A Determinant Formula for Sums of Powers of Integers

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Abstract

In this note, we first derive a recursive formula for the sum of powers $S_k(n) = 1^k + 2^k + \cdots + n^k$, with k and n non-negative integers. We then apply it to establish, via Cramer's rule, an explicit determinant formula for $S_k(n)$ involving the Bernoulli numbers and the binomial $(n + \frac{1}{2})$. Evaluating the determinant gives us directly $S_k(n)$ in the form of the so-called Faulhaber polynomial, namely as a sum of even or odd powers of $(n + \frac{1}{2})$. Furthermore, a connection with Hessenberg matrices is shown.

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1 Introduction

For k and n non-negative integers, define the sum of powers

$$S_k(n) = 1^k + 2^k + \cdots + n^k,$$

and set $S_k(0) = 0$ for all k . In this note (Section 2), we derive the following recursive formula for $S_k(n)$:

$$S_k(n) = \frac{1}{k+1} \left[k \left(n + \frac{1}{2} \right) S_{k-1}(n) - \sum_{r=1}^{k-2} \binom{k}{r} B_{k-r} S_r(n) \right], \quad k \geq 3, \quad (1)$$

where B_j is the j -th Bernoulli number. Next, in Section 3, we apply (1) to obtain, by Cramer's rule, an explicit determinant formula for $S_k(n)$ in terms of the Bernoulli numbers and the binomial $(n + \frac{1}{2})$. Evaluating the corresponding determinant yields $S_k(n)$ in the form of a polynomial in $n + \frac{1}{2}$. This type of polynomials are known in the literature as Faulhaber polynomials, after the German mathematician Johann Faulhaber (1580–1635) who expressed for the first time the $S_k(n)$'s as polynomials in $S_1(n) = \frac{1}{2}n(n+1)$ [6, 7, 11, 1] (see Theorem 4.1). As will be shown in Section 4, the latter is equivalent to the fact that the $S_k(n)$'s are the sums only of even or odd powers of $(n + \frac{1}{2})$ depending on whether k is odd or even (Theorem 4.2). In Section 5, we define an $n \times n$ Hessenberg matrix H_n and write down the recurrence relation satisfied by the sequence of determinants $\{\det(H_n), n \geq 1\}$. Then, formula (1) is re-derived from said recurrence relation. Some concluding remarks are provided in Section 6.

We point out that this note could be of interest for use in introductory courses on number theory or discrete mathematics as enrichment material related to the sums of powers of integers.

2 Proof of the recursive formula

The proof of formula (1) is based on the following lemma.

Lemma 2.1. *For $n, k \geq 1$, we have*

$$S_k(n) = (n+1)S_{k-1}(n) - \sum_{i=1}^n S_{k-1}(i). \quad (2)$$

Proof. Next, we give the most simple and direct proof of (2) one can conceivably construct. For that, it suffices to display $(n+1)S_{k-1}(n)$ as the sum of the $n+1$ rows

$$\left. \begin{array}{l} 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \\ 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \\ \vdots \\ 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \end{array} \right\} n+1,$$

and then note that this sum can be decomposed as the sum of the following two pieces:

$$\left. \begin{array}{l} 1^{k-1} \\ 1^{k-1} + 2^{k-1} \\ 1^{k-1} + 2^{k-1} + 3^{k-1} \\ \vdots \\ 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \end{array} \right\} n + \left. \begin{array}{l} 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \\ 2^{k-1} + 3^{k-1} + \cdots + n^{k-1} \\ 3^{k-1} + \cdots + n^{k-1} \\ \vdots \\ n^{k-1} \end{array} \right\} n.$$

Clearly, summing the rows of the piece on the left gives $\sum_{i=1}^n S_{k-1}(i)$. Further, the entries in the i -th column of the piece on the right sum to i^k , and hence the sum of all these columns amounts to $S_k(n)$. Thus, the lemma follows. $\square$

