -module because $k[T]/(T-$

$\lambda) \cong k$ as algebras. Since the set $\{N_j^i : j = 1, \dots, n; i = 1, \dots, m_j\}$ are finite, and so A is a tame algebra.

According to the Drozd's tame-and-wild theorem and Lemma 2.3 and Proposition 3.1, we obtain our main result.

Theorem 3.3 *Let B be an excellent extension of A , then A and B have the same representation type.*

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