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Fixed Point Results in Dislocated Quasi Metric Spaces

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Abstract

The aim of this article is to prove some fixed point theorems in the context of dislocated quasi metric spaces. We have established a new fixed point theorem in complete dislocated quasi metric space using some new type of rationl contraction conditions. We have also proved a unique fixed point for generalized contraction by omitting the condition of continuity imposed by C. T. Aage and J. N. Salunke[2].

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1. Introduction

The concept of dislocated metric space was introduced by P. Hitzler [5] in which the self distance of points need not to be zero necessarily. They also generalized famous Banach's contraction principle in dislocated metric space. Dislocated metric space play a vital rule in topology, logical programming and electronic engineering. F. M. Zeyada et al.[9] develops the notation of complete dislocated quasi metric spaces and generalized the result of Hitzler[5] in dislocated quasi metric space. After F. M. Zeyada et al.[9] many papers have been published containing fixed point results in dislocated quasi metric spaces (see [1],[2],[6],[7]).

In this manuscript we have proved a new fixed point theorem in complete dislocated quasi metric space using some new type of rational contraction conditions also we have obtained a unique fixed point for generalized contraction by dropping the restriction of continuity. Our results generalizes some existing fixed point results in the literature.

2. Preliminaries

Throughout this paper $\mathbb{R}^+$ will represent the set of non negative real numbers.

Definition 2.1. [9]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions,

d_1) $d(x, x) = 0$;
d_2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;
d_3) $d(x, y) = d(y, x)$;
d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfy the conditions from d_1 to d_4 then it is called metric on X, if d satisfy conditions d_2 to d_4 then it is called dislocated metric (d-metric) on X, and if d satisfy conditions d_2 and d_4 then it is called dislocated quasi metric(dq-metric) on X.

Clearly every metric space is a dislocated metric space but the converse is not necessarily true as clear form the following example.

Example 2.1. Let X=[0,1] define the distance function $d : X \times X \to \mathbb{R}^+$ by,

$$d(x, y) = max\{x, y\}.$$

Clearly X is dislocated metric space but not a metric space.
Also every metric space is dislocated quasi metric space but the converse is not true, and every dislocated metric space is dislocated quasi metric space but the converse is not true as clear from the following example.

Example 2.2. Let $X = [0, 1]$ we define the function $d : X \times X \to \mathbb{R}^+$ as,

$$d(x, y) = |x - y| + |x| \quad \text{for all } x, y \in X.$$

Clearly X is dq-metric space but not a metric space nor dislocated metric space.
In our main work we will use the following definitions which can be found in [9].

Definition 2.2. A sequence $\{x_n\}$ in dq-metric space is called Cauchy sequence if for $\epsilon > 0$ there exist a positive integer N such that for $m, n \geq$ N, we have $d(x_m, x_n) < \epsilon$.

Definition 2.3. A sequence $\{x_n\}$ is called dq-convergent in X if for $n \geq$ N, we have $d(x_n, x) < \epsilon$ where x is called the dq-limit of the sequence $\{x_n\}$.

Definition 2.4. A dq-metric space (X, d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.5. Let (X, d) be a dq-metric space, a mapping $T : X \to X$ is called contraction if there exist $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \ \text{ for all } \ x, y \in X \text{ and } \alpha \in [0, 1).$$

Definition 2.6. [3]. Let (X, d) be a metric space a self mapping $T : X \to X$ is called generalized contraction if and only if for all $x, y \in X$, there exist c_1, c_2, c_3, c_4 such that $\sup\{c_1 + c_2 + c_3 + 2c_4\} < 1$ and

$$d(Tx, Ty) \leq c_1.d(x, y) + c_2.d(x, Tx) + c_3.d(y, Ty) + c_4.[d(x, Ty) + d(y, Tx)]$$

Lemma 1. [9]. *Limit in dq-metric space is unique.*

Theorem 2.1. [9]. *Let (X, d) be a complete dq-metric space $T : X \to X$ be a continuous contraction then T has a unique fixed point in X.*

3. Main Results

Theorem 3.1. *Let (X, d) be a complete dq-metric space $T : X \to X$ be a continuous self mapping satisfying the condition,*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(x, Ty)d(y, Ty)}{d(x, y) + d(y, Ty)} + \gamma \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \mu \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} \tag{1}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu \geq 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, ..., x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X, consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Now by use of (1) we have,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, Tx_n)d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_n, Tx_n)}$$

$$+\gamma \frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{1 + d(x_{n-1}, x_n)} + \mu \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}Tx_n)}{1 + d(x_{n-1}, x_n)}$$

$$= \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_{n+1})d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n-1})} + \gamma \frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} + \mu \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_n)}$$

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_n, x_{n+1}) + \mu d(x_{n-1}, x_{n+1})$$

by simplification we have,

$$d(x_n, d_{n+1}) \leq \frac{\alpha + \mu}{1 - (\beta + \gamma + \mu)} d(x_{n-1}, x_n)$$

$$\text{let} \qquad h = \frac{\alpha + \mu}{1 - (\beta + \gamma + \mu)} < 1$$

$$\text{so} \qquad d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n)$$

$$\text{also} \qquad d(x_{n-1}, x_n) \leq hd(x_{n-2}, x_{n-1})$$

$$\text{thus} \qquad d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1})$$

similarly proceeding we get

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$$

taking limit $n \to \infty$, as $h < 1$ so $h^n \to 0$ implies

$$d(x_n, x_{n+1}) \to 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete dq-metric space. So there must exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Now to show that u is a fixed point of T, since T is continuous therefore,
$\lim_{n\to\infty} Tx_n = Tu \quad \Rightarrow \quad \lim_{n\to\infty} x_{n+1} = Tu \quad \Rightarrow \quad Tu = u.$
Hence u is the fixed point of T.
Uniqueness: Let $u \neq v$ are two distinct fixed points of T then by (1),

$$d(u,v) = d(Tu,Tv) \leq \alpha d(u,v) + \beta \frac{d(u,Tv)d(v,Tv)}{d(u,v)d(v,Tv)} + \gamma \frac{d(u,Tu)d(v,Tv)}{1+d(u,v)} + \mu \frac{d(u,Tu)d(u,Tv)}{1+d(u,v)}.$$

