On Certain Sums Extended over Prime Factors

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Abstract

Let $P(n)$ be the greatest prime factor of a positive integer $n$. In this note we obtain asymptotic formulae for the sums

$$\sum_{k=1}^{n} \frac{P(k)^m}{k^s} \quad (s = 0, 1, \ldots, m),$$

where $m$ is an arbitrary but fixed positive integer.

We also obtain asymptotic formulae for the sum

$$\sum_{k=1}^{n} \frac{p(k)^m}{k^s} \quad (s = 0, 1, \ldots, m),$$

where $p(n)$ denotes the least prime factor of $n$.

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1 Introduction

Let $P(n)$ be the greatest prime factor of a positive integer $n$ and let $m$ be an arbitrary but fixed positive integer. For completeness, let $P(1) = 1$. Let us consider the $m + 1$ sums

$$F_m^s(n) = \sum_{k=1}^{n} \frac{P(k)^m}{k^s} \quad (s = 0, 1, \ldots, m).$$
The case $m = 1$ and $s = 0$ was studied by K. Alladi and P. Erdős [1]. These authors obtained the following asymptotic formula

$$F_0^1(n) = \sum_{k=1}^{n} P(k) \sim \frac{\pi^2}{12} \frac{n^2}{\log n} = \frac{\zeta(2)}{2} \frac{n^2}{\log n},$$

(2)

where $\zeta(s)$ denotes the Riemann’s zeta function.

The case $m = 1$ and $s = 1$ was studied by J. Kemeny [3]. This author obtained the following asymptotic formula

$$F_1^1(n) = \sum_{k=1}^{n} \frac{P(k)}{k} \sim \frac{\pi^2}{6} \frac{n}{\log n} = \frac{\zeta(2)}{n} \frac{n}{\log n}.$$

(3)

J. Kemeny claims that formula (2) can be obtained from his formula (3). However, we shall see that formula (3) can be obtained from formula (2).

The case $s = 0$ was studied by R. Jakimczuk [2]. This author obtained the following asymptotic formula

$$F_m^0(n) = \sum_{k=1}^{n} P(k)^m \sim \frac{\zeta(m+1)}{m+1} \frac{n^{m+1}}{\log n} \quad (m \geq 1).$$

(4)

If $m = 1$ then equation (4) becomes equation (2). In this note we study the more general sums (1).

Let $p(n)$ be the least prime factor of a positive integer $n$. In this note we also obtain asymptotic formulae for the sums

$$f_m^s(n) = \sum_{k=1}^{n} \frac{p(k)^m}{k^s} \quad (s = 0, 1, \ldots, m),$$

where $m$ is an arbitrary but fixed positive integer.

2 Main Results

**Lemma 2.1** Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $a_i \sim b_i$. If $\sum_{i=1}^{\infty} b_i$ is divergent then we have the following asymptotic formula

$$\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i.$$

Proof. See for example ([4], page 332).

**Theorem 2.2** Let $m$ be an arbitrary but fixed positive integer. We have the following asymptotic formulae

$$F_m^s(n) = \sum_{k=1}^{n} \frac{P(k)^m}{k^s} \sim \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s+1}}{\log n} \quad (s = 0, 1, \ldots, m).$$

(5)
Proof. Let \( m \) be a fixed positive integer. If \( s = 0 \) then formula (5) is true (see formula (4)). Suppose that formula (5) is true for \( s \in \{0, \ldots, m - 1\} \). We shall prove that formula (5) is also true for \( s + 1 \).

We have

\[
F_{m}^{s+1}(n) = F_{m}^{s}(1) + \frac{1}{2} (F_{m}^{s}(2) - F_{m}^{s}(1)) + \frac{1}{3} (F_{m}^{s}(3) - F_{m}^{s}(2)) + \cdots \\
+ \frac{1}{n} (F_{m}^{s}(n) - F_{m}^{s}(n - 1)) = \left( 1 - \frac{1}{2} \right) F_{m}^{s}(1) + \left( \frac{1}{2} - \frac{1}{3} \right) F_{m}^{s}(2) \\
+ \left( \frac{1}{3} - \frac{1}{4} \right) F_{m}^{s}(3) + \cdots + \left( \frac{1}{n - 1} - \frac{1}{n} \right) F_{m}^{s}(n - 1) + \frac{1}{n} F_{m}^{s}(n) \\
= \frac{1}{1.2} F_{m}^{s}(1) + \frac{1}{2.3} F_{m}^{s}(2) + \frac{1}{3.4} F_{m}^{s}(3) + \cdots + \frac{1}{(n - 1)n} F_{m}^{s}(n - 1) \\
+ \frac{1}{n} F_{m}^{s}(n) = \frac{1}{2} F_{m}^{s}(1) + \frac{1}{3} F_{m}^{s}(2) + \frac{1}{4} F_{m}^{s}(3) + \cdots + \frac{1}{n} F_{m}^{s}(n) \\
+ \frac{1}{n + 1} = \frac{F_{m}^{s}(n)}{n + 1}.
\]

That is,

\[
F_{m}^{s+1}(n) = \frac{1}{2} F_{m}^{s}(1) + \frac{1}{3} F_{m}^{s}(2) + \frac{1}{4} F_{m}^{s}(3) + \cdots + \frac{1}{n} F_{m}^{s}(n) + \frac{F_{m}^{s}(n)}{n + 1}. (6)
\]

Now, we have (see (5))

\[
F_{m}^{s}(n) = f(n) \frac{\zeta(m + 1) \ n^{m-s+1}}{m-s+1 \ \log n} \quad (n \geq 2),
\]

where \( f(n) \to 1 \).

