The Fine Spectrum of Upper Triangular Double-Band Matrices $U(r,s)$ over the Class of Convergent Series

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Abstract

The main purpose of this paper is to determine the fine spectrum of the upper triangular double-band matrices $U(r,s)$ over the class of convergent series and we also determined the the approximate point spectrum, defect spectrum and compression spectrum.

Keywords: Spectrum of an operator, Infinite matrices, Sequence spaces.

1. Introduction

In the existing literature of spectrum, there are many studies concerning with some particular limitation matrices over some sequence spaces. We give a short summary about the existing literature. For example;

Wenger [7] have studied the fine spectra of Hölder summability operators. The spectrum of the Cesàro operator over some sequence spaces such as l_p , c_0 , bv were calculated by González [8], Reade [9], Okutoyi [10], and Akhmedov and Başar [11], respectively. Altay and Başar [12] studied the fine spectrum of the difference operator Δ over c_0 and c and later Akhmedov and Başar also determined the fine spectrum of the difference operator Δ over bv_p where $1 \leq p < \infty$ in [14]. The fine spectrum of the generalized difference operator over the sequence spaces c_0 and c were computed by Altay and Başar [13]. Besides, Karakaya and Altun [15] investigated the fine spectra of upper triangular double-band matrices on c_0 and c . Later, Karaısa and Başar [17] also determined the fine spectra of upper triangular triple-band matrix on l_p , $(0 < p < \infty)$. Finally, Dutta and Tripathy [16] have determined fine spectrum of the generalized difference operator over the class of convergent series.

2. Preliminaries, Background and Notations

Let X and Y be the Banach spaces and $T: X \rightarrow Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T , i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\varphi)(x) = \varphi(Tx)$ for all $\varphi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\theta\}$ be a complex normed space and $T: \mathcal{D}(T) \rightarrow X$ be a linear operator with domain $\mathcal{D}(T) \subset X$. By T , associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $\mathcal{D}(T)$. If T_λ has an inverse, which is linear, we denote it by T_λ^{-1} , that is

$$T_{\lambda}^{-1} = (T - \lambda I)^{-1},$$

and it is called to be the resolvent operator of T .

The name *resolvent* is appropriate, since T_{λ}^{-1} helps to solve the equation $T_{\lambda}x = y$. Thus, $x = T_{\lambda}^{-1}y$ provided T_{λ}^{-1} exist. More important, the investigation of properties of T_{λ}^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_{λ} and T_{λ}^{-1} depend on λ , and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_{λ}^{-1} exist. Boundedness of T_{λ}^{-1} is another property that will be essential. We shall also ask for what λ 's the domain of T_{λ}^{-1} is dense in X , to name just a few aspects. For our investigation of T , T_{λ} and T_{λ}^{-1} , we need some basic concepts in spectral theory which are given as follows (see [1 pp. 370-371]):

Definition 1.1. Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathcal{D}(T) \rightarrow X$ also be a linear operator with domain $\mathcal{D}(T) \subset X$. A regular value of λ of T is a complex number such that

(R1) T_{λ}^{-1} exist

(R2) T_{λ}^{-1} is bounded

(R3) T_{λ}^{-1} is defined on a set which is dense in X .

The resolvent set $\rho(T)$ of T is the set of all regular values λ of T . Its complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T . The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

The point spectrum $\sigma_p(T)$ is the set such that T_{λ}^{-1} does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T .

The continuous spectrum $\sigma_c(T)$ is the set such that T_{λ}^{-1} exist and satisfies (R3) but not (R2).

The residual spectrum $\sigma_r(T)$ is the set such that T_{λ}^{-1} exist (and may be bounded or not) but not satisfy (R3).

From Goldberg (see [5, pp. 58–71]), if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1} :

$$(I) R(T) = X,$$

$$(II) R(T) \neq \overline{R(T)} = X,$$

$$(III) \overline{R(T)} \neq X$$

and

$$(1) T^{-1} \text{ exists and is continuous,}$$

$$(2) T^{-1} \text{ exists but is discontinuous,}$$

$$(3) T^{-1} \text{ does not exist.}$$

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2$ and III_3 . If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in II_1$ then λ is in the resolvent set $\rho(T, X)$ of T is the set of all regular values λ of T on X . The other classifications give rise to the fine spectrum of T . If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\sigma(T, X) II_2$.

Furthermore, following Appell in [6], we define the three more subdivisions of the spectrum called as the approximate point spectrum, defect spectrum, and compression spectrum.

Given a bounded linear operator T in a Banach space X , we call a sequence x_k in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \rightarrow 0$, as $k \rightarrow \infty$.

