International Mathematical Forum, Vol. 9, 2014, no. 18, 857 - 869
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.4480

Natural Lifts and Curvatures, Arc-Lengths of the Spherical Indicatries of the Evolute Curve in E^3

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Abstract

In this paper, we investigated the natural lifts and curvatures of the spherical indicatries of the evolute in E^3. Firstly, it is shown that the Darboux vektor of evolute curve and binormal vektor of the evolute curve are linearly dependent, secondly the relations among the geodesic curvatures and arc-lengths with respect to $E^3$and S^2 of the fixed pole curve (C^*) and the spherical indicartix curves $(T^*), (N^*), (B^*)$ generated by the unit vektor (C^*) on S^2 have been obtained and finally, the condition being the natural lifts of the spherical indicatrix of the evolute curve (α^*) are an integral curve of geodesic spray has expressed.

Mathematics Subject Classification: 53A04

Keywords: Evolute curve, Natural lift of spherical indicatrix curve, Geodesic spray, Geodesic curvatures

1 Introduction and Preliminary notes

We now review some basic concepts on classical differantial geometry of space curves and surfaces in three dimensional Euclidean space E^3.For any two vectors $X = \{x_{1,}x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, we denote $\langle X, Y \rangle = \sum x_i y_i$ as standart inner product. Let $\alpha : I \subset IR \longrightarrow E^3$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be a curve with $\dot{\alpha}(s) \neq \overrightarrow{0}$. We also denote the norm of X by $\|X\|$. The arc-length of a curve a measured from $\alpha(s_0)$, $s_0 \in I$ is

$$s = \int_{s_0}^{s} \left\| \dot{\alpha}(s) \right\| du$$

We say that a curve α is parametrical by the arc-length if it satisfies $\left\|\alpha^{'}(s)\right\| = 1$. Throughout this paper we denote the arc-length of space curves. Let us denote $T(s) = \alpha^{'}(s)$ and we call a unit tangent vector of α at s . We define curvature of α by $\kappa(s) = \left\|\alpha^{''}(s)\right\|$. If $\kappa(s) \neq 0$, then the unit principal normal vector $N(s)$ of the curve α by $\alpha^{''}(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called a unit binormal vector of the curve α at s. Then the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame of the curve α at the point $\alpha(s)$ Then the following Serret-Frenet formula holds:

$$\begin{aligned} T^{'}(s) &= \kappa(s)N(s) \\ N^{'}(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ B^{'}(s) &= -\tau(s)N(s) \end{aligned}$$

where $\tau(s)$ is the torsion of the curve α.For any unit speed curve $\alpha : I \to E^3$, the vector W is called Darboux vector defined by

$$W = \tau(s)T(s) + \kappa(s)B(s) \tag{1}$$

If we consider the normalization of the Darboux $C = \frac{1}{\|W\|}W$, we have

$$\cos\phi = \frac{\kappa(s)}{\|W\|} \qquad \sin\phi = \frac{\tau(s)}{\|W\|}$$

and

$$C = \sin\phi T(s) + \cos\phi B(s)$$

where $\measuredangle(W, B) = \phi$.

$$\kappa(t) = \frac{\left\|\alpha^{'}(t) \times \alpha^{''}(t)\right\|}{\left\|\alpha^{'}(t)\right\|^3}, \quad \tau(t) = \frac{\det(\alpha^{'}(t), \alpha^{''}(t), \alpha^{'''}(t))}{\left\|\alpha^{'}(t) \times \alpha^{''}(t)\right\|^2} \tag{2}$$

