On Derivations of Semi-Girard Quantales

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Abstract

In this paper we will define semi-Girard quantales whose properties are weaker than the properties of Girard quantales. We will give an example which is a semi-Girard quantale but not a Girard quantale. We will show that the right(or left) divisor of the dualizing element by itself is the identity of a semi-Girard Quantale. We will define some derivations on a semi-Girard quantale whose dualizing element is the bottom element 0, and investigate some properties of such derivations.

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1 Introduction

Quantales were introduced by C.J. Mulvey [9] in 1986 with the purpose of studying the spectrum of C^* -algebras and the foundations of quantum mechanics. The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth [4, 12] in the 1930s. It has become a

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useful tool in studying noncommutative topology, linear logic and C*-algebra theory [1, 2, 5]. Following Mulvey, various types and aspects of quantales have been considered by many researchers [7, 8, 13]. The importance of quantales for linear logic is revealed in Yetter's work [14]. Yetter has clarified the use of quantales in linear logic and he has introduced the term "Girard quantale". In [10], J. Paseka and D. Kruml have shown that any quantale can be embedded into a unital quantale. In [11], K.I. Rosenthal has proved that every unital quantale can be embedded into a Girard quantale.

In this paper we will define semi-Girard quantales whose properties are weaker than the properties of Girard quantales. Of course a Girard quantale is a semi-Girard quantale. We will give an example which is a semi-Girard quantale but not a Girard quantale. We will show that the right(or left) divisor of the dualizing element by itself is the identity of a semi-Girard Quantale. We will define some derivations on a semi-Girard quantale whose dualizing element is the bottom element 0, and investigate some properties of such derivations.

2 Preliminary Notes

Definition 2.1. [3, 6, 9] A *quantale* is an algebraic structure $Q = \langle Q, \vee, \otimes, 0 \rangle$ such that

(Q1) $\langle Q, \vee, 0 \rangle$ is sup-lattice,

(Q2) $\langle Q, \otimes \rangle$ is a semigroup,

(Q3) $x \otimes \bigvee y_i = \bigvee (x \otimes y_i)$ and $(\bigvee y_i) \otimes x = \bigvee (y_i \otimes x)$ for all $x \in Q$, $\{y_i\} \subseteq Q$, $i \in I$.

By the property of (Q1), every quantale has the smallest element and the greatest element. In this paper, the smallest element in a quantale is denoted by 0 and the greatest element in a quantale is denoted by 1. A quantale Q is said to be unital if there exist the identity $e \in Q$ for the multiplication such that $e \otimes x = x \otimes e = x$ for every $x \in Q$. A quantale Q is said to be commutative if $a \otimes b = b \otimes a$ for all $a, b \in Q$. Since $\langle Q, \vee, 0 \rangle$ is a sup-lattice, it is possible to define the following operations

$$x \backslash y = \bigvee \{z \in Q \mid x \otimes z \leq y\},$$

$$x / y = \bigvee \{z \in Q \mid z \otimes x \leq y\}.$$

The meaning of $x \backslash y$ is almost same to the right divisor of y by x (divided in left by x). So we call $x \backslash y$ a right divisor of y by x . (Some others call it a left division.) Obviously, if Q is commutative then the right divisor and the left divisor coincide $x \backslash y = x / y$. The followings are well known facts.

Proposition 2.2. [6, 11] *Let Q be a quantale. Then, for any $x, y, z \in Q$ and $\{y_i\}_{i \in I} \subseteq Q$, the followings are always true.*

- (1) $x \otimes 0 = 0 \otimes x = 0$,
- (2) if $x \leq y$ then $x \otimes z \leq y \otimes z$ and $z \otimes x \leq z \otimes y$,
- (3) if $x \leq y$ then $z \backslash x \leq z \backslash y$ and $z / x \leq z / y$, $y \backslash z \leq x \backslash z$ and $y / z \leq x / z$
- (4) $(x / y) \otimes x \leq y$ and $x \otimes (x \backslash y) \leq y$,
- (5) $(\bigvee y_i) / x = \bigwedge (y_i / x)$ and $(\bigvee y_i) \backslash x = \bigwedge (y_i \backslash x)$, $i \in I$
- (6) $x / (\bigwedge y_i) = \bigwedge (x / y_i)$, $x \backslash (\bigwedge y_i) = \bigwedge (x \backslash y_i)$, $i \in I$
- (7) $y \backslash (x \backslash z) = x \otimes y \backslash z$ and $x / (y / z) = x \otimes y / z$,
- (8) $x \backslash (y / z) = y / (x \backslash z)$,
- (9) $(x \backslash y) \otimes (y \backslash z) \leq x \backslash z$,
- (10) $x \backslash y \leq (y \backslash z) / (x \backslash z)$.