Therefore

\[
F_{m}^{s+1}(n) = f(n) \frac{n}{n + 1} \frac{\zeta(m + 1) \ n^{m-s}}{m-s+1 \ \log n} = g(n) \frac{\zeta(m + 1) \ n^{m-s}}{m-s+1 \ \log n} \quad (n \geq 2),
\]

where \( g(n) \to 1 \), and

\[
\frac{1}{i + 1} \frac{F_{m}^{s}(i)}{i} = f(i) \frac{\zeta(m + 1) \ i^{m-s-1}}{i + 1 \ m-s+1 \ \log i} \\
= h(i) \frac{\zeta(m + 1) \ i^{m-s-1}}{m-s+1 \ \log i} \quad (i \geq 2),
\]

where \( h(i) \to 1 \). That is,

\[
\frac{1}{i + 1} \frac{F_{m}^{s}(i)}{i} \sim \frac{\zeta(m + 1) \ i^{m-s-1}}{m-s+1 \ \log i} \quad (i \geq 2).
\]
Consequently (9) and lemma 2.1 give
\[
\sum_{i=2}^{n} \frac{1}{i+1} \frac{F_m(i)}{i} \sim \sum_{i=2}^{n} \frac{\zeta(m+1)}{m-s+1} \frac{i^{m-s-1}}{\log i}.
\]  

We have (L’Hospital’s rule)
\[
\lim_{x \to \infty} \frac{\int_a^x \frac{t^{m-s-1}}{\log t} \, dt}{\frac{x^{m-s}}{(m-s) \log x}} = \lim_{x \to \infty} \frac{\frac{x^{m-s-1}}{\log x}}{\frac{d}{dx} \left( \frac{x^{m-s}}{(m-s) \log x} \right)} = 1 \quad (s = 0, \ldots, m - 1).
\]  

Note that if \( s = 0, \ldots, m - 2 \) then the function \( \frac{x^{m-s-1}}{\log x} \) is strictly increasing from a certain \( n_0 \). Consequently (see (11))
\[
\sum_{i=2}^{n} \frac{i^{m-s-1}}{\log i} = \sum_{i=2}^{n_0-1} \frac{i^{m-s-1}}{\log i} + \int_{n_0}^{n} \frac{x^{m-s-1}}{\log x} \, dx \\
+ O \left( \frac{n^{m-s}}{\log n} \right) \sim \frac{n^{m-s}}{(m-s) \log n}.
\]  

Since the left side of the equality is a sum of rectangles of basis 1.

On the other hand if \( s = m - 1 \) then the function \( \frac{x^{m-s-1}}{\log x} \) is strictly decreasing. Consequently (see (11))
\[
\sum_{i=2}^{n} \frac{i^{m-s-1}}{\log i} = \int_{2}^{n} \frac{x^{m-s-1}}{\log x} \, dx + O(1) \sim \frac{n^{m-s}}{(m-s) \log n}.
\]  

Since the left side of the equality is a sum of rectangles of basis 1.

Equations (10), (12) and (13) give
\[
\sum_{i=2}^{n} \frac{1}{i+1} \frac{F_m(i)}{i} = k(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{(m-s) \log n},
\]  

where \( k(n) \to 1 \).

Equations (6), (8) and (14) give
\[
F_{m+1}(n) = \frac{1}{2} F_m(1) + k(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{(m-s) \log n} + g(n) \frac{\zeta(m+1)}{m-s+1} \frac{n^{m-s}}{\log n} \\
\sim \frac{\zeta(m+1)}{m-s} \frac{n^{m-s}}{\log n} = \frac{\zeta(m+1)}{m-(s+1)+1} \frac{n^{m-(s+1)+1}}{\log n}.
\]  

That is, equation (5). The theorem is proved.

If \( s = m \) then we obtain the following corollary.
Corollary 2.3 The following asymptotic formula holds

\[ F_m(n) = \sum_{k=1}^{n} \left( \frac{P(k)}{k} \right)^m \sim \zeta(m+1) \frac{n}{\log n} \quad (m \geq 1). \] (15)

If \( m = 1 \) then equation (15) becomes the Kemeny’s formula (3).

Note that if \( k \) is prime then we have \( P(k) = k \). Consequently each prime contributes 1 to the sum \( F_m(n) \) (see (15)).

Theorem 2.4 Let \( m \) be an arbitrary but fixed positive integer. We have the following asymptotic formulae

\[ f_{sm}^m(n) = \sum_{k=1}^{n} \frac{p(k)^m}{k^s} \sim \frac{1}{m-s+1} n^{m-s+1} \log n \quad (s = 0, 1, \ldots, m). \]

Proof. If \( s = 0 \) then the theorem is true (see [2]). Now, the proof is the same as Theorem 2.2. The theorem is proved.

If \( s = m \) then we obtain the following corollary.

Corollary 2.5 The following asymptotic formula holds

\[ f_{m}^m(n) = \sum_{k=1}^{n} \left( \frac{p(k)}{k} \right)^m \sim \frac{n}{\log n} \quad (m \geq 1). \] (16)

Note that if \( k \) is prime then we have \( p(k) = k \). Consequently each prime contributes 1 to the sum \( f_{m}^m(n) \) (see (16)).

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References


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