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Convergence Theorems for Pettis Integral of Functions Taking Values in Locally Convex Spaces

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Abstract

In this paper we present convergence theorems for Pettis integral of functions defined on a complete probability space and taking values in a complete locally convex topological vector space.

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1 Introduction

We present convergence theorems for Pettis integral of functions defined on a complete probability space and taking values in a complete locally convex topological vector space, Theorems 2.2 and 2.3, which are the analogues of the known convergence theorems for Pettis integral of functions taking values in a Banach space. For that purpose, we apply a technique which is based on the limit projective of Banach spaces, Lemma 2.1.

Theorem 2.2 is the analogue of Vitali Theorem for Pettis integral of functions taking values in a Banach space, Theorem 8.1 in [6]. Theorem 2.2 has been proved by another approach in [7]. Theorem 2.3 is the analogue of Theorem 2.8 in [9]. For more information on convergence theorems for Pettis integral, see [6], [7], [8], [4] and [2].

Throughout this paper (Ω, Σ, μ) is a complete probability space and V is a complete locally convex space with its topology τ and topological dual V'. By P we denote the family of all continuous semi-norms in this space; for every $p \in \mathrm{P}$, $\widetilde{V}^p$ denotes the quotient vector space of the vector space V with respect to the equivalence relation $x \sim_p y \Leftrightarrow p(x-y) = 0$; the map $\phi_p : V \to \widetilde{V}^p$ is the canonical quotient map, thus is the equivalence class of an element with respect to the relation "$\sim_p$"; the quotient normed space $(\widetilde{V}^p, \widetilde{p})$ is called the normed component of the space V, where $\widetilde{p}(\phi_p(x)) = p(x)$, for each $x \in V$; the Banach space $(\overline{V}^p, \overline{p})$ which is the completion of the space $(\widetilde{V}^p, \widetilde{p})$ is called the Banach component of the space V; $\widetilde{V}'_p$ and $\overline{V}'_p$ are the topological duals of $(\widetilde{V}^p, \widetilde{p})$ and $(\overline{V}^p, \overline{p})$ respectively. It is easy to see that

$$V' = \{\widetilde{v}'_p \circ \phi_p / \widetilde{v}'_p \in \widetilde{V}'_p, p \in \mathrm{P}\}, \tag{1}$$

because for every $v' \in V'$, we have that $|v'(.)| \in \mathrm{P}$. For every $p, q \in \mathrm{P}$ such that $p \leq q$, we define the map $\widetilde{g}_{pq} : \widetilde{V}^q \to \widetilde{V}^p$ by $\widetilde{g}_{pq}(w_q) = w_p$, for all $w_q \in \widetilde{V}^q$, where $w_p = \phi_p(x)$, for some vector $x \in w_q$. We define also the map $\overline{g}_{pq} : \overline{V}^q \to \overline{V}^p$ as the continuous linear extension of $\widetilde{g}_{pq}$, for every $p, q \in \mathrm{P}$ such that $p \leq q$.

The following definition is given by [1], Definition 1.

Definition 1.1 *A function $f : S \to V$ is called Pettis integrable if the function $v' \circ f$ is Lebesgue integrable for each $v' \in V'$ and if for every measurable set E of Ω, there is a vector $x_E \in V$ such that $v'(x_E) = \int v'(f(s))$ for every $v' \in V'$. The vector x_E is said to be Pettis integral of function f on E and we denote:*

$$x_E = (P) \int_E f.$$

If V is a Banach space, this definition is the same with Definition 2, [3], p.52.

Definition 1.2 *A family H of real-valued integrable functions defined on Ω is said to be uniformly integrable if it satisfies the conditions*

1. $\sup_{h \in H} \int_\Omega |h| < \infty$,

2. *For each $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that the inequality*

$$\sup_{h \in H} \int_E |h| < \epsilon$$

holds for every $E \in \Sigma$ satisfying $\mu(E) \leq \delta(\epsilon)$.

