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Warped Product Submanifolds of an S-Manifold

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Abstract

In this paper we prove that there does not exist warped product of the type $N_\theta \times_\lambda N_T$ and $N_T \times_\lambda N_\theta$ of an S-manifold , where N_θ and N_T are the slant and invariant submanifold of an S-manifold .We observe that only warped product of the type $N_T \times_\lambda N_\perp$ exist,with $N_\perp$ as a anti-invariant submanifolds of an S-manifold.Some basic results are discussed for the warped product of the type $N_T \times_\lambda N_\perp$ and finally ,we prove an inequality for the squared norm of second fundamental form and equality case is also discussed.

Mathematics Subject Classification: 53C25, 53C40, 53C42, 53D15

Keywords: warped product, semi-slant, S-manifold

1 Introduction

The study of semi-invariant submanifold or contact CR-submanifolds of almost contact metric manifolds was initiated by A.Bejancu [3] .In particular ,semi-invariant submanifolds of different classes of almost contact metric manifolds have also been studied (c.f.,[4] , [18]).Later N.Papaghuic[18] generalized the notion further and introduced semi-slant submanifolds of almost Hermition manifolds which includes the class of slant , holomorphic ,totally real and CR-submanifolds. J.L.Cabrerizo et al. [15] have studied semi-slant submanifolds of Sasakian manifolds. For manifolds with an f-structure ,D.E Blair [9] introduced S-manifolds as the analogue of Kaehler structure in almost complex case and of Sasakian structure in the almost contact case .

R.L.Bishop and B.O'Neil [20] introduced the notion of warped product manifolds. These manifolds are generaliztion of Riemannian product manifolds and occur naturally e.g.,surface of revolution is a warped product manifold ([16] , [21]).Due to wide applications of warped product submanifolds this becomes a fascinating and interesting topic for research and many articles are available in literature .CR-warped product was introduced by B.Y.chen [7, 8], he studied warped products CR-submanifolds in the setting of Kaehler manifolds and showed that there does not exist warped product CR-submanifolds of the form $M_\perp \times_f M_T$, therefore he considered warped product CR-submanifolds of the types $M_T \times_f M_\perp$ and established a relation ship between the warping fuction$f$ and the squared norm of the second fundamental form of the CR-warped product submanifolds in Kaehler manifolds .In the available literature,many geometers have studied warped products in the setting of almost contact metric manifolds (c.f. , [13] , [21] , [22]).

2 Preliminaries

Let $(\overline{M}, g)$ be a Riemannian manifold with dim $(\overline{M}) = 2n + s$. $\overline{M}$ is said to be an S-manifolds if there exists on $\overline{M}$ an f-structure f of rank 2n and s global vector fields $\xi_1, \xi_2, ..., \xi_s$(structure vector fields) such that (c.f.,[9]).

(i) If $\eta_1, \eta_2, ...\eta_s$ are the dual 1-form of $\xi_1, \xi_2, ..., \xi_s$, then

$$f\xi_\alpha = 0\ , \eta_\alpha \circ f = 0\ , f^2 = -\bar{I} + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha \tag{2.1}$$

$$g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y) \tag{2.2}$$

for any $X, Y \in T\overline{M}$, $\alpha = 1, 2, ...s$

ii)The f-structure f is normal ,that is

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$$

where $[f,f]$ is the Nijenhuis tensor of f.

iii) $\eta_1 \wedge \eta_2 \wedge ..., \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ and $d\eta_1 = d\eta_2 = ... = F$,for any α , where F is the fundamental 2-form defined by

$$F(X,Y) = g(X, fY), \ for \ \ any \ \ X, Y \in T\overline{M}.$$

In the case s=1, an S-manifold is a Sasakian manifold . For s⩾ 2 , examples of S-manifolds are given in [9, 10, 11, 12] .