In what follows, we call the set

$$\sigma_{ap}(T, X) = \{\lambda \in \mathbb{C} : \text{there exist a Weyl sequence for } \lambda I - T\} \quad (2.1)$$

the approximate point spectrum of T . Moreover, the subspectrum

$$\sigma_\delta(T, X) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\} \quad (2.2)$$

is called defect spectrum of T .

The two subspectra given by (2.1) and (2.2) form a (not necessarily disjoint) subdivision

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{\delta}(T, X) \quad (2.3)$$

of the spectrum. There is another subspectrum

$$\sigma_{co}(T, X) = \left\{ \lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X \right\}, \quad (2.4)$$

which is often called compression spectrum in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X) \quad (2.5)$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_{\delta}(T, X)$. Moreover, comparing these subspectra with

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X) \quad (2.6)$$

we note that

$$\begin{aligned} \sigma_r(T, X) &= \sigma_{co}(T, X) \setminus \sigma_c(T, X) \\ \sigma_c(T, X) &= \sigma(T, X) \setminus [\sigma_p(T, X) \cup \sigma_{co}(T, X)]. \end{aligned} \quad (2.7)$$

Propositions. (see [6, Proposition 1.3, p. 28]). Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

- (a) $\sigma(T^*, X^*) = \sigma(T, X)$,
- (b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.
- (c) $\sigma_{ap}(T^*, X^*) = \sigma_{\delta}(T, X)$.
- (d) $\sigma_{\delta}(T^*, X^*) = \sigma_{ap}(T, X)$.
- (e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.
- (f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.
- (g) $\sigma(T, X) = \sigma_p(T^*, X^*) \cup \sigma_{ap}(T, X) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite-dimensional spaces (see [6]).

By w , we shall denote the space of all real valued sequences. Any vector subspace of w is called as a sequence space. We shall write l_∞ , c , c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N} = \{1, 2, 3, \dots\}$. Then, we say that A defines a matrix mapping from X into Y , and we denote it by writing $A: X \rightarrow Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n \in \mathbb{N}}$, the A -transform of x , is in Y , where

$$(Ax)_n = \sum_k a_{nk} x_k \quad n \in \mathbb{N}. \quad (2.8)$$

By $(X:Y)$, we denote the class of all matrices A such that $A: X \rightarrow Y$. Thus, $A \in (X:Y)$ if and only if the series on the right side of (2.8) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$.

By well-established convention, we define the space γ of all convergent series is defined by

$$\gamma = \left\{ x = (x_k) \in w : \left(\sum_{k=0}^n x_k \right) \in c \right\}. \quad (2.9)$$

An upper triangular double-band infinite matrix is of the form

$$U(r, s) = \begin{bmatrix} r & s & 0 & 0 & 0 & \dots \\ 0 & r & s & 0 & 0 & \dots \\ 0 & 0 & r & s & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If $T: \gamma \rightarrow \gamma$ is a bounded linear operator with the matrix A , then its adjoint operator $T^*: \gamma^* \rightarrow \gamma^*$ is defined by transpose of the matrix A and γ is isomorphic to l_1 with the norm

$$\|x\|_{(l_1, l_1)} = \sum_{k=0}^{\infty} |x_k|.$$

Lemma 2.1. ([4] p.253) A matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.

Lemma 2.2. ([5] p.59) T has a dense range if and only if T^* is one to one.

Lemma 2.3. ([5] p.60) T has a bounded inverse if and only if T^* is onto.

3. Fine spectra of the upper triangular double-band matrices $U(r, s)$ over sequence space γ

In this section, we give the main result about the spectrum and the subdivisions of spectrum. The spectral results are clear when $s = 0$, so in what follows we will have $s \neq 0$.

Theorem 3.1. $U(r, s): \gamma \rightarrow \gamma$ is a bounded linear operator with $|r + s| \leq \|U(r, s)\|_{(\gamma, \gamma)} \leq |r| + |s|$.

Proof. It is not hard to prove that $U(r, s)$ is linear from $U(r, s): \gamma \rightarrow \gamma$ to itself and so we omit it.

Since $U(r, s)x = (rx_k + sx_{k+1})$ for any $x \in \gamma$ and for all $k \in \mathbb{N}$ then we have

$$\begin{aligned} \|U(r, s)x\|_{(\gamma, \gamma)} &= \sup_n \left| \sum_{k=0}^n rx_k + sx_{k+1} \right| = \sup_n \left| r \sum_{k=0}^n x_k + s \sum_{k=0}^n x_{k+1} \right| \\ &\leq |r| \sup_n \left| \sum_{k=0}^n x_k \right| + |s| \sup_n \left| \sum_{k=0}^n x_{k+1} \right| = (|r| + |s|) \|x\|_{\gamma} . \end{aligned} \quad (3.1)$$

Let us take $e^{(2)} \in \gamma$. Then, since $U(r, s)e^{(2)} = (s, r, 0, 0, \dots)$, we find that

$$\|U(r, s)\|_{(\gamma, \gamma)} \geq \frac{\|U(r, s)e^{(2)}\|_{(\gamma, \gamma)}}{\|e^{(2)}\|_{(\gamma, \gamma)}} = |s + r|. \quad (3.2)$$

Thus, we have the result from (3.1) and (3.2).

