On Certain Sums Extended over Prime Factors

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Abstract

Let $P(n)$ be the greatest prime factor of a positive integer n . In this note we obtain asymptotic formulae for the sums

$$\sum_{k=1}^n \frac{P(k)^m}{k^s} \quad (s = 0, 1, \dots, m),$$

where m is an arbitrary but fixed positive integer.

We also obtain asymptotic formulae for the sum

$$\sum_{k=1}^n \frac{p(k)^m}{k^s} \quad (s = 0, 1, \dots, m),$$

where $p(n)$ denotes the least prime factor of n .

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1 Introduction

Let $P(n)$ be the greatest prime factor of a positive integer n and let m be an arbitrary but fixed positive integer. For completeness, let $P(1) = 1$. Let us consider the $m + 1$ sums

$$F_m^s(n) = \sum_{k=1}^n \frac{P(k)^m}{k^s} \quad (s = 0, 1, \dots, m). \quad (1)$$

The case $m = 1$ and $s = 0$ was studied by K. Alladi and P. Erdős [1]. These authors obtained the following asymptotic formula

$$F_1^0(n) = \sum_{k=1}^n P(k) \sim \frac{\pi^2}{12} \frac{n^2}{\log n} = \frac{\zeta(2)}{2} \frac{n^2}{\log n}, \quad (2)$$

where $\zeta(s)$ denotes the Riemann's zeta function.

The case $m = 1$ and $s = 1$ was studied by J. Kemeny [3]. This author obtained the following asymptotic formula

$$F_1^1(n) = \sum_{k=1}^n \frac{P(k)}{k} \sim \frac{\pi^2}{6} \frac{n}{\log n} = \zeta(2) \frac{n}{\log n}. \quad (3)$$

J. Kemeny claims that formula (2) can be obtained from his formula (3). However, we shall see that formula (3) can be obtained from formula (2).

The case $s = 0$ was studied by R. Jakimczuk [2]. This author obtained the following asymptotic formula

$$F_m^0(n) = \sum_{k=1}^n P(k)^m \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n} \quad (m \geq 1). \quad (4)$$

If $m = 1$ then equation (4) becomes equation (2). In this note we study the more general sums (1).

Let $p(n)$ be the least prime factor of a positive integer n . In this note we also obtain asymptotic formulae for the sums

$$f_m^s(n) = \sum_{k=1}^n \frac{p(k)^m}{k^s} \quad (s = 0, 1, \dots, m),$$

where m is an arbitrary but fixed positive integer.

2 Main Results

Lemma 2.1 *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $a_i \sim b_i$. If $\sum_{i=1}^{\infty} b_i$ is divergent then we have the following asymptotic formula*

$$\sum_{i=1}^n a_i \sim \sum_{i=1}^n b_i.$$

Proof. See for example ([4], page 332).

Theorem 2.2 *Let m be an arbitrary but fixed positive integer. We have the following asymptotic formulae*

$$F_m^s(n) = \sum_{k=1}^n \frac{P(k)^m}{k^s} \sim \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s+1}}{\log n} \quad (s = 0, 1, \dots, m). \quad (5)$$

Proof. Let m be a fixed positive integer. If $s = 0$ then formula (5) is true (see formula (4)). Suppose that formula (5) is true for $s \in \{0, \dots, m-1\}$. We shall prove that formula (5) is also true for $s+1$.

We have

$$\begin{aligned}
 F_m^{s+1}(n) &= F_m^s(1) + \frac{1}{2}(F_m^s(2) - F_m^s(1)) + \frac{1}{3}(F_m^s(3) - F_m^s(2)) + \dots \\
 &+ \frac{1}{n}(F_m^s(n) - F_m^s(n-1)) = \left(1 - \frac{1}{2}\right)F_m^s(1) + \left(\frac{1}{2} - \frac{1}{3}\right)F_m^s(2) \\
 &+ \left(\frac{1}{3} - \frac{1}{4}\right)F_m^s(3) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right)F_m^s(n-1) + \frac{1}{n}F_m^s(n) \\
 &= \frac{1}{1.2}F_m^s(1) + \frac{1}{2.3}F_m^s(2) + \frac{1}{3.4}F_m^s(3) + \dots + \frac{1}{(n-1)n}F_m^s(n-1) \\
 &+ \frac{1}{n}F_m^s(n) = \frac{1}{2}F_m^s(1) + \frac{1}{3}\frac{F_m^s(2)}{2} + \frac{1}{4}\frac{F_m^s(3)}{3} + \dots + \frac{1}{n}\frac{F_m^s(n-1)}{n-1} \\
 &+ \frac{1}{n+1}\frac{F_m^s(n)}{n} + \frac{F_m^s(n)}{n+1}.
 \end{aligned}$$

That is,

$$F_m^{s+1}(n) = \frac{1}{2}F_m^s(1) + \frac{1}{3}\frac{F_m^s(2)}{2} + \frac{1}{4}\frac{F_m^s(3)}{3} + \dots + \frac{1}{n+1}\frac{F_m^s(n)}{n} + \frac{F_m^s(n)}{n+1}. \quad (6)$$

Now, we have (see (5))

$$F_m^s(n) = f(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s+1}}{\log n} \quad (n \geq 2), \quad (7)$$

where $f(n) \rightarrow 1$.