For a historical account of equation (2), the reader is referred to Katz's paper [10]. It is a well-known result, on the other hand, that $S_{k-1}(n)$ is expressible as a polynomial in n of degree k with zero constant term:

$$S_{k-1}(n) = a_{k-1,1} n + a_{k-1,2} n^2 + \cdots + a_{k-1,k-2} n^{k-2} + a_{k-1,k-1} n^{k-1} + a_{k-1,k} n^k, \quad (3)$$

where $a_{k-1,k-1} = \frac{1}{2}$ and $a_{k-1,k} = \frac{1}{k}$. Moreover, the remaining coefficients $a_{k-1,r}$ are connected to the Bernoulli numbers through the relation (see, for example, [13])

$$a_{k-1,r} = \frac{1}{k} \binom{k}{r} B_{k-r}, \quad k \geq 3, \quad r = 1, 2, \dots, k-2. \quad (4)$$

Equipped with relations (2), (3), and (4), the proof of (1) is straightforward. For that, we first evaluate $S_{k-1}(i)$ for each $i = 1, 2, \dots, n$. Using (3) we have

$$\begin{aligned} S_{k-1}(1) &= a_{k-1,1} 1 + a_{k-1,2} 1^2 + \cdots + a_{k-1,k-2} 1^{k-2} + \frac{1}{2} 1^{k-1} + \frac{1}{k} 1^k, \\ S_{k-1}(2) &= a_{k-1,1} 2 + a_{k-1,2} 2^2 + \cdots + a_{k-1,k-2} 2^{k-2} + \frac{1}{2} 2^{k-1} + \frac{1}{k} 2^k, \\ &\vdots \\ S_{k-1}(n) &= a_{k-1,1} n + a_{k-1,2} n^2 + \cdots + a_{k-1,k-2} n^{k-2} + \frac{1}{2} n^{k-1} + \frac{1}{k} n^k. \end{aligned}$$

Then summing terms corresponding to the same coefficient $a_{k-1,r}$ in all these equations yields

$$\sum_{i=1}^n S_{k-1}(i) = \sum_{r=1}^{k-2} a_{k-1,r} S_r(n) + \frac{1}{2} S_{k-1}(n) + \frac{1}{k} S_k(n).$$

Substituting this expression into (2) and recalling (4), one finally gets formula (1).

3 A determinant formula for sums of powers of integers

Unless otherwise stated, in what follows we drop the explicit dependence of $S_k(n)$ on n and write this merely as S_k . It is easily seen that the well-known formulas for S_1 and S_2

$$\begin{aligned} S_1 &= \frac{1}{2} n(n+1), \\ S_2 &= \frac{1}{6} n(n+1)(2n+1), \end{aligned}$$

can be put in the equivalent form

$$\begin{aligned} 2S_1 &= N^2 - \frac{1}{4}, \\ -2NS_1 + 3S_2 &= 0, \end{aligned} \quad (5)$$

where $N = n + \frac{1}{2}$. Now, by letting $k = 3, 4, 5, \dots, p$ in (1), we obtain the system of equations

$$\begin{aligned} \binom{3}{1} B_2 S_1 - 3NS_2 + 4S_3 &= 0, \\ \binom{4}{1} B_3 S_1 + \binom{4}{2} B_2 S_2 - 4NS_3 + 5S_4 &= 0, \\ \binom{5}{1} B_4 S_1 + \binom{5}{2} B_3 S_2 + \binom{5}{3} B_2 S_3 - 5NS_4 + 6S_5 &= 0, \\ \vdots \\ \binom{p}{1} B_{p-1} S_1 + \binom{p}{2} B_{p-2} S_2 + \dots + \binom{p}{p-2} B_2 S_{p-2} - pNS_{p-1} + (p+1)S_p &= 0. \end{aligned} \quad (6)$$

The sets of equations (5) and (6) can be combined in matrix form as

$$\begin{pmatrix} 2 & 0 & 0 & \dots & 0 & 0 \\ -2N & 3 & 0 & \dots & 0 & 0 \\ \binom{3}{1} B_2 & -3N & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ \binom{p-1}{1} B_{p-2} & \binom{p-1}{2} B_{p-3} & \binom{p-1}{3} B_{p-4} & \dots & p & 0 \\ \binom{p}{1} B_{p-1} & \binom{p}{2} B_{p-2} & \binom{p}{3} B_{p-3} & \dots & -pN & p+1 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{p-1} \\ S_p \end{pmatrix} = \begin{pmatrix} N^2 - \frac{1}{4} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \quad (7)$$