Theorem 3.2. $\sigma_p(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}$.

Proof. Consider $U(r, s)x = \alpha x$ for $U(r, s)x = \alpha x$. Then, by solving the system of equations,

$$\left. \begin{array}{l} rx_0 + sx_1 = \alpha x_0 \\ rx_1 + sx_2 = \alpha x_1 \\ rx_2 + sx_3 = \alpha x_2 \\ \vdots \\ rx_n + sx_{n+1} = \alpha x_n \\ \vdots \end{array} \right\}. \quad (3.3)$$

$$\text{we find that } x_n = \left(\frac{\alpha - r}{s} \right)^n x_0 \text{ for } n \in \mathbb{N}. \quad (3.4)$$

If x_0 is the first non-zero term of the sequence $x = (x_n)$, then $r = \alpha$ and $x_i = 0$ for all $i \in \mathbb{N}$ which contradicts to the fact that $x \neq \theta$.

Now, let $\alpha \neq r$. Due to the fact that we take $x \in \gamma$, $\left(\sum_{k=0}^n x_k \right)$ is in the convergent sequence space c . That is, $\lim_{n \rightarrow \infty} \sum_{k=0}^n x_k = \sum_{k=0}^{\infty} x_k < \infty$.

On the other hand, since $\sum_{k=0}^{\infty} |x_k| = \sum_{k=0}^{\infty} \left| \frac{\alpha - r}{s} \right| < \infty$ if and only if $|r - \alpha| < |s|$, we can say that $\left(\sum_{k=0}^n x_k \right) \in c$ for $|r - \alpha| < |s|$. Thus, $\alpha \in \sigma_p(U(r, s), \gamma)$ if and only if $|r - \alpha| < |s|$. This completes the proof.

Theorem 3.3. $\sigma_p(U^*(r, s), \gamma^*) = \emptyset$.

Proof. Suppose that $U^*(r, s)x = \alpha x$ for $\theta \neq x \in \gamma^* \cong l_1$. Then, consider the system of the linear equations

$$\left. \begin{array}{l} rx_0 = \alpha x_0 \\ sx_0 + rx_1 = \alpha x_1 \\ sx_1 + rx_2 = \alpha x_2 \\ \vdots \\ sx_{k-1} + rx_k = \alpha x_k \\ \vdots \end{array} \right\}. \quad (3.5)$$

If x_0 is the no-zero term of the sequence $x = (x_n)$, then $\alpha = r$ and we obtain from (3.5) $x_{i-1} = 0$ for any $i \in \mathbb{N}$ which is a contradiction.

If x_j is the non-zero term of the sequence $x = (x_n)$ for $1 \leq j \leq n$ and $n \in \mathbb{N}$, then we obtain from (3.5) $x_j = \left(\frac{s}{\alpha - r} \right) x_{j-1}$ which implies that $x_j = 0$. This contradicts the fact that $x_j \neq 0$. Thus, the proof is completed.

Corollary 3.4. $\sigma_r(U(r, s), \gamma) = \emptyset$.

Theorem 3.5. $\sigma(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

Proof. Let $y = (y_k) \in \gamma^* \cong l_1$ and by consider the equation $(U^*(r, s) - \alpha I)x = y$, we have

$$\left. \begin{aligned} (r - \alpha)x_0 &= y_0 \\ sx_0 + (r - \alpha)x_1 &= y_1 \\ sx_1 + (r - \alpha)x_2 &= y_2 \\ &\vdots \end{aligned} \right\} \quad (3.6)$$

and we obtain from (3.6) $x_k = \frac{1}{r - \alpha} \sum_{i=0}^k \left(\frac{-s}{r - \alpha} \right)^{k-i} y_i$ for all $k \in \mathbb{N}$. One can see that

$$\sum_{i=0}^k |x_k| \leq \frac{1}{|r - \alpha| - |s|} \sum_{i=0}^k |y_k| \text{ for } |s| < |r - \alpha|$$

and all $k \in \mathbb{N}$. If k goes to infinity, then we get

$$\|x\|_{l_1} \leq \frac{1}{|r - \alpha| - |s|} \|y\|_{l_1}.$$

Thus, $(U^*(r, s) - \alpha I)$ is onto for $|s| < |r - \alpha|$ and by Lemma 2.3, $(U(r, s) - \alpha I)$ has a bounded inverse, that is, $\sigma_c = (U(r, s), \gamma) \subset \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$ for $|s| < |r - \alpha|$. Combining this with Theorem 3.2 and Corollary 3.4, we get

$$\{\alpha \in \mathbb{C} : |\alpha - r| < |s|\} \subset \sigma(U(r, s), \gamma) \subset \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}. \quad (3.7)$$

Since the spectrum of any bounded linear operator is closed, we have the result from (3.7).