Let $f : \Omega \to V$ be a function at let $p \in \mathrm{P}$. We put

$$Z_f^p = \{\widetilde{v}_p' \circ (\phi_p \circ f) / \widetilde{v}_p' \in B(\widetilde{V}_p')\},$$

where $B(\widetilde{V}_p')$ is the closed unit ball in $\widetilde{V}_p'$. It is clear to see that

$$Z_f^p = \{\overline{v}_p' \circ (\phi_p \circ f) / \overline{v}_p' \in B(\overline{V}_p')\}$$

where $B(\overline{V}_p')$ is the closed unit ball in $\overline{V}_p'$.

2 Convergence theorems for Pettis integral

The proof of the following auxiliary lemma is similar in spirit to the proof of Lemma 4.1 in [5].

Lemma 2.1 *Let $f : \Omega \to V$ be a function. The function f is Pettis integrable if and only if for every $p \in \mathrm{P}$ the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$. In this case, we have*

$$\phi_p((P) \int_E f) = (P) \int_E \phi_p \circ f \quad \textit{for all} \quad p \in \mathrm{P},\ E \in \Sigma. \tag{2}$$

Proof. Suppose that the function f is Pettis integrable and let $p \in \mathrm{P}$. Then, by Definition 1.1 and (1), the function $\widetilde{v}_p' \circ (\phi_p \circ f) = v' \circ f$ is the Lebesgue integrable, for each $\widetilde{v}_p' \in B(\widetilde{V}_p')$. We have also that for each $E \in \Sigma$, there is a vector $x_E \in V$ such that $v'(x_E) = \int_E v' \circ f$, for all $v' \in V'$. Hence, we obtain by (1) that

$$(\widetilde{v}_p' \circ \phi_p)(x_E) = \int_E (\widetilde{v}_p' \circ \phi_p) \circ f \quad \text{for each} \quad \widetilde{v}_p' \in B(\widetilde{V}_p').$$

Therefore, the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$ and $\phi_p(x_E) = (P) \int_E \phi_p \circ f$, for all $E \in \Sigma$.

Conversely, assume that for every $p \in \mathrm{P}$ the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$, and let $v' \in V'$ and $E \in \Sigma$ are given. Then, by the equality (1), there exist $p \in \mathrm{P}$ and $\widetilde{v}_p' \in B(\widetilde{V}_p')$ such that $v' = \widetilde{v}_p' \circ \phi_p$ and the function $v' \circ f = \widetilde{v}_p' \circ (\phi_p \circ f)$ is Lebesgue integrable. We have also that for each $p \in \mathrm{P}$ there exists $\overline{I}_p(E) \in \overline{V}^p$ such that

$$\overline{v}_p'(\overline{I}_p(E)) = \int_E \overline{v}_p' \circ (\phi_p \circ f) = \int_E \widetilde{v}_p' \circ (\phi_p \circ f) \quad \text{for all} \quad \overline{v}_p' \in \overline{V}_p' \tag{3}$$

where $\widetilde{v}_p' = \overline{v}_p'|_{\widetilde{V}^p}$.

Assume that two arbitrary continuous seminorms p and q such that $p \leq q$ are given. Since $\overline{I}_q(E) \in \overline{V}^q$, there is a sequence $(w_q^n) \subset \widetilde{V}^q$ such that $\lim_{n\to\infty} w_q^n = \overline{E}_q(E)$. Then, $\lim_{n\to\infty} w_p^n = \overline{g}_{pq}(\overline{I}_q(E))$, where $w_p^n = \widetilde{g}_{pq}(w_q^n)$, for all $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} \widetilde{v}'_p(w_p^n) = \overline{v}'_p(\overline{g}_{pq}(\overline{I}_q(E))) \quad \text{for all} \quad \overline{v}'_p \in \overline{V}'_p, \tag{4}$$

where $\widetilde{v}'_p = \overline{v}'_p|_{\widetilde{V}^p}$. By virtue of (3), we have