Then, solving for S_p and applying Cramer's rule to the above $p \times p$ lower triangular system of linear equations yields

$$S_p = \frac{1}{(p+1)!} \begin{vmatrix} 2 & 0 & 0 & \dots & 0 & N^2 - \frac{1}{4} \\ -2N & 3 & 0 & \dots & 0 & 0 \\ \binom{3}{1} B_2 & -3N & 4 & \dots & 0 & 0 \\ \binom{4}{1} B_3 & \binom{4}{2} B_2 & -4N & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots \\ \binom{p-1}{1} B_{p-2} & \binom{p-1}{2} B_{p-3} & \binom{p-1}{3} B_{p-4} & \dots & p & 0 \\ \binom{p}{1} B_{p-1} & \binom{p}{2} B_{p-2} & \binom{p}{3} B_{p-3} & \dots & -pN & 0 \end{vmatrix}.$$

By expanding the determinant with respect to the last column, this simplifies to

$$S_p = \frac{(-1)^{p+1}}{(p+1)!} \left(N^2 - \frac{1}{4} \right) \begin{vmatrix} -2N & 3 & 0 & \dots & 0 \\ \binom{3}{1}B_2 & -3N & 4 & \dots & 0 \\ \binom{4}{1}B_3 & \binom{4}{2}B_2 & -4N & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ \binom{p-1}{1}B_{p-2} & \binom{p-1}{2}B_{p-3} & \binom{p-1}{3}B_{p-4} & \dots & p \\ \binom{p}{1}B_{p-1} & \binom{p}{2}B_{p-2} & \binom{p}{3}B_{p-3} & \dots & -pN \end{vmatrix}. \quad (8)$$

Evaluating the determinant of order $p-1$ in (8) gives us S_p as a polynomial in N . So, for the first values of p we find

$$\begin{aligned} S_1 &= \frac{N^2}{2} - \frac{1}{8}, \\ S_2 &= \frac{2N(N^2 - \frac{1}{4})}{3!} = \frac{N^3}{3} - \frac{N}{12}, \\ S_3 &= \frac{N^2 - \frac{1}{4}}{4!} \begin{vmatrix} -2N & 3 \\ \frac{1}{2} & -3N \end{vmatrix} = \frac{N^4}{4} - \frac{N^2}{8} + \frac{1}{64}, \\ S_4 &= \frac{\frac{1}{4} - N^2}{5!} \begin{vmatrix} -2N & 3 & 0 \\ \frac{1}{2} & -3N & 4 \\ 0 & 1 & -4N \end{vmatrix} = \frac{N^5}{5} - \frac{N^3}{6} + \frac{7N}{240}, \\ S_5 &= \frac{N^2 - \frac{1}{4}}{6!} \begin{vmatrix} -2N & 3 & 0 & 0 \\ \frac{1}{2} & -3N & 4 & 0 \\ 0 & 1 & -4N & 5 \\ -\frac{1}{6} & 0 & \frac{5}{3} & -5N \end{vmatrix} = \frac{N^6}{6} - \frac{5N^4}{24} + \frac{7N^2}{96} - \frac{1}{128}, \\ S_6 &= \frac{\frac{1}{4} - N^2}{7!} \begin{vmatrix} -2N & 3 & 0 & 0 & 0 \\ \frac{1}{2} & -3N & 4 & 0 & 0 \\ 0 & 1 & -4N & 5 & 0 \\ -\frac{1}{6} & 0 & \frac{5}{3} & -5N & 6 \\ 0 & -\frac{1}{2} & 0 & \frac{5}{2} & -6N \end{vmatrix} = \frac{N^7}{7} - \frac{N^5}{4} + \frac{7N^3}{48} - \frac{31N}{1344}, \end{aligned}$$

and so on. Note that S_1 , S_3 , and S_5 involve only even powers of N , whereas S_2 , S_4 , and S_6 involve only odd ones. As we will see in the next section, this feature holds generally so that S_p ($p \geq 1$) is expressible as an even or odd polynomial in N depending on whether p is odd or even [8].

On the other hand, it is clear from (8) that the term of maximum degree in N corresponds to the product of the elements on the main diagonal of the

determinant. The term of maximum degree of S_p will be thus given by

$$\frac{(-1)^{p+1}}{(p+1)!}(-1)^{p-1}p!N^2N^{p-1} = \frac{N^{p+1}}{p+1}.$$

Indeed, the simple formula

$$\sum_{r=1}^n r^p \simeq \frac{(n + \frac{1}{2})^{p+1}}{p+1},$$

proves a fairly good approximation to the sums of powers [3].