Furtermore, a curve $\alpha : I \to E^3$ with $\kappa(s) \neq 0$ is called a cylindrical helix if the tangent line of α makes a constant angle with a fixed direction. It has been known that the curve $\alpha(s)$ is a cylindrical helix if and only if $\frac{\tau(s)}{\kappa(s)}$ =constant. If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is, of course,cylindrical helix. We call it a circular helix. Let M be a surface in E^3 and the curve $\alpha : I \to M$ be an unit speed.Then, the curve $\alpha : I \to M$ is called an integral curve of the vector field X if

$$\frac{d\alpha(s)}{ds} = X(\alpha(s)). \tag{3}$$

The curve $\overline{\alpha} : I \to TM$, $\overline{\alpha}(s) = (\alpha(s), \alpha^{'}(s))$ is called the natural lift of the curve $\alpha : I \to E^3$, where $TM = \cup_{P\in M} T_M P$ [1] Then, for all $X \in TM$ the map $S : TM \to TM$ is called Weingarten map (or shape operator) defined by

$$S(X) = D_X N$$

where D is affine connection on M. Accordingly, for all $V \in TM$ the vector field X is called geodesic spray on TM defined by [1],

$$D_X V = -\langle V, S(V)\rangle N_P, \ P \in M$$

Then, the equation as know Gauss equation is

$$\overline{D}_X Y = D_X Y + \langle S(X), Y\rangle N \tag{4}$$

where $N \in TM^{\perp}$. Here $\overline{D}$ become a covariant derivative operator sense of Gauss of a Riemannian connection on M. Let T be the unit tangent vector of the curve α on M. The curve α is called a geodesic curve in E^3 satisfied by

$$D_T T = 0.$$

Similarly, the curve α is called a geodesic curve in M satisfied by

$$\overline{D}_T T = 0.$$

Then, we know that the geodesic curvature of t9he curve α on M defined by

$$k_g = \left\|\overline{D}_T T\right\|$$

where T is the unit tangent vector of the curve α on M. The geodesic curvature of the curve α on M at $P \in M$ is the vector projection of the curvature vector k of α at P onto the tangent plane $T_M(P)$ .Similarly, the geodesic curvature of the curve α in E^3 defined by

$$\gamma_g = \|D_T T\|.$$

Let us consider the Frenet frame $\{T, N, B\}$ and the vector C. Accordingly, the arc-lengths and the geodesic curvatures of the spherical indicatrix curves

$(T),(N)$ and (B) with the fixed pole curve (C) with respect to E^3, respectively, generated by the vectors T,N and B with the vector C on S^2 are as follows:

$$\begin{cases} s_T = \int_0^s \kappa ds \\ s_N = \int_0^s \|W\| ds \\ s_B = \int_0^s \tau ds \\ s_C = \int_0^s \phi' ds \end{cases} \tag{5}$$

$$\begin{cases} k_T = \int_0^s \frac{1}{\cos\phi} ds \\ k_N = \int_0^s [1 + (\frac{\phi'}{\|W\|})^2]^{\frac{1}{2}} ds \\ k_B = \int_0^s \frac{1}{\sin\phi} ds \\ k_C = \int_0^s [1 + (\frac{\|W\|}{\phi'})^2]^{\frac{1}{2}} ds \end{cases} . \tag{6}$$

Similarly, the geodesic curvatures of the geodesic curvatures of the spherical indicatrix curves $(T),(N)$ and (B) with the fixed pole curve (C) with respect to S^2, respectively, generated by the vectors T,N and B with the vector C on S^2 are as follows:

$$\begin{cases} \gamma_T = \int_0^s \tan\phi ds \\ \gamma_N = \int_0^s \frac{\phi'}{\|W\|} ds \\ \gamma_B = \int_0^s \cot\phi ds \\ \gamma_C = \int_0^s \frac{\|W\|}{\phi'} ds \end{cases} \tag{7}$$

[4] thus we can give the following theorem:

Theorem 1.1 *Let M be a surface in E^3 and the curve $\alpha : I \to M$ be a unit speed. The natural lift $\overline{\alpha} : I \to TM$ is an integral curve of the geodesic spray X if and only if the natural lift $\overline{\alpha}$ is a geodesic on M .*

Result of this theorem, the following corollaries can be given.