Moreover if Q is a unital quantale with e , then

- (11) $e / x = e \backslash x = x$.

We introduce the notion of Semi-Girard quantale and investigate several properties.

Definition 2.3. [11] A quantale Q is said to be a Girard quantale if Q has an element t having the following properties.

- (1) For all $x \in Q$, $(x / t) \backslash t = (x \backslash t) / t = x$,
- (2) For all $x \in Q$, $x \backslash t = x / t$.

An element which has the property (1) is called *dualizing element* and an element which has the property (2) is called *cyclic*. So a Girard quantale is a quantale which has a cyclic dualizing element. Here we have a question what is the structure of a quantale which has a dualizing element being not cyclic.

We will give the definition of a semi-Girard quantale as the following.

Definition 2.4. A quantale Q is called a semi-Girard quantale if there exists a dualizing element $t \in Q$ such that $(x \backslash t) / t = (x / t) \backslash t = x$ for all $x \in Q$.

Clearly a Girard quantale is a semi-Girard quantale. Here is an example which is a semi-Girard quantale but not a Girard quantale.

Example 2.5. Let $Q = \{0, e, f, d, c, a, b, 1\}$, the partial order on Q as $0 < e < d < c < a < 1$ and $0 < f < d < c < b < 1$ (i.e $e \wedge f = 0$, $e \vee f = d$ and $a \wedge b = 0$, $a \vee b = 1$) and the operator $\otimes$ on Q be defined as follows :

$\otimes$	0	e	f	d	c	a	b	1
0	0	0	0	0	0	0	0	0
e	0	0	0	0	0	0	e	e
f	0	0	0	0	0	f	0	f
d	0	0	0	0	0	f	e	d
c	0	0	0	0	c	c	c	c
a	0	e	0	e	c	a	c	a
b	0	0	f	f	c	c	b	b
1	0	e	f	d	c	a	b	1

It is easy to check that Q is semi-Girard quantale. But Q is not Girard quantale because of $e \setminus 0 = a \neq b = e/0$.

Lemma 2.6. *Let Q be a semi-Girard quantale with its dualizing element as t . Then, for any $x, y, z \in Q$,*

$$(1) \{x \otimes (y \setminus t)\} / t = x / y,$$

$$(2) \{(y/t) \otimes x\} \setminus t = x \setminus y,$$

$$(3) x \otimes y = \{x / (y/t)\} \setminus t,$$

$$(4) y \otimes x = \{x \setminus (y \setminus t)\} / t.$$

Proof. (1). $z \leq \{x \otimes (y \setminus t)\} / t \Leftrightarrow z \otimes \{x \otimes (y \setminus t)\} = t \Leftrightarrow (z \otimes x) \otimes (y \setminus t) = t \Leftrightarrow (z \otimes x) \leq (y \setminus t) / t = y$. Therefore $z \leq x / y$. (2). $z \leq \{(y/t) \otimes x\} \setminus t \Leftrightarrow \{(y/t) \otimes x\} \otimes z \leq t \Leftrightarrow (y/t) \otimes \{x \otimes z\} \leq t \Leftrightarrow (x \otimes z) \leq (y/t) \setminus t = y$. Therefore $z \leq x \setminus y$. (3). By (1) and $y = (y/t) \setminus t$, $x \otimes y = \{x \otimes \{(y/t) \setminus t\} / t\} \setminus t = \{x / (y/t)\} \setminus t$. (4). By (2) and $y = (y/t) \setminus t$, $y \otimes x = \{\{(y/t) \setminus t\} \otimes x\} \setminus t / t = \{x \setminus (y \setminus t)\} / t$. $\square$

Theorem 2.7. *If Q is a semi-Girard quantale with its dualizing element t , then $t \setminus t$ is the right identity and t/t is the left identity of Q . Thus $t \setminus t = t/t$ is the identity of Q .*

Proof. By lemma 3.3 (3), $x \otimes (t \setminus t) = \{x / \{(t \setminus t) / t\}\} \setminus t = (x/t) \setminus t = x$ and $(t/t) \otimes x = \{x \setminus \{(t/t)\} \setminus t\} / t = (x \setminus t) / t = x$ $\square$

Corollary 2.8. *Let Q be a semi-Girard quantale with its dualizing element 0. Then*

$$(1) 1 \setminus x = 1/x = x,$$

(2) $x \leq y$ if and only if $x \setminus y = 1$ if and only if $x/y = 1$.