Unfortunately, the determination of the trailing coefficients of S_p using (8) is considerably more difficult to achieve than it was for the leading coefficient. Upon examination of the above formulas for $S_2, \dots, S_6$, however, one can correctly guess that the next highest term of S_p ($p > 1$) is given by $-\frac{p}{24}N^{p-1}$. A complete characterization of the coefficients of S_p expressed as a polynomial in N is given elsewhere using the technique of generating functions [5]. Finally, we note that, for any given p , the set of power sums $\{S_1, S_2, \dots, S_p\}$ can alternatively be obtained by inverting the square matrix in equation (7).

4 Symmetry and Faulhaber polynomials

An immediate consequence of formula (8) is that, for all $p \geq 1$, S_p is a multiple of S_1 since the factor $N^2 - \frac{1}{4}$ is proportional to S_1 . Furthermore, for odd $p \geq 3$, S_1^2 divides S_p , the quotient S_p/S_1^2 being a polynomial in S_1 . On the other hand, for even $p \geq 2$, S_2 divides S_p , the quotient S_p/S_2 being a polynomial in S_1 . These results were essentially known to Faulhaber by 1615, and can be formally established in the following theorem [1, 7, 11].

Theorem 4.1 (Faulhaber). (i) For $p = 3, 5, 7, \dots$ there exists a polynomial F_p in S_1 of degree $\frac{1}{2}(p-3)$ such that $S_p = S_1^2 F_p(S_1)$. (ii) For $p = 2, 4, 6, \dots$ there exists a polynomial F_p in S_1 of degree $\frac{1}{2}(p-2)$ such that $S_p = S_2 F_p(S_1)$.

Let us now focus on the symmetry properties of S_p . For this purpose, it is convenient to extend the definition of S_p to view it as a polynomial function of the continuous real variable x . Consider first the case of odd p . Regarding S_1 , from the expression $S_1(x) = \frac{1}{2}x^2 + \frac{1}{2}x$ it is quickly verified that

$$S_1(x - \frac{1}{2}) = S_1(-x - \frac{1}{2}). \quad (9)$$

For odd $p \geq 3$, from Theorem 4.1 we have that $S_p(x) = [S_1(x)]^2 F_p(S_1(x))$, and then, noting (9), it follows that

$$\begin{aligned} S_p(x - \frac{1}{2}) &= [S_1(x - \frac{1}{2})]^2 F_p(S_1(x - \frac{1}{2})) \\ &= [S_1(-x - \frac{1}{2})]^2 F_p(S_1(-x - \frac{1}{2})) \\ &= S_p(-x - \frac{1}{2}). \end{aligned}$$

So, for odd $p \geq 1$ and for all values of x , it is the case that $S_p(x - \frac{1}{2}) = S_p(-x - \frac{1}{2})$ revealing the fact that $S_p(x)$ is symmetric with respect to reflection in the line $x = -\frac{1}{2}$. This amounts to saying that, for odd p , $S_p(x)$ can be expressed as an even polynomial in $x + \frac{1}{2}$.

Similarly, from the expression $S_2(x) = \frac{1}{3}(2x+1)S_1(x)$, one can easily check that

$$S_2(x - \frac{1}{2}) = -S_2(-x - \frac{1}{2}). \quad (10)$$

Now, for even $p \geq 2$, from Theorem 4.1 we have that $S_p(x) = S_2(x)F_p(S_1(x))$. Then, using (9) and (10), it is readily seen that, for even $p \geq 2$ and for all values of x , $S_p(x - \frac{1}{2}) = -S_p(-x - \frac{1}{2})$. This means that $S_p(x)$ is symmetric with respect to reflection in the point $(-\frac{1}{2}, 0)$, and thus $S_p(x)$ can be expressed as an odd polynomial in $x + \frac{1}{2}$.

Therefore, we can state the following version of Faulhaber's theorem.