Theorem 3.6. $\sigma_c(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}$.

Proof. Since $\sigma(U, \gamma) = \sigma_p(U, \gamma) \cup \sigma_c(U, \gamma) \cup \sigma_r(U, \gamma)$, then we have

$$\sigma_c(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}.$$

Goldberg Classifications for the operator $U(r, s)$ are given at following two theorems:

Theorem 3.7. If $|r - \alpha| < |s|$ then $\alpha \in \sigma(U, \gamma)I_3$.

Proof. To prove this theorem, we must show that $U_\alpha^{-1} = (U(r, s) - \alpha I)^{-1}$ does not exist and $U_\alpha^{-1} = (U(r, s) - \alpha I)^{-1}$ is surjective for $|r - \alpha| < |s|$, that is, $R(U_\alpha) = \gamma$.

Let $\alpha \in S$. Then, since we know that $\alpha \in \sigma_p(U, \gamma)$ by theorem 2, U_α^{-1} does not exist.

Now, let us prove that U_α^{-1} is surjective for $|r - \alpha| < |s|$.

Let $y = (y_0, y_1, y_2, \dots) \in \gamma$ and by considering the equation $U_\alpha x = y$, we have

$$\left. \begin{aligned} (r - \alpha)x_0 + sx_1 &= y_0 \\ (r - \alpha)x_1 + sx_2 &= y_1 \\ (r - \alpha)x_2 + sx_3 &= y_2 \\ &\vdots \end{aligned} \right\}. \quad (3.8)$$

If we choose $x = (x_0, x_1, x_2, \dots)$ such that $x_0 = 0$, then we obtain from (3.8)

$$x_k = \frac{1}{s} \sum_{i=0}^{k-1} \left(\frac{r - \alpha}{s} \right)^{k-i-1} y_i \text{ for } k \geq 1.$$

One can see that

$$\sum_{i=0}^{k-1} |x_i| \leq \frac{1}{|s| - |r - \alpha|} \sum_{i=0}^{k-1} |y_i| \quad (3.9)$$

because of $\sum_{i=0}^{k-1} \left| \frac{r - \alpha}{s} \right|^{k-i-1} \leq \frac{|s|}{|s| - |r - \alpha|}$ for all $k \geq 1$ and $|r - \alpha| < |s|$.

Taking limits of both sides of the inequality in (3.9) for $k \rightarrow \infty$, we get

$\lim_{k \rightarrow \infty} \left(\sum_{i=0}^{k-1} |x_i| \right) \leq \frac{1}{|s| - |r - \alpha|} \lim_{k \rightarrow \infty} \left(\sum_{i=0}^{k-1} |y_i| \right) < \infty$ and so $x \in \gamma$. This completes the proof.

Theorem 3.8. If $|r - \alpha| = |s|$ then $\alpha \in \sigma(U, \gamma) III_2$.

Proof. Let $|r - \alpha| = |s|$. It is easy to show that U_α^{-1} is not surjective.

On the other hand, if $|r - \alpha| = |s|$ then $\alpha \in \sigma_c(U, \gamma)$. Hence, we can write $\overline{R(U_\alpha)} = \gamma$ for $|r - \alpha| = |s|$. This completes the proof.

Finally, we give the approximate spectrum, the defect spectrum and the compression spectrum as a corollary:

Corollary 3.9. The following statements hold:

- i) $\sigma_{ap}(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\},$
- ii) $\sigma_\delta(U(r, s), \gamma) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\},$
- iii) $\sigma_{co}(U(r, s), \gamma) = \emptyset.$

Proof. (i) By [17], since $\sigma_{ap}(U(r, s), \gamma) = \sigma(U(r, s), \gamma) \setminus \sigma(U(r, s), \gamma) III_1$

from Table 1 and we have $\sigma(U(r, s), \gamma) III_1 = \sigma(U(r, s), \gamma) III_2 = \emptyset$ by Corollary 3.4 then we reach the result.

At the same way, since

$$\sigma_\delta(U(r, s), \gamma) = \sigma(U(r, s), \gamma) \setminus \sigma(U(r, s), \gamma) I_3 \quad (3.10)$$

and

$$\sigma_{co}(U(r, s), \gamma) = \sigma(U(r, s), \gamma) III_1 \cup \sigma(U(r, s), \gamma) III_2 \cup \sigma(U(r, s), \gamma) III_3, \quad (3.11)$$

it is seen the results in (b) and (c) by helping Corollary 3.4, Theorem 3.7 and Theorem 3.3, respectively.

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