Therefore

$$\frac{F_m^s(n)}{n+1} = f(n) \frac{n}{n+1} \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{\log n} = g(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{\log n} \quad (n \geq 2), \quad (8)$$

where $g(n) \rightarrow 1$, and

$$\begin{aligned}
 \frac{1}{i+1} \frac{F_m^s(i)}{i} &= f(i) \frac{i}{i+1} \frac{\zeta(m+1)}{m-s+1} \frac{i^{m-s-1}}{\log i} \\
 &= h(i) \frac{\zeta(m+1)}{m-s+1} \frac{i^{m-s-1}}{\log i} \quad (i \geq 2),
 \end{aligned}$$

where $h(i) \rightarrow 1$. That is,

$$\frac{1}{i+1} \frac{F_m^s(i)}{i} \sim \frac{\zeta(m+1)}{m-s+1} \frac{i^{m-s-1}}{\log i} \quad (i \geq 2). \quad (9)$$

Consequently (9) and lemma 2.1 give

$$\sum_{i=2}^n \frac{1}{i+1} \frac{F_m^s(i)}{i} \sim \sum_{i=2}^n \frac{\zeta(m+1)}{m-s+1} \frac{i^{m-s-1}}{\log i}. \quad (10)$$

We have (L'Hospital's rule)

$$\lim_{x \rightarrow \infty} \frac{\int_a^x \frac{t^{m-s-1}}{\log t} dt}{\frac{x^{m-s}}{(m-s) \log x}} = \lim_{x \rightarrow \infty} \frac{\frac{x^{m-s-1}}{\log x}}{\frac{d}{dx} \left(\frac{x^{m-s}}{(m-s) \log x} \right)} = 1 \quad (s = 0, \dots, m-1). \quad (11)$$

Note that if $s = 0, \dots, m-2$ then the function $\frac{x^{m-s-1}}{\log x}$ is strictly increasing from a certain n_0 . Consequently (see (11))

$$\begin{aligned} \sum_{i=2}^n \frac{i^{m-s-1}}{\log i} &= \sum_{i=2}^{n_0-1} \frac{i^{m-s-1}}{\log i} + \int_{n_0}^n \frac{x^{m-s-1}}{\log x} dx \\ &+ O\left(\frac{n^{m-s-1}}{\log n}\right) \sim \frac{n^{m-s}}{(m-s) \log n}. \end{aligned} \quad (12)$$

Since the left side of the equality is a sum of rectangles of basis 1.

On the other hand if $s = m-1$ then the function $\frac{x^{m-s-1}}{\log x}$ is strictly decreasing. Consequently (see (11))

$$\sum_{i=2}^n \frac{i^{m-s-1}}{\log i} = \int_2^n \frac{x^{m-s-1}}{\log x} dx + O(1) \sim \frac{n^{m-s}}{(m-s) \log n}. \quad (13)$$

Since the left side of the equality is a sum of rectangles of basis 1.

Equations (10), (12) and (13) give

$$\sum_{i=2}^n \frac{1}{i+1} \frac{F_m^s(i)}{i} = k(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{(m-s) \log n}, \quad (14)$$

where $k(n) \rightarrow 1$.

Equations (6), (8) and (14) give

$$\begin{aligned} F_m^{s+1}(n) &= \frac{1}{2} F_m^s(1) + k(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{(m-s) \log n} + g(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{\log n} \\ &\sim \frac{\zeta(m+1)}{m-s} \frac{n^{m-s}}{\log n} = \frac{\zeta(m+1)}{m-(s+1)+1} \frac{n^{m-(s+1)+1}}{\log n}. \end{aligned}$$

That is, equation (5). The theorem is proved.

If $s = m$ then we obtain the following corollary.

Corollary 2.3 *The following asymptotic formula holds*

$$F_m^m(n) = \sum_{k=1}^n \left(\frac{P(k)}{k} \right)^m \sim \zeta(m+1) \frac{n}{\log n} \quad (m \geq 1). \quad (15)$$

If $m = 1$ then equation (15) becomes the Kemeny's formula (3).

Note that if k is prime then we have $P(k) = k$. Consequently each prime contributes 1 to the sum $F_m^m(n)$ (see (15)).

Theorem 2.4 *Let m be an arbitrary but fixed positive integer. We have the following asymptotic formulae*

$$f_m^s(n) = \sum_{k=1}^n \frac{p(k)^m}{k^s} \sim \frac{1}{m-s+1} \frac{n^{m-s+1}}{\log n} \quad (s = 0, 1, \dots, m).$$

Proof. If $s = 0$ then the theorem is true (see [2]). Now, the proof is the same as Theorem 2.2. The theorem is proved.

If $s = m$ then we obtain the following corollary.

Corollary 2.5 *The following asymptotic formula holds*

$$f_m^m(n) = \sum_{k=1}^n \left(\frac{p(k)}{k} \right)^m \sim \frac{n}{\log n} \quad (m \geq 1). \quad (16)$$

Note that if k is prime then we have $p(k) = k$. Consequently each prime contributes 1 to the sum $f_m^m(n)$ (see (16)).

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References

- [1] K. Alladi and P. Erdős, On an additive arithmetic function, *Pacific Journal of Mathematics*, **71** (1977), 275 - 294.
- [2] R. Jakimczuk, A note on sums of greatest (least) prime factors, *International Journal of Contemporary Mathematical Sciences*, **8** (2013), 423 - 432.
- [3] J. Kemeny, Largest prime factor, *Journal of Pure and Applied Algebra*, **89** (1993), 181 - 186.
- [4] J. Rey Pastor, P. Pi Calleja y C. A. Trejo, *Análisis Matemático*, Kapelusz, 1969.

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