Proof. (1) Since $1 \setminus 0 = 0$, we have $0/0 = 1$. Thus $\forall t \in Q, t \leq 1 \setminus x \Leftrightarrow 1 \otimes t \leq x \Leftrightarrow t \leq x$. Therefore $1 \setminus x = x$. (2) $x = x \otimes 1 \leq y \Leftrightarrow 1 \leq x \setminus y$ (i.e. $x \setminus y = 1$), $x = 1 \otimes x \leq y \Leftrightarrow 1 \leq x/y$ (i.e. $x/y = 1$) $\square$

Theorem 2.9. Let Q be a semi-Girard quantale and $x, y, z \in Q$. Then

$$(1) x \setminus y = (y \setminus t)/(x \setminus t),$$

$$(2) x/y = (y/t) \setminus (x/t).$$

Proof. (1) $\forall t \in Q, t \leq x \setminus y \Leftrightarrow x \otimes t \leq (y \setminus t)/t \Leftrightarrow x \otimes t \otimes (y \setminus t) \leq t \Leftrightarrow t \otimes (y \setminus t) \leq x \setminus t \Leftrightarrow t \leq (y \setminus t)/(x \setminus t)$. Therefore $x \setminus y = (y \setminus t)/(x \setminus t)$. (2) Similarly to (1). $\square$

3 On derivations of semi-Girard Quantales

In this section we will investigate some properties of some derivations on a semi-Girard quantale whose dualizing element is the bottom element 0.

So we assume that a quantale Q in this section is a semi-Girard quantale with the dualizing element as 0.

Definition 3.1. Let Q be a semi-Girard quantale. A map $\delta : Q \rightarrow Q$ is a $(\setminus, \vee)$ -derivation of Q if it satisfies the identity

$$\delta(x \setminus y) = (\delta(x) \setminus y) \vee (x \setminus \delta(y))$$

for all $x, y \in Q$.

And A map $\delta : Q \rightarrow Q$ is a $(/, \vee)$ -derivation of Q if it satisfies the identity

$$\delta(x/y) = (\delta(x)/y) \vee (x/\delta(y))$$

for all $x, y \in Q$.

In this paper, we will call $(\setminus, \vee)$ -derivation of Q a *right derivation* of Q and $(/, \vee)$ -derivation of Q a *left derivation* of Q unless otherwise specified.

The identity function of Q clearly satisfy the condition of such two derivations.

Example 3.2. Let $Q = \{0, e, f, d, c, a, b, 1\}$ be a semi-Girard quantale given by example 2.5. Then we can get the following operation table by the right derivation $\setminus$ on Q .

$\backslash$	0	e	f	d	c	a	b	1
0	1	1	1	1	1	1	1	1
e	a	1	a	1	1	1	1	1
f	b	b	1	1	1	1	1	1
d	c	b	a	1	1	1	1	1
c	d	d	d	d	1	1	1	1
a	f	d	f	d	b	1	b	1
b	e	e	d	d	a	a	1	1
1	0	e	f	d	c	a	b	1

By example 2.5, it is clear that Q is a semi-Girard quantale whose dualizing element is 0. Now we define a map $\delta : Q \rightarrow Q$ as follows;

$$\delta(x) = \begin{cases} e & \text{if } x = 0, e \\ d & \text{if } x = f, d \\ b & \text{if } x = c, b \\ 1 & \text{if } x = a, 1 \end{cases}$$

$$\delta(x) = \begin{cases} f & \text{if } x = 0, f \\ d & \text{if } x = e, d \\ a & \text{if } x = c, a \\ 1 & \text{if } x = b, 1 \end{cases}$$

$$\delta(x) = \begin{cases} d & \text{if } x = 0, e, f, d \\ 1 & \text{if } x = c, b, a, 1 \end{cases}$$

$$\delta(x) = \begin{cases} c & \text{if } x = 0 \\ a & \text{if } x = e \\ b & \text{if } x = f \\ 1 & \text{if } x = d, c, b, a, 1 \end{cases}$$

$$\delta(x) = \begin{cases} a & \text{if } x = 0, e \\ 1 & \text{if } x = f, d, c, b, a, 1 \end{cases}$$

$$\delta(x) = \begin{cases} b & \text{if } x = 0, f \\ 1 & \text{if } x = e, d, c, b, a, 1 \end{cases}$$

$d(x) = 1$ for all $x \in Q$.