Theorem 4.2 (Faulhaber). *(i) For $p = 1, 3, 5, \dots$ there exists an even polynomial $\mathcal{F}_p$ in $N = n + \frac{1}{2}$ of degree $p + 1$ such that $S_p = \mathcal{F}_p(N)$. (ii) For $p = 2, 4, 6, \dots$ there exists an odd polynomial $\mathcal{F}_p$ in N of degree $p + 1$ such that $S_p = \mathcal{F}_p(N)$.*

We have just shown that Theorem 4.1 implies Theorem 4.2. It can be shown, conversely, that Theorem 4.2 implies Theorem 4.1, so that these are actually equivalent to each other. Interestingly, the implication Theorem 4.2 $\Rightarrow$ Theorem 4.1 is nicely seen by using the formula (8). For that, let us denote the determinant appearing in (8) as $\det(H_{p-1})$ and consider separately the cases of odd and even p :

- For odd p , from Theorem 4.2 we have that S_p is expressible as an even polynomial in N of degree $p + 1$. So, from (8) we deduce that $\det(H_{p-1})$ must be an even polynomial in N of degree $p - 1$, and thus

$$S_p \propto \left(N^2 - \frac{1}{4}\right) \left[a + bN^2 + cN^4 + \dots + dN^{p-1}\right].$$

Since $N^2 = 2S_1 + \frac{1}{4}$, this can be written as

$$S_p \propto S_1 \left[a' + b'S_1 + c'S_1^2 + \dots + d'S_1^{(p-1)/2}\right]. \quad (11)$$

Clearly, for $p = 1$ we have that $a' \neq 0$ and $b' = c' = \dots = d' = 0$. Now the key point to observe is that, for odd $p \geq 3$, $a' = 0$. This follows from the fact that for this case, namely for odd $p \geq 3$, we have that $S'_p(x) = pS_{p-1}(x)$, and then $S'_p(x = 0) = 0$ [14]. Hence, differentiating (11) with respect to x gives $S'_p(x) \propto (x + \frac{1}{2})[a' + 2b'S_1 + 3c'S_1^2 + \dots + \frac{p+1}{2}d'S_1^{(p-1)/2}]$, and then $S'_p(x = 0) \propto a'$. Thus putting $a' = 0$ in (11) yields the Faulhaber form, $S_p \propto S_1^2[b' + c'S_1 + c''S_1^2 + \dots + d'S_1^{(p-3)/2}]$. Moreover, it can be proved that $b', c', c'', \dots, d' \neq 0$.

- For even p , from Theorem 4.2 we have that S_p is expressible as an odd polynomial in N of degree $p+1$. Hence, from (8) it follows that $\det(H_{p-1})$ must be an odd polynomial in N of degree $p-1$, and thus

$$S_p \propto \left(N^2 - \frac{1}{4}\right) \left[aN + bN^3 + cN^5 + \cdots + dN^{p-1}\right].$$

Factorizing N and noting that $(N^2 - \frac{1}{4})N \propto S_2$, we have

$$S_p \propto S_2 \left[a + bN^2 + cN^4 + \cdots + dN^{p-2}\right].$$

Then, as $N^2 = 2S_1 + \frac{1}{4}$, we retrieve the Faulhaber form, $S_p \propto S_2 \left[a' + b'S_1 + c'S_1^2 + \cdots + d'S_1^{(p-2)/2}\right]$. Likewise, it turns out that $a', b', c', \dots, d' \neq 0$.

5 A connection with Hessenberg matrices

An $n \times n$ matrix $H_n = (h_{ij})$ is called a (lower) Hessenberg matrix if all entries above the superdiagonal are zero but the matrix is not lower triangular, i.e.,

$$H_n = \begin{pmatrix} h_{11} & h_{12} & 0 & \cdots & 0 \\ h_{21} & h_{22} & h_{23} & \cdots & 0 \\ h_{31} & h_{32} & h_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & h_{n-1,n} \\ h_{n,1} & h_{n,2} & h_{n,3} & \cdots & h_{n,n} \end{pmatrix},$$

where $h_{i,i+1} \neq 0$ for some $i = 1, 2, \dots, n-1$. Consider the sequence of determinants $\{\det(H_n), n \geq 1\}$ and define $\det(H_0) = 1$. Then it can be shown (see, for example, [4]) that $\{\det(H_n), n \geq 1\}$ satisfies the recurrence relation

$$\det(H_n) = \sum_{r=1}^n (-1)^{n-r} q_{n,r} \det(H_{r-1}), \quad (12)$$

where

$$q_{n,r} = \begin{cases} h_{n,n}, & \text{if } r = n; \\ h_{n,r} \prod_{i=r}^{n-1} h_{i,i+1}, & \text{if } r = 1, 2, \dots, n-1. \end{cases}$$