Moreover, for the Riemannian conection $\overline{\nabla}$ of g on an S-manifold $\overline{M}$, the following formulas were also proved in [9] .

$$(\overline{\nabla}_X f)Y = \sum_{\alpha=1}^{s} \{g(X,Y)\xi_\alpha - X\eta_\alpha(Y)\} \tag{2.3}$$

$$\overline{\nabla}_X \xi_\alpha = -fX \tag{2.4}$$

for any $X, Y \in T\overline{M}$, and $\alpha = 1, ..., s$.

Let $\mathcal{L}$ denotes the distribution determined by $-f^2$and μ the complementary distribution.μ is determined by $f^2 + \bar{I}$ and spanned by $\xi_1, \xi_2, ..., \xi_s$.If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$ for any α, and if $X \in \mu$, then $fX = 0$.

Throughout ,we denote by $\overline{M}$ an S-manifold ,M a submanifold of $\overline{M}$ with structure vector fields $\xi_1, \xi_2, ..., \xi_s$ tangent to M. h and A denote the second fundamental form and the shape opreator of the immersion of M into $\overline{M}$ respectively .If ∇ is the induced connection on M, the Gauss and Weingarten formulae of M into $\overline{M}$ are then given as follows

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^\perp V, \tag{2.6}$$

for all vector fields X, Y on M and normal vector fields V on M ,$\nabla^\perp$denotes the connection on the normal bundle $TM^\perp$of M. h and A are related by

$$g(A_V X, Y) = g(h(X,Y), V) \tag{2.7}$$

where the induced Riemannian metric on M is denoted by the same symbol g. Now , for any $x \in M, X \in T_xM$ and $V \in T_xM^{\perp}$,we put

$$fX = TX + NX \tag{2.8}$$

$$fV = tV + nV \tag{2.9}$$

where TX and NX are the tangential and normal parts of fX respectively and tV and nV are the tangential and normal parts of fV.

The covariant derivatives ∇T and ∇N are defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.10}$$

$$(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y \tag{2.11}$$

Now, Let M be an n-dimensional submanifold immersed in $\overline{M}$.M is said to be invariant submanifold if $\xi_\alpha \in TM$ for any α and $fX \in TM$, for any $X \in TM$ otherwise ,M is said to be anti-invariant submanifold if $fX \in TM^{\perp}$, for any $X \in TM$.

For a submanifold M of an S-manifold $\overline{M}$ by equation (2.4),(2.5) and (2.8), we get

$$\nabla_X \xi_\alpha = -TX \tag{2.12}$$

$$h(X, \xi_\alpha) = -NX \tag{2.13}$$

for each $X \in TM$.

Now using equations (2.3),(2.5),(2.8),(2.10),(2.11) we get

$$(\nabla_X T)Y = A_{NY}X + th(X,Y) + \sum_{\alpha=1}^{s} g(X,Y)\xi_\alpha - \sum_{\alpha=1}^{s} X\eta_\alpha(Y) \tag{2.14}$$

$$(\nabla_X N)Y = -h(X, TY) + nh(X, Y) \tag{2.15}$$

For any $x \in M$ and $X \in T_X M$ if the vectors X and ξ_α $(\alpha = 1, ..., s)$ are linearly independent , the angle $\theta(X) \in \left[0, \frac{\pi}{2}\right]$ between fX and $T_X M$ is well defined . If $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_X M$,we say that M is slant in $\overline{M}$ .The constant angle θ is called the slant angle of M in $\overline{M}$. If $\theta = \frac{\pi}{2}$ then M is an anti-invariant submanifold and if $\theta = 0$ M is invariant submanifold .

Carriazo et al.[1] give the following characterization for slant submanifold of S-manifolds.

Theorem 2.1 [1] *Let M be a submanifold of an S-manifold $\overline{M}$ such that $\xi_\alpha \in TM$.Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda \sum_{\alpha=1}^{s} (\bar{I} - \eta_\alpha \otimes \xi_\alpha) \tag{2.16}$$

Furthermore, in such case ,if θ is the slant angle of M, then it verifies that $\lambda = \cos^2 \theta$.