It is easily checked that all of δ 's are a right derivations of Q .

Example 3.3. Let $R = Z_{2^n}$ and Q be the set of all ideals of R . It is easily verified that Q is a commutative Girard quantale where 0 is the cyclic dualizing element. Define a map $\delta : R \rightarrow R$ by $\delta(\langle 2^i \rangle) = \langle 2^{i-1} \rangle$ and $\delta(0) = \langle 2^{n-1} \rangle$. Then δ satisfies the condition of two derivations.

Theorem 3.4. Let δ be a right(or left) derivation of Q . Then $\delta(1) = 1$.

Proof. Since $1 \setminus 1 = 1$ and $\delta(1) \leq 1 \Leftrightarrow \delta(1) \setminus 1 = 1$, $\delta(1) = \{(\delta(1) \setminus 1) \vee (1 \setminus \delta(1))\} = 1$. Similarly we can get the same result if δ is a left derivations. $\square$

Theorem 3.5. *Let δ be a right(or left) derivation of Q . Then $\delta(x) = x \vee \delta(x)$.*

Proof. Since $1 \setminus x = x$, $\delta(x) = \delta(1 \setminus x) = (\delta(1) \setminus x) \vee (1 \setminus \delta(x)) = x \vee \delta(x)$. $\square$

Corollary 3.6. *Let δ be a right derivation of Q . Then $x \leq \delta(x)$ for all $x \in Q$.*

Lemma 3.7. *Let f be a function with $f : Q \rightarrow Q$ such that $x \leq f(x)$ for all $x \in Q$. Then $f(x) \setminus y \leq x \setminus f(y)$ and $f(x)/y \leq x/f(y)$.*

Proof. Since $a \leq b \Rightarrow b \setminus c \leq a \setminus c$, we can get $f(x) \setminus y \leq x \setminus y$. And by the fact $x \setminus y \leq x \setminus f(y)$ we can get the result. $\square$

Theorem 3.8. *Let δ be a map on a semi-Girard quantale Q . Then the following identities are equivalent :*

- (1) δ is a right(left) derivation of Q ,
- (2) $\delta(x \setminus y) = x \setminus (\delta y)$ ($\delta(x/y) = x/\delta(y)$).

Proof. (1) $\Rightarrow$ (2). By the Lemma 4.7, $(\delta(x) \setminus y) \vee (x \setminus \delta(y)) = x \setminus \delta(y)$ for all $x, y \in Q$ if δ is a right derivation. Also $\delta(x/y) = x/\delta(y)$ if δ is a left derivation. (2) $\Rightarrow$ (1). Suppose that δ is a map satisfying $\delta(x \setminus y) = x \setminus \delta(y)$ for all $x, y \in Q$. Let $x = \delta(1), y = 1$. Then $\delta(1) = \delta(\delta(1) \setminus 1) = \delta(1) \setminus \delta(1) = 1$. Hence $\delta(1) = 1$ and we have $1 = \delta(1) = \delta(x \setminus x) = x \setminus \delta(x)$ for all $x \in Q$. This implies that $x \leq \delta(x)$ for all $x \in Q$, and so $x \leq \delta(x) \Rightarrow \delta(x) \setminus y \leq x \setminus \delta(y)$. In left case, the proof is similarly to the right case. Hence $\delta(x \setminus y) = x \setminus \delta(y) = (\delta(x) \setminus y) \vee (x \setminus \delta(y))$ for all $x, y \in Q$, and so δ is a right derivation of Q . $\square$

Corollary 3.9. *Let Q be a semi-Girard quantale. If $\delta_1, \delta_2, \dots, \delta_k$ are right(left) derivations of Q , then $\delta_1 \circ \delta_2 \circ \dots \circ \delta_k$ is also a right(left) derivation.*

Proof. It is sufficient to prove the theorem in case of right(left) derivation. Let δ_1 and δ_2 be right(left) derivations of Q and $x, y \in Q$. Then $(\delta_1 \circ \delta_2)(x \setminus y) = \delta_1(x \setminus \delta_2(y)) = x \setminus \delta_1(\delta_2(y)) = x \setminus (\delta_1 \circ \delta_2)(y)$. Thus the theorem is proved. $\square$

Let Q be a semi-Girard quantale. Then, for each $a \in Q$, we define a map $\delta_a^r : Q \rightarrow Q$ by $\delta_a^r(x) = a \setminus x$ for all $x \in Q$ and define a map $\delta_a^l : Q \rightarrow Q$ by $\delta_a^l(x) = a/x$ for all $x \in Q$.