Now, it is easy to see that formula (8) can be rewritten as

$$\det(H_{p-1}) = \frac{(p+1)!}{2(-1)^{p+1}S_1} S_p, \quad p \geq 1, \quad (13)$$

where H_{p-1} is (whenever $p \geq 3$) a lower Hessenberg matrix of order $p-1$ with

$$q_{p-1,r} = \begin{cases} -pN, & \text{if } r = p-1; \\ \binom{p}{r} \frac{p!B_{p-r}}{(r+1)!}, & \text{if } r = 1, 2, \dots, p-2. \end{cases} \quad (14)$$

Then, letting $n = p-1$ and replacing $\det(H_{p-1})$ and $\det(H_{r-1})$ by the corresponding expression in equation (13), recurrence (12) becomes

$$\frac{(p+1)!}{2(-1)^{p+1}S_1}S_p = \sum_{r=1}^{p-1} (-1)^{p-1-r} q_{p-1,r} \frac{(r+1)!}{2(-1)^{r+1}S_1} S_r.$$

Multiplying both sides of this equation by $2(-1)^{p+1}S_1$, we get

$$(p+1)!S_p = - \sum_{r=1}^{p-1} q_{p-1,r} (r+1)!S_r,$$

from which, taking into account (14), we obtain

$$(p+1)!S_p = pNp!S_{p-1} - \sum_{r=1}^{p-2} \binom{p}{r} p!B_{p-r}S_r.$$

Thus dividing both sides of this equation by $(p+1)!$, and after renaming the index p as k , we come full circle and find again formula (1).

6 Conclusion

In this note, we have derived a recursive formula for the sums of powers $S_p(n) = \sum_{r=1}^n r^p$. Using Cramer's rule, this formula allows us to obtain a determinant formula for $S_p(n)$ in terms of the Bernoulli numbers and the binomial $(n + \frac{1}{2})$. Evaluating the determinant gives us $S_p(n)$ as a sum of even or odd powers of $(n + \frac{1}{2})$. We have also shown the equivalence between the two forms of Faulhaber's polynomials for expressing $S_p(n)$. Furthermore, the relationship between the recursion formula (1) for $S_p(n)$ and the recurrence (12) for the sequence of determinants of a Hessenberg matrix has been worked out.

It should be mentioned that there are other determinant formulas giving $S_p(n)$ as a polynomial in n or in $n+1$ [2]. Such formulas for $S_p(n)$, however, involve determinants of order $p+1$, unlike our determinant which is of order $p-1$. Moreover, to the best of our knowledge, both the recursive formula (1) and the resulting determinant formula (8) for $S_p(n)$ appear to be new.

Finally, it is worth writing out explicitly a closed formula stated in [9] expressing $S_p(n)$ in terms of $n + \frac{1}{2}$, namely,

$$S_p(n) = \sum_{i=0}^{\lfloor p/2 \rfloor} \binom{p}{2i} B_{2i} \left(\frac{1}{2} \right) g(p-2i), \quad (15a)$$

where $B_j(x)$ is the j -th Bernoulli polynomial, and

$$g(p) = \frac{1}{p+1} \left[\left(n + \frac{1}{2} \right)^{p+1} - \left(\frac{1}{2} \right)^{p+1} \right]. \quad (15b)$$

Since $B_j(\frac{1}{2}) = (2^{1-j} - 1)B_j$, using (15a) and (15b) we can get the coefficients corresponding to the Faulhaber polynomial encapsulated in (8). As a by-product, by equating to zero the constant term for the case in which p is even (say, $p = 2k$, $k \geq 1$), from (15a) and (15b) we readily obtain the following recursive relation for the Bernoulli numbers:

$$\sum_{i=0}^k \binom{2k+1}{2i} (2^{2i} - 2) B_{2i} = 0, \quad k \geq 1,$$

which corresponds to the identity (5) in the paper [12]. Incidentally, the constant term for the case in which p is odd (say, $p = 2k + 1$, $k \geq 0$), is given by

$$c_{2k+1} = \frac{1}{(2k+2)2^{2k+2}} \sum_{i=0}^k \binom{2k+2}{2i} (2^{2i} - 2) B_{2i}.$$

From this expression we deduce that, for example, $c_5 = -\frac{1}{128}$. Of course, this agrees with the constant term of the Faulhaber polynomial for S_5 derived from the determinant (8) in Section 3.

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