A Characterization of Duo-Rings in which Every Dedekind Finite Module is Finitely Generated

Sidy Demba Touré, Khady Diop, Sidy Mohamed Ould Mohamed and Mamadou Sangharé

Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
de l’Université Cheikh Anta Diop de Dakar, Sénégal

Abstract

Let \( R \) be an associative ring with \( 1 \neq 0 \) and \( M \) an unitary \( R \)-module. \( M \) is said to be Dedekind finite if \( M \) is not isomorphic to any proper direct summand of itself. The ring \( R \) is called \( FGDF \) ring if every Dedekind finite module is finitely generated. In this note we will prove that artinian principal ideal duo-rings characterize \( FGDF \)-duo-rings.

Keywords: Dedekind finite modules, hopfian modules, Artinian principal ideal rings, Duo-rings

1 Introduction

Let \( R \) be an associative ring with \( 1 \neq 0 \). An \( R \)-module \( M \) is said to be hopfian (generalized-hopfian) if every surjective endomorphism of \( M \) is an automorphism (resp. superfluous). The ring \( R \) is called \( FGS \)-ring if every hopfian \( R \)-module is finitely generated. It have been proved in [1] that artinian principal ideal duo-rings characterize \( FGS \)-duo-rings. A module \( M \) is
Dedekind finite if $M$ is not isomorphic to any proper direct summand of itself. Obviously any indecomposable module is Dedekind finite and a vector space is Dedekind finite if and only if it is finite dimensional. Vasconcelos proved that in a commutative ring every finitely generated module is hopfian (see [5]). On the other hand Haghany proved that any left hopfian module is Dedekind finite (see [4]). Thus, in a commutative ring every finitly generated module is Dedekind finite module. The converse of this property is not true in genral because the $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Q}$ is Dedekind finite but it is not finitely generated.

The ring $R$ is called $FGDF$-ring if any Dedekind finite module is finitely generated. $R$ is called a duo-ring if every one sided ideal is two sided. The purpose of this note is to give a characterization of $FGDF$-duo-rings.

## 2 The main result

The main result of this note is the following theorem.

**Theorem 2.1.**

Let $R$ be a duo-ring. Then the following statements are equivalent

(i) $R$ is an artinian principal ideal ring

(ii) $R$ is a $FGDF$-ring

To prove this theorem we need some results:

**Proposition 2.2.** [1] theorem 3.4

A ring $R$ is a $FGS$-duo-ring if and only if $R$ is an artinian principal ideal duo-ring.

**Proposition 2.3.** [2] proposition 3.2 (2)

If an $R$-module $M$ is a Dedeking finite module, then so is any direct summand of $M$.

**Proposition 2.4.** [2] proposition 3.2 (3)

If an $R$-module $M$ is a direct sum of an infinite family $(M_i)_{i \in I}$ of nonzero submodules of $M$ such that any two of them are isomorphic, then $M$ is not a Dedekind finite module.

**Lemma 2.5.** [4] corollary 1.4

If $M$ is a left hopfian module, then $M$ is a Dedeking finite module.

**Proposition 2.6.**

If $R$ is a left $FGDF$-ring then $R$ is a left $FGS$-ring.
Proof. Assume that $R$ is a left $FGDF$-ring. Let $M$ be a left hopfian module; then by lemma 2.4 $M$ is a Dedekind finite module and consequently $M$ is finitely generated. 

**Proof of the main theorem**

Proof. (i) $\implies$ (ii) Assume that $R$ is an artinian principal ideal duo-ring. Following [3] every left $R$-module is a direct sum of cyclic submodules. Let now $M$ be a Dedekind finite module which is not finitely generated. We can write $M = \bigoplus_{i \in I} M_i$ where $(M_i)_{i \in I}$ is an infinite family of cyclic submodules of $M$. Since there is only a finite number of non isomorphic cyclic submodules, then there is an infinite sub-family $(M'_j)_{j \in J}$ of the family $(M_i)_{i \in I}$ such that any two of them are isomorphic. Therefore, we can write $M = K \oplus L$ where $L = \bigoplus_{j \in J} M'_j$. By proposition 2.3 $L$ is a Dedekind finite module. By proposition 2.4 $L$ is not a Dedekind finite module. This is a contradiction.

(ii) $\implies$ (i) Assume that $R$ is $FGDF$-duo-ring. Then by proposition 2.6 $R$ is a $FGS$-duo-ring and consequently, by proposition 2.2 $R$ is an artinian principal ideal duo-ring. 

**Corollary 2.7.** Let $R$ be a duo-ring. Then the following statements are equivalent

(i) $R$ is an artinian principal ideal ring

(ii) $R$ is a $FGS$-ring

(iii) $R$ is a $FGDF$-ring.

**References**


Received: March 17, 2014