Corollary 1.2 *If the curve α is a unit circle, then the tangents indicatrix of the curve α is a great circle on the sphere S^2. Then, the natural lift $(\overline{T})$ is an integral curve of the geodesic spray on the tangent bundle $T(S^2)$.*

Corollary 1.3 *If the curve α is a circular helix, then the principal normals indicatrix of the curve α is a great circle on the sphere S^2 . Then, the natural lift $(\overline{N})$ is an integral curve of the geodesic spray on the tangent bundle $T(S^2)$.*

Corollary 1.4 *There is not exist any curve such that the binormals indicatrix (B) is a great circle on the sphere S^2. Then, the natural lift $(\overline{B})$ is not an integral curve of the geodesic spray on the tangent bundle $T(S^2)$.*

Corollary 1.5 *If the curve α is a helix, then the fixed pole curve (C) is a great circle on the sphere S^2. Then, the natural lift $(\overline{C})$ is an integral curve of the geodesic spray on the tangent bundle $T(S^2)$.*

Definition 1.6 *Let $\alpha : I \to E^3$ and $\alpha^* : I \to E^3$be the C^2-class differantiable unit speed two curves of class C^2 and let $\{T(s), N(s), B(s)\}$and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^* , respectively. For $\forall s \in I$, if tangent of curve α^* at point of $\alpha^*(s)$ pass through point of $\alpha(s)$ and if $\langle T^*(s), T(s)\rangle = 0$, curve α^* is called a evolute of curve α.*

The relations between the Frenet frames $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ are as follows:

$$\begin{cases} T^* = \cos(\varphi + c)N - \sin(\varphi + c)B \\ N^* = -T \\ B^* = \sin(\varphi + c)N + \cos(\varphi + c)B \end{cases} \tag{8}$$

Additionally,the relation between curvatures and the torsions are:

$$\begin{cases} \kappa^* = \frac{\kappa^3 \cos^3(\varphi+c)}{\kappa\tau \sin(\varphi+c) - \kappa' \cos(\varphi+c)} \\ \\ \tau^* = \frac{-\kappa^3 \sin(\varphi+c)\cos^2(\varphi+c)}{\kappa\tau \sin(\varphi+c) - \kappa' \cos(\varphi+c)} \end{cases} \tag{9}$$

where $\measuredangle(\alpha^*(s) - \alpha(s), N(s)) = \varphi(s) + c$ and for $c \in IR$

$$\varphi(s) = \int_0^s \tau(u)du$$

and

$$\alpha^*(s) = \alpha(s) + \rho(s)N(s) - \rho(s)\tan[\varphi(s) + c]B(s) \quad [3], [6].$$

2 Natural Lifts and Curvatures of the Spherical Indicatrices of the Evolute Curve

Theorem 2.1 *Let α^* be evolute of curve α .There are following relation between B binormal vector of curve α at point $\alpha(s)$ and W^* Darboux vector of curve α^* at point $\alpha^*(s)$.*

$$W^* = \frac{\kappa^3 \cos(\varphi + c)}{\kappa\tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)} B \tag{10}$$

Proof 2.2 *Substituting* (8) *and* (9) *into* $W^* = \tau^* T + \kappa^* B^*$ *we get*

$$W^* = \frac{\kappa^3 \cos(\varphi + c)}{\kappa\tau \sin(\varphi + c) - \kappa' \cos(\varphi + c)} B$$

Corollary 2.3 *Let α^* be evolute of curve α .The unit vector C^* in direction of the Darboux vector W of curve α^* and binormal vector B of curve α are linearly dependent on.*

$$C^* = B \tag{11}$$

Proof 2.4 *Substituting* (8) *and* (9) *into* $C^* = \sin\phi^* T^* + \cos\phi^* B^*$ *and rearranging we get*

$$C^* = B.$$

Theorem 2.5 *Let α^* be evolute of curve α .The angle in between Darboux vector W^* with binormal vector B^* of curve α^* is ϕ^* and there is following connection with each other.*

$$\begin{cases} \sin\phi^* = -\sin(\varphi + c) \\ \\ \cos\phi^* = \cos(\varphi + c). \end{cases} \tag{12}$$