Theorem 3.10. *For each $a \in Q$, the map δ_a^r is a left derivation and δ_a^l is a right derivation.*

Proof. $\delta_a^r(x/y) = a \setminus (x/y) = x/(a \setminus y) = x/\delta_a^r(y)$. Similarly, $\delta_a^l(x \setminus y) = x \setminus \delta_a^l(y)$. $\square$

Theorem 3.11. *Let Q be a semi-Girard quantale. If δ is a left derivation of Q and $\delta(0) = 0$, then δ is an identity derivation.*

Proof. $\delta(x) = \delta((x \setminus 0)/0) = (x \setminus 0)/\delta(0) = (x \setminus 0)/0 = x$. (i.e δ is a identity derivation $\Leftrightarrow \delta(0) = 0$). $\square$

Corollary 3.12. *Let δ be a right(or left) derivation. Then $\delta(x) \leq \delta(y)$ if $x \leq y$. That is δ is a order preserving function.*

Proof. If $x \leq y$, then $y \setminus 0 \leq x \setminus 0$ and $\delta(x) = (x \setminus 0)/\delta(0) \leq (y \setminus 0)/\delta(0) = \delta(y)$. $\square$

Generally a right derivations may not be a left derivation. Also a left derivations may not be a right derivation. We can get the following theorem.

Theorem 3.13. *Let Q be a semi-Girard quantale. If every right derivation is a left derivation, then Q is a Girard quantale.(i.e 0 is cyclic)*

Proof. Since d_a^l is a right derivation, it is also a left derivation. So $a/(x/y) = d_a^l(x/y) = x/d_a(y) = x/(a/y)$, that is $a/(x/y) = x/(a/y)$. It implies that for every $x, y, z \in Q$, $x/(y/z) = y/(x/z)$. Thus $l \otimes x \otimes y \leq z \Leftrightarrow l \otimes x \leq y/z \Leftrightarrow l \leq x/(y/z) \Leftrightarrow l \leq y/(x/z) \Leftrightarrow l \otimes y \leq x/z \Leftrightarrow l \otimes y \otimes x \leq z$ for all $x, y, z \in Q$. Let $x \otimes y = 0$ and $y \otimes x \neq 0$. Then there are exist z s.t $z \not\leq y \otimes x$. Since $x \otimes y = 0$, $1 \otimes (x \otimes y) = 0 \leq z$. But it is contradiction to the above identity by the fact $1 \otimes (y \otimes z) \neq 0 > z$. Hence $x \otimes y = 0$ implies $y \otimes x = 0$. Thus 0 is cyclic. $\square$

Definition 3.14. Let δ be a right(left) derivation of Q . Define a set $Fix_\delta(Q) := \{x \in Q \mid \delta(x) = x\}$ for all $x \in Q$.

Remark 3.15. 1. If $y \in Fix_\delta(Q)$, then $\delta(x \setminus y) = x \setminus \delta(y) = x \setminus y$. Hence $x \setminus y \in Fix_\delta(Q)$ for all $x \in Q$.

2. $Fix_\delta(Q)$ is closed under the operation $\setminus$.

Definition 3.16. Let Q be a semi-Girard quantale and let δ be a right derivation of Q . Define a $Ker\delta$ by $Ker\delta = \{x \in Q \mid \delta(x) = 1\}$.

Theorem 3.17. *Let Q be a semi-Girard quantale and let δ be a right derivation of Q . Then $Ker\delta = \{x \in Q \mid \delta(0) \setminus 0 \leq x\}$.*

Proof. Since $x = (x/0) \setminus 0$, $1 = \delta(x) = \delta((x/0) \setminus 0) = (x/0) \setminus \delta(0) \Leftrightarrow (x/0) \leq \delta(0) \Leftrightarrow \delta(0) \setminus 0 \leq (x/0) \setminus 0 = x$. Thus $Ker\delta = \{x \in Q \mid \delta(0) \setminus 0 \leq x\}$. $\square$

Corollary 3.18. *For all $x \in ker\delta$ and $x \leq y \Rightarrow y \in ker\delta$.*

Proof. Clear. $\square$

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