$$\begin{aligned}\lim_{n\to\infty} \widetilde{v}'_p(w_p^n) &= \lim_{n\to\infty} (\widetilde{v}'_p \circ \widetilde{g}_{pq})(w_q^n) = \int_E (\widetilde{v}'_p \circ \widetilde{g}_{pq}) \circ (\phi_q \circ f) \\ &= \int_E \widetilde{v}'_p \circ (\widetilde{g}_{pq} \circ \phi_q) \circ f = \int_E \widetilde{v}'_p \circ (\phi_p \circ f) = \overline{v}'_p(\overline{I}_p(E))\end{aligned}$$

for all $\overline{v}'_p \in \overline{V}'_p$, where $\widetilde{v}'_p = \overline{v}'_p|_{\widetilde{V}^p}$. The last result together with (4) yields

$$\overline{v}'_p(\overline{g}_{pq}(\overline{I}_q(E))) = \overline{v}'_p(\overline{I}_p) \quad \text{for all} \quad \overline{v}'_p \in \overline{V}'_p.$$

Therefore by Corollary IV.6.2 in [11], we obtain $\overline{g}_{pq}(\overline{I}_q(E)) = \overline{I}_p(E)$. Hence, by Theorem II.5.4 in [10], there is a vector $I_f(E) \in V$ such that

$$\phi_p(I_f(E)) = \overline{I}_p(E) \quad \text{for each} \quad p \in \mathrm{P}.$$

Thus, since $v' = \widetilde{v}'_p \circ \phi_p$, we have

$$v'(I_f(E)) = \widetilde{v}'_p(\phi_p(I_f(E))) = \widetilde{v}'_p(\overline{I}_p(E)) = \int_E \widetilde{v}'_p \circ (\phi_p \circ f) = \int_E (\widetilde{v}'_p \circ \phi_p) \circ f.$$

Hence

$$v'(I_f(E)) = \int_E v' \circ f,$$

and since v' and E are arbitrary, the last equality holds for each $v' \in V'$ and $E \in \Sigma$. This means that f is Pettis integrable and $I_f(E) = (P)\int_E f$, for all $E \in \Sigma$. Hence, the equality (2) holds and the proof is finished.

Theorem 2.2 *Let (f_n) be a sequence of Pettis integrable functions $f_n : \Omega \to V$ and let $f : \Omega \to V$ be a function such that*

(i) for every $v' \in V'$, we have $v' \circ f_n \to v' \circ f$ μ- a.e.;

(ii) for every $p \in \mathrm{P}$, we have $\bigcup_{n\in\mathbb{N}} Z^p_{f_n}$ is uniformly integrable.

Then, f is Pettis integrable and, for each $E \in \Omega$, we have

$$\lim_{n\to\infty} (P)\int_E f_n = (P)\int_E f$$

in the weak topology $\sigma(V, V')$.

Proof. By Lemma 2.1, each function $\phi_p \circ f_n$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$ and

$$(P)\int_E \phi_p \circ f_n = \phi_p((P)\int_E f_n) \quad \text{for each} \quad E \in \Omega.$$

By hypothesis and (1), for each $\widetilde{v}'_p \in \widetilde{V}'_p$, we have $\widetilde{v}'_p \circ (\phi_p \circ f_n) \to \widetilde{v}'_p \circ (\phi_p \circ f)$, μ- a.e. Thus, the conditions of Theorem 8.1 in [6] are satisfied. Therefore, the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$ and

$$\lim_{n\to\infty} \overline{v}'_p((P)\int_E \phi_p \circ f_n) = \overline{v}'_p((P)\int_E \phi_p \circ f) \quad \text{for each} \quad E \in \Omega,\ \overline{v}'_p \in \overline{V}'_p.$$

Hence, by Lemma 2.1, we obtain that f is Pettis integrable and

$$\lim_{n\to\infty} (\widetilde{v}'_p \circ \phi_p)((P)\int_E f_n) = (\widetilde{v}'_p \circ \phi_p)((P)\int_E f) \quad \text{for all} \quad E \in \Omega,\ \widetilde{v}'_p \in \widetilde{V}'_p,$$

where $\widetilde{v}'_p = \overline{v}'_p|_{\widetilde{V}^p}$. Further, we obtain by (1) that

$$\lim_{n\to\infty} v'((P)\int_E f_n) = v'((P)\int_E f) \quad \text{for all} \quad E \in \Omega,\ v' \in V'$$

and the proof is finished.