Proof 2.6 *Writing numerical values at* (9) *into* $\sin\phi^* = \frac{\tau^*}{\|W^*\|}$ *and* $cos\phi^* = \frac{\kappa^*}{\|W^*\|}$, *with calculation we obtain*

$$\sin\phi^* = -\sin(\varphi + c)$$

$$\cos\phi^* = \cos(\varphi + c)$$

Now, let compute the arc-lengths with the geodesic curvatures of spherical indicatrix curves with the $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to E^3 and S^2 : Primarly, by the (5), for the arc-length s_{T^*} of tangents indicatrix (T^*) of the evolute curve α^* we can write

$$s_{T^*} = \int_0^s \left\| \frac{dT^*}{ds} \right\| ds.$$

$$T^* = \cos(\varphi + c)N - \sin(\varphi + c)B$$

differentiating we get

$$\frac{dT^*}{ds} = -\kappa \cos(\varphi + c)T + (\tau - \varphi^{'}) \sin(\varphi + c)N + (\tau - \varphi^{'}) \cos(\varphi + c)B.$$

$\varphi^{'} = \tau$ and normalizing we obtain

$$s_{T^*} = \int_0^s \kappa \cos(\varphi + c) ds. \tag{13}$$

For the arc-length s_{N^*} of principal normals indicatrix (N^*)

$$s_{N^*} = \int_0^s \left\| \frac{dN^*}{ds} \right\| ds.$$

Compiting, we get

$$s_{N^*} = \int_0^s \kappa ds \tag{14}$$

Similarly, for the arc-length s_{B^*} of binormals indicatrix (B^*) of the evolute curve α^* we can write

$$s_{B^*} = \int_0^s \left\| \frac{dB^*}{ds} \right\| ds.$$

$$s_{B^*} = \int_0^s \kappa \sin(\varphi + c) ds. \tag{15}$$

Finally, for the arc-length s_{C^*} of the fixed pole curve (C^*) we can write

$$s_{C^*} = \int_0^s \left\| \frac{dC^*}{ds} \right\| ds.$$

$$s_{C^*} = \int_0^s \tau ds \tag{16}$$

Corollary 2.7 *Let α^* be evolute of curve α in E^3 and $\{T^*, N^*, B^*\}$ be Frenet-frame of the evolute curve α^* at $\alpha^*(s)$. For the arc-lengths of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to E^3we have*

$$1)\ s_{T^*} = \int_0^s \kappa \cos(\varphi + c) ds$$

$$2)\ s_{N^*} = \int_0^s \kappa ds$$

$$3)\ s_{B^*} = \int_0^s \kappa \sin(\varphi + c) ds$$

$$4)\ s_{C^*} = \int_0^s \tau ds$$

Now let us compute the geodesic curvatures of spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to E^3:

For the geodesic curvature k_{T^*} of the tangents indicatrix curve (T^*) of the evolute curve α^* we can write

$$k_{T^*} = \|D_{T_{T^*}} T_{T^*}\|$$

Differentiating the curve $\alpha_{T^*}(s_{T^*}) = T^*(s^*)$ with the respect to s_{T^*} and normalizing we obtain

$$T_{T^*} = -T$$

Computing the vector $D_{T_{T^*}} T_{T^*}$ we get

$$D_{T_{T^*}} T_{T^*} = -\kappa N . \frac{1}{\kappa \cos(\varphi + c)} = - \sec(\varphi + c) N$$

$$k_{T^*} = \sec(\varphi + c) \tag{17}$$

Similarly, we have

$$k_{N^*} = \frac{\|W\|}{\kappa} \tag{18}$$

$$k_{B^*} = \cos ec(\varphi + c) \tag{19}$$

$$k_{C^*} = \frac{\|W\|}{\tau}. \tag{20}$$

Then the following corollary can be given.