Now, we have the following Corollary,which can be proved directly by (2.16).

Corollary 2.1 *Let M be a slant submanifold of an S-manifold $\overline{M}$ such that $\xi_\alpha \in TM$.Then,*

$$g(TX, TY) = \cos^2 \theta \left[g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y) \right]$$

$$g(NX, NY) = \sin^2 \theta \left[g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y) \right]$$

for any $X, Y \in TM$.

A semi-slant submanifold M of an almost contact metric manifold $\overline{M}$ is a submanifold which contains two orthogonal complementary distribution D

and D_θ,such that D is invariant under f and D_θ is slant with slant angle $\theta \neq 0$, i.e ,$fD = D$and fZ makes a constant angle θ with TM for each $Z \in D_\theta$.In particular if $\theta = \frac{\pi}{2}$,then semi-slant submanifolds reduced to CR-submanifold defined in [3] .For a semi-slant submanifold M of an S-manifold , we have

$$TM = D \oplus D_\theta \oplus \langle \xi \rangle$$

The orthogonal complement of ND_θ in the normal bundle $TM^\perp$, is an invariant subbundle of $TM^\perp$and is denoted by μ .Thus, we have

$$TM^\perp = ND_\theta \oplus \mu \tag{2.17}$$

A semi-slant submanifold M is called a semi-slant product if the distributions D and D_θ are involutive and parallel on M. In this case M is foliated by the leaves of these distributions.

As a generalization of the product manifolds and in particular of a semi-slant product submanifold , one can consider warped product of manifolds which are defined as

Definition 2.1 Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and λ positive differentiable function on B .The warped product of B and F is the Riemannian manifold $(B \times F, g)$,where

$$g = g_B + \lambda^2 g_F$$

For a warped product manifold $N_1 \times_\lambda N_2$,we denote by $D_1$and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively .

In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift .In case of semi-slant warped product submanifolds $D_1$and D_2 are replaced by D and D_θ respectively .

The warped product manifold $(B \times F, g)$ is denoted by $B \times_\lambda F$.If X is the tangent vector field to $M = B \times_\lambda F$ at (p, q) then

$$\|X\|^2 = \|d\pi_1 X\|^2 + \lambda^2(p) \|d\pi_2 X\|^2 .$$

R.L.Bishop and B.O'Neill [20] proved the following

Theorem 2.2 *Let $M = B \times_\lambda F$ be a warped product manifolds .If $X, Y \in TB$ and $V, W \in TF$ then*

(i) $\nabla_X Y \in TB$,
(ii) $\nabla_X V = \nabla_V X = (\frac{X\lambda}{\lambda})V$,
(iii) $\nabla_V W = \frac{-g(V,W)}{\lambda}\nabla\lambda$.
$\nabla\lambda$ *is the gradient of* λ *and is defined as*

$$g(\nabla\lambda, X) = X\lambda, \tag{2.18}$$

for all $X \in TM$.

Corollary 2.2 *On a warped product manifold* $M = N_1 \times_\lambda N_2$*, the following statements hold*
(i) N_1 *is totally geodesic in* M,
(ii) N_2 *is totally umbilical in* M,

3 Warped Product Submanifolds of S-manifolds

Let $\overline{M}$ be an S-manifold.Throughout this section,we denote by N_T an invariant submanifold of $\overline{M}$ and N_θ a slant submanifold of $\overline{M}$,with slant angle θ.

Matsumoto and Mihai [17] proved the following theorem .