Theorem 2.3 *Let (f_n) be a sequence of Pettis integrable functions $f_n : \Omega \to V$ converging point-wise to a function $f : \Omega \to V$ in (V, τ). Then the following are equivalent.*

(i) for every $p \in \mathrm{P}$, the family $\bigcup_{n\in N} Z^p_{f_n}$ is uniformly integrable;

(ii) the function f is Pettis integrable and, for each $E \in \Omega$, we have

$$\lim_{n\to\infty} (P)\int_E (f_n) = (P)\int_E f \quad \textit{in } (V, \tau).$$

Proof. Note that the sequence (f_n) converges point-wise to f in (V, τ), if and only if for every $p \in \mathrm{P}$ the sequence $\phi_p \circ f_n$ converges point-wise to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$.

$(i) \Rightarrow (ii)$ Let p be an element of P. We have that the sequence $\phi_p \circ f_n$ converges point-wise to $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$. we have also by Lemma 2.1 that each function $\phi_p \circ f_n$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$ and

$$(P)\int_E (\phi_p \circ f_n) = \phi_p((P)\int_E f_n) \in \widetilde{V}^p \quad \text{for all} \quad E \in \Omega. \tag{5}$$

So, the conditions of Theorem 2.8 in [9] are satisfied. Therefore, the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$ and the equality

$$\lim_{n\to\infty}(P)\int_E \phi_p \circ f_n = (P)\int_E \phi_p \circ f, \tag{6}$$

holds in $(\overline{V}^p, \overline{p})$, for each $E \in \Omega$.

Let E be an element of Σ. We have that for every $p \in \mathrm{P}$, the function $\phi_p \circ f$ is Pettis integrable in the Banach component $(\overline{V}^p, \overline{p})$. Therefore, by Lemma 2.1, the function f is Pettis integrable and

$$(P)\int_E \phi_p \circ f = \phi_p((P)\int_E f) \in \widetilde{V}^p. \tag{7}$$

Hence, inserting the right-hand-sides of (5) and (7) to (6), we obtain

$$\lim_{n\to\infty}\phi_p((P)\int_E f_n) = \phi_p((P)\int_E f) \quad \text{for each} \quad p \in \mathrm{P}.$$

Therefore, $\lim_{n\to\infty}(P)\int_E f_n = (P)\int_E f$ in (V, τ).

$(ii) \Rightarrow (i)$ Assume that (ii) holds and let $E \in \Sigma$ and $p \in \mathrm{P}$. We have that the equality

$$\lim_{n\to\infty}\phi_p((P)\int_E f_n) = \phi_p((P)\int_E f), \tag{8}$$

holds in the Banach space $(\overline{V}^p, \overline{p})$. According to Lemma 2.1, the functions $\phi_p \circ f$ and $\phi_p \circ f_n$ are Pettis integrable in $(\overline{V}^p, \overline{p})$ and

$$\phi_p((P)\int_E f) = (P)\int_E \phi_p \circ f \quad \phi_p((P)\int_E f_n) = (P)\int_E \phi_p \circ f_n.$$

Hence, inserting the right-hand-sides of these equalities to (8), we obtain that the equality

$$\lim_{n\to\infty}(P)\int_E \phi_p \circ f_n = (P)\int_E \phi_p \circ f$$

holds in $(\overline{V}^p, \overline{p})$. Therefore, by Theorem 2.8 in [9], the family $\bigcup_{n\in N} Z^p_{f_n}$ is uniformly integrable and the proof is finished.

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