Corollary 2.8 *Let α^* be evolute of curve α in E^3 and $\{T^*, N^*, B^*\}$ be Frenet-frame of the evolute curve α^* at $\alpha^*(s)$. For the geodesic curvatures of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to E^3 we have*

$$1)\quad k_{T^*} = \sec(\varphi + c)$$

$$2)\quad k_{N^*} = \frac{\|W\|}{\kappa}$$

$$3)\quad k_{B^*} = cosec(\varphi + c)$$

$$4)\quad k_{C^*} = \frac{\|W\|}{\tau}.$$

Now let us compute the geodesic curvatures of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to S^2:

For the geodesic curvature γ_{T^*} of the tangents indicatrix curve (T^*) of the evolute curve α^* we can write

$$\gamma_{T^*} = \left\|\overline{D}_{T_{T^*}} T_{T^*}\right\|$$

By (4), (8) and (17) we obtain

$$\overline{D}_{T_{T^*}} T_{T^*} = \left(\frac{\sin^2(\varphi + c)}{\cos(\varphi + c)}\right) N - \sin(\varphi + c) B$$

Normalizing

$$\gamma_{T^*} = \tan(\varphi + c) \tag{21}$$

If the curve $\left(\overline{T^*}\right)$ is an integral curve of the geodesic spray then $\overline{D}_{T_{T^*}} T_{T^*} = 0$. Thus $\sin(\varphi + c) = 0$. So, we can give the following corollary.

Corollary 2.9 *If the tangent vector of evolute curve α^* is equal principal normal of curve α, then the natural lift $\left(\overline{T^*}\right)$ of the tangent indicatrix (T^*) is an integral curves of the geodesic spray on tangent bundle $T(S^2)$.*

For the geodesic curvature γ_{N^*} of the principal normals indicatrix curve (N^*) of the evolute curve α^* with respect to S^2 we can write

$$\gamma_{N^*} = \left\|\overline{D}_{T_{N^*}} T_{N^*}\right\|.$$

By direct calculation, the vector $\overline{D}_{T_{N^*}}T_{N^*}$ is obtain as

$$\overline{D}_{T_{N^*}}T_{N^*} = -\frac{\tau}{\kappa}B$$

Normalizing

$$\gamma_{N^*} = \frac{\tau}{\kappa} \tag{22}$$

If the curve $\left(\overline{N^*}\right)$ is an integral curve of the geodesic spray then $\overline{D}_{T_{N^*}}T_{N^*} = 0$. From here, we get $\tau = 0$ and $\kappa = 0$. So, we can give the following corollary.

Corollary 2.10 *If curve α^* is evolute of a planary curve α, then the natural lift $\left(\overline{N^*}\right)$ of the principal normals indicatrix (N^*) is an integral curves of the geodesic spray on tangent bundle $T(S^2)$(see example 1).*

For the geodesic curvature γ_{B^*} of the binormals indicatrix curve (B^*) of the evolute curve α^* we can write

$$\gamma_{B^*} = \left\|\overline{D}_{T_{B^*}}T_{B^*}\right\|.$$

$$\overline{D}_{T_{B^*}}T_{B^*} = \left(\frac{\cos^2(\varphi + c)}{\sin(\varphi + c)}\right)N + \cos(\varphi + c)B$$

Normalizing

$$\gamma_{B^*} = \cot(\varphi + c) \tag{23}$$

If the curve $\left(\overline{B^*}\right)$ is an integral curve of the geodesic spray then $\overline{D}_{T_{B^*}}T_{B^*} = 0$. Thus $\cos(\varphi + c) = 0$. So, we can give the following corollary.

Corollary 2.11 *If the tangent vector of evolute curve α^* is equal binormal vector of curve α, the natural lift $\left(\overline{B^*}\right)$ of the binormals indicatrix (B^*) is an integral curves of the geodesic spray on tangent bundle $T(S^2)$.*

For the geodesic curvature γ_{C^*} of the fixed pole curve (C^*) of the evolute curve α^* with respect to S^2 we can write

$$\gamma_{C^*} = \left\|\overline{D}_{T_{C^*}}T_{C^*}\right\|.$$

By direct calculation, the vector $\overline{D}_{T_{C^*}}T_{C^*}$ is obtain as

$$\overline{D}_{T_{C^*}}T_{C^*} = \frac{\kappa}{\tau}T$$

Normalizing

$$\gamma_{C^*} = \frac{\kappa}{\tau} \tag{24}$$

If the curve $\left(\overline{C^*}\right)$ is an integral curve of the geodesic spray then $\overline{D}_{T_{C^*}}T_{C^*} = 0$. From here, we get $\kappa = 0$ and $\tau \neq 0$. So, we can give the following corollary.