Theorem 3.1 *If* $N_1 \times_\lambda N_2$ *is a warped product submanifold of a Sasakian manifold* $\overline{M}$, *where* N_1 *and* N_2 *are any submanifold of a Sasakian manifold* $\overline{M}$ *with* ξ *tangential to* N_2. *Then* M *is a Riemannian product .*

On the line of Theorem 3.1 , we can prove the following theorem

Theorem 3.2 *If* $N_1 \times_\lambda N_2$ *is a warped product submanifold of an S-manifold* $\overline{M}$,*where* N_1*and* N_2 *are any submanifold of an S-manifold* $\overline{M}$ *with* ξ_α *tangential to* N_2.*Then* M *is a Riemannian product .*

Hence we can consider only nontrivial semi-slant warped product submanifolds as $N_\theta \times_\lambda N_T$ and $N_T \times_\lambda N_\theta$ with $\xi_1, \xi_2, ..., \xi_s$ tangential to N_θ and N_T respectively .If $\theta = \frac{\pi}{2}$ these warped products are known as warped product contact CR-submanifolds and contact CR-warped product submanifolds respectively .

Theorem 3.3. *Let* $\overline{M}$ *be a* $(2m+s)$*-dimensional S-manifold .Then there do not exists warped product submanifolds* $N_\theta \times_\lambda N_T$ *on* $\overline{M}$ *such that* N_θ *is slant* *submanifold tangent to $\xi_1, \xi_2, ..., \xi_s$ and N_T is an invariant submanifold of $\overline{M}$.*

Proof. For warped product submanifold $N_\theta \times_\lambda N_T$ of $\overline{M}$ with ξ_α tangential to N_θ.Then by Theorem 2.2

$$\nabla_X Z = \nabla_Z X = Z \ln \lambda X, \tag{3.1}$$

for any $X \in TN_T$ and $Z \in TN_\theta$.In particular for $Z = \xi_\alpha$, from equations (2.12) and (3.1) ,

$$\xi_\alpha \ln \lambda X = 0,$$

this mean $\xi_\alpha \ln \lambda = 0$ i.e.,λ is constant , hence warped product does not exists.

On the same line of Theorem 3.3 of [22] ,we can prove the following theorem.

Theorem 3.4 *There does not exists a warped product semi-slant submanifold of the type $N_T \times_\lambda N_\theta$ in an S-manifold other than a contact CR-warped submanifold.*

The above theorem motivates us to study the warped product of the type $N_T \times_\lambda N_\perp$ in S-manifolds .

Let $M = N_T \times_\lambda N_\perp$ be a contact CR-warped product submanifold of an S-manifold $\overline{M}$. In view of decomposition (2.17) , we may write

$$h(X, Y) = h_{fD^\perp}(X, Y) + h_\mu(X, Y) \tag{3.2}$$

for each $X, Y \in TM$,where $h_{fD^\perp}(X, Y) \in fD^\perp$and $h_\mu(X, Y) \in \mu$.If $\{e_1, e_2, ..., e_n\}$ be a local orthonormal frame of vector fields on M then we define

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Now we have the following proposition for the warped product of the type $N_T \times_\lambda N_\perp$.

Proposition 3.1. *Let $M = N_T \times_\lambda N_\perp$ be a contact CR-warped product submanifold of an S-manifold $\overline{M}$. Then*

(i) $h_{fD^\perp}(X, Z) = -fX \ln \lambda fZ - \sum_{\alpha=1}^{s} \eta_\alpha(X) fZ,$

(ii) $g(h(fX, Z), fW) = X \ln \lambda g(Z, W),$

(iii) $g(h(fX, Z), fh(X, Z)) = \|h(X, Z)\|^2,$

for any $X \in TN_T$ and $Z, W \in TN_\perp$.