Corollary 2.12 *There is no a evolute of curve α^* is such that the fixed pole (C^*) is a great circle on the unit sphere S^2. Hence the natural lift $\left(\overline{C^*}\right)$ of the fixed pole (C^*) is never an integral curves of the geodesic spray on tangent bundle $T(S^2)$.*

Corollary 2.13 *Let α^* be evolute of curve α in E^3 and $\{T^*, N^*, B^*\}$ be Frenet-frame of the evolute curve α^* at $\alpha^*(s)$. For the geodesic curvatures of the spherical indicatrix curves $(T^*), (N^*)$ and (B^*) with the fixed pole curve (C^*) with the respect to S^2 we have*

$$1)\quad \gamma_{T^*} = \tan(\varphi + c)$$

$$2)\quad \gamma_{N^*} = \frac{\tau}{\kappa}$$

$$3)\quad \gamma_{B^*} = \cot(\varphi + c)$$

$$4)\quad \gamma_{C^*} = \frac{\kappa}{\tau}\quad .$$

Example 2.14 *The evolute of a planary curve $\alpha : I \to E^2$, $\alpha(t) = (t^2, t)$ is $\alpha^* : I \to E^2, \alpha^*(t) = \left(\frac{1}{2}(1 + 6t^2), -4t^3\right)$ semi-cubic parabola. The principal normals indicatrix of the curve α^* is $N^*(t) = -\frac{(2t,1)}{\sqrt{1+4t^2}}$ and the natural lift $\overline{N^*}(t) = \frac{(-2,4t)}{(4t^2+1)^{\frac{3}{2}}}$. Then the natural lift $\left(\overline{N^*}\right)$ of the principal normals indicatrix (N^*) is an integral curves of the geodesic spray*

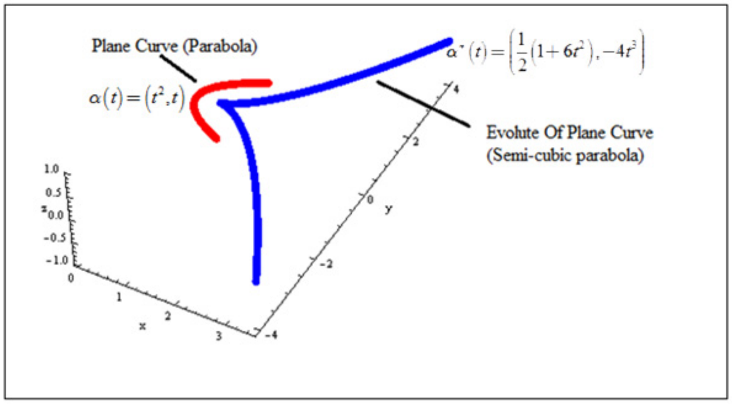

Figure 1: Evolute of Plane Curve

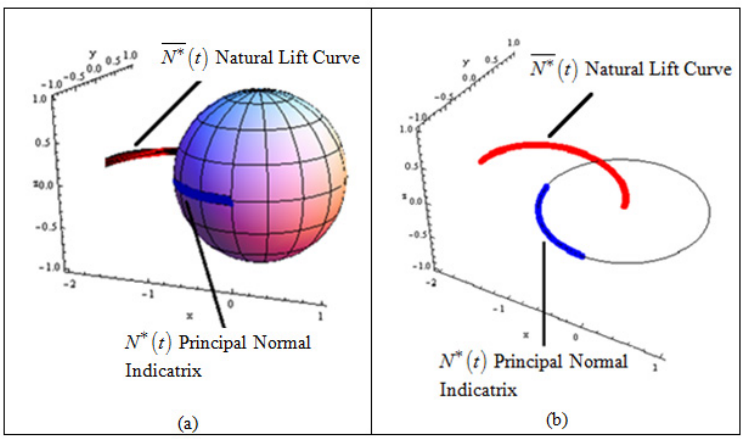

Figure 2: (a), (b): Principal Normal Indicatrix and Its Natural Lift

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Received: April 27, 2014