Proof. By Gauss formula

$$h(fX, Z) = (\overline{\nabla}_Z f)X + f\nabla_Z X + fh(X, Z) - \nabla_Z fX,$$

Using equations (2.3),(2.5)and (3.1), we have

$$h(fX,Z) = -\sum_{\alpha=1}^{s} Z\eta_\alpha(X) + X\ln\lambda fZ + fh(X,Z) - fX\ln\lambda Z$$

Comparing tangential parts on above equation , we get

$$\sum_{\alpha=1}^{s} Z\eta_\alpha(X) = fh_{fD^\perp}(X,Z) - fX\ln\lambda Z,$$

Or

$$fh_{fD^\perp}(X,Z) = fX\ln\lambda Z + \sum_{\alpha=1}^{s} Z\eta_\alpha(X),$$

taking inner product with $W \in D^\perp$ on both side , we find

$$-g(h(X,Z),fW) = fX\ln\lambda g(Z,W) + \sum_{\alpha=1}^{s} \eta_\alpha(X)g(Z,W),$$

Or

$$h_{fD^\perp}(X,Z) = -fX\ln\lambda fZ - \sum_{\alpha=1}^{s} \eta_\alpha(X)fZ,$$

Which proves the part (i) of the proposition .

Now on comparing the normal parts

$$h(fX,Z) = X\ln\lambda fZ + fh_\mu(X,Z) \tag{3.3}$$

Or

$$h(fX,Z) - fh_\mu(X,Z) = X\ln\lambda fZ,$$

Taking inner product with fW,the above equation yields

$$g(h(fX,Z),fW) = X\ln\lambda g(Z,W)$$

Again taking inner product with $fh(X,Z)$ in (3.3),we find

$$g(h(fX,Z),fh(X,Z)) = \|h(X,Z)\|^2,$$

which is the part (iii) of the proposition .

For contact CR-warped product submanifold $M = N_T \times_\lambda N_\perp$ of an S-manifold $\overline{M}$, we have the following theorem.

Theorem 3.5 *Let $M = N_T \times_\lambda N_\perp$ be a contact CR-warped product submanifold of an S-manifold $\overline{M}$ then*

(i) The squared norm of the second fundamental form satisfies

$$\|h\|^2 \geqslant 2q \|\nabla \ln \lambda\|^2 + qs,$$

where $\nabla \ln \lambda$ is the gradient of $\ln \lambda$ and q is the dimension of anti-invariant distribution .

(ii) The equality holds if $h(D, D) = 0$ and $h(D^\perp, D^\perp) = 0$.

Proof. Let $\{\xi_1, \xi_2, ..., \xi_s, X_1, X_2, ..., X_P, X_{P+1} = fX_1, ..., X_{2P} = fX_P\}$ be a local orthonormal frame of vector fields on N_T and $\{Z_1, Z_2, ..., Z_q\}$ be the local orthonormal frame of vector fields on $N_\perp$.Then by the definition of squared norm of the mean curvature vector

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^{2P} g(h(X_i, X_j), h(X_i, X_j)) + \sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)) \\ &+ \sum_{r,s=1}^{q} g(h(Z_r, Z_s), h(Z_r, Z_s)) + \sum_{i=1}^{2p}\sum_{\alpha=1}^{s} g((h(X_i, \xi_\alpha), h(X_i, \xi_\alpha)) \\ &+ \sum_{r=1}^{q}\sum_{\alpha=1}^{s} g(h(Z_r, \xi_\alpha), h(Z_r, \xi_\alpha)) \end{aligned} \tag{3.4}$$

Using equation (2.12), the above equation provides us the following inequality

$$\|h\|^2 \geqslant \sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)) + \sum_{r=1}^{q}\sum_{\alpha=1}^{s} g(h(Z_r, \xi_\alpha), h(Z_r, \xi_\alpha))$$

On using part (i) of the proposition (3.1) and equation (3.3), the above inequality takes the form

$$\|h\|^2 \geqslant \sum_{i=1}^{2p}\sum_{r=1}^{q} (fX_i \ln \lambda)^2 g(Z_r, Z_r) + qs,$$

Or

$$\|h\|^2 \geqslant 2q \|\nabla \ln \lambda\|^2 + qs,$$

Which is a required inequality .

It is evident from (3.4), that if inequality holds identically, then $h(D, D) = 0$, $h(D^\perp, D^\perp) = 0$.

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