Since u and v are fixed points of T so the above inequality becomes,

$$d(u,v) \leq \alpha d(u,v) + \beta \frac{d(u,v)d(v,v)}{d(u,v)d(v,v)} + \gamma \frac{d(u,u)d(v,v)}{1+d(u,v)} + \mu \frac{d(u,u)d(u,v)}{1+d(u,v)}$$

by use of (1)and using the fact that u, v are the fixed points of T we get,

$$d(u,u) = d(v,v) = 0$$

thus the above inequality take the form,

$$d(u,v) \leq \alpha d(u,v)$$

the above inequality is possible if $d(u,v) = 0$ similarly we can show that $d(v,u) = 0$ hence $u = v$.
Therefore fixed point of T is unique. □

In [2] C. T. Aage and J. N. Salunke proved that every continuous generalized contraction on a complete dq-metric space has a unique fixed point. The following result shows that the assumption of continuity can be omitted to obtain the theorem under less restrictive conditions.

Theorem 3.2. *Let (X,d) be a complete dq-metric space and $T : X \to X$ is a self mapping satisfying,*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \mu[d(x,Ty) + d(y,Tx)] \tag{2}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu \geq 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary in X We define a sequence $\{x_n\}$ in X by the rule,

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

by using (2) we have

$$\leq \alpha d(x_{n-1} + x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) + \mu[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$

$$= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) + \mu[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]$$

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) + \mu d(x_{n-1}, x_n) + \mu d(x_n, x_{n+1}).$$

Simplification yields,

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta + \mu}{1 - (\gamma + \mu)} d(x_{n-1}, x_n) \quad \text{let} \quad h = \frac{\alpha + \beta + \mu}{1 - (\gamma + \mu)} < 1$$

$$\text{so} \qquad d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n)$$

$$\text{also} \qquad d(x_{n-1}, x_n) \leq h d(x_{n-2}, x_{n-1})$$

$$\text{thus} \qquad d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1})$$

similarly proceeding we have,

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$$

since $h < 1$ taking limit $n \to \infty$ so $h^n \to 0$ and $d(x_n, x_{n+1}) \to 0$ hence $\{x_n\}$ is a Cauchy sequence in complete dq-metric space so there must exist $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now to show that u is the fixed point of T consider,

$$d(u, Tu) \leq d(u, x_n) + d(x_n, Tu) = d(u, x_n) + d(Tx_{n-1}, Tu)$$

$$\leq d(u, x_n) + \alpha d(x_{n-1}, u) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(u, Tu) + \mu[d(x_{n-1}, Tu) + d(u, Tx_{n-1})]$$

$$\leq d(u, x_n) + \alpha d(x_{n-1}, u) + \beta d(x_{n-1}, x_n) + \gamma d(u, Tu) + \mu[d(x_{n-1}, Tu) + d(u, x_n)]$$

taking limit $n \to \infty$ we have,

$$d(u, Tu) \leq (\gamma + \mu) d(u, Tu)$$

which is possible only if $d(u, Tu) = 0$ also

$$d(Tu, u) \leq d(Tu, x_n) + d(x_n, u) = d(Tu, Tx_{n-1}) + d(x_n, u)$$

$$\leq \alpha d(u, x_{n-1}) + \beta d(u, Tu) + \gamma d(x_{n-1}, Tx_{n-1}) + \mu[d(u, Tx_{n-1}) + d(x_{n-1}, Tu)] + d(x_n, u)$$

taking limit $n \to \infty$ we have,

$$d(Tu, u) \leq \beta d(u, Tu) + \mu d(u, Tu)$$

since $d(u, Tu) = 0$ therefore $d(Tu, u) \leq 0$ which is possible only if $d(Tu, u) = 0$ hence $d(u, Tu) = d(Tu, u) = 0 \Rightarrow Tu = u$.

Thus u is the fixed point of T.

Uniqueness: Let $u \neq v$ are two distinct fixed points of T then consider,

$$d(u, v) = d(Tu, Tv) \leq \alpha d(u, v) + \beta d(u, Tu) + \gamma d(v, Tv) + \mu[d(u, Tv) + d(v, Tu)]$$

since u and v are fixed points of T and using (2) we have,

$$d(u, u) = d(v, v) = 0$$

$$d(u, v) \leq \alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) + \mu[d(u, v) + d(v, u)]$$

$$d(u, v) \leq (\alpha + \mu) d(u, v) + \mu d(v, u) \tag{3}$$

$$d(v,u) \leq (\alpha+\mu)d(v,u) + \mu d(u,v) \tag{4}$$

subtracting (4) from (3) we have,

$$|d(u,v) - d(v,u)| \leq |(\alpha+u) - u||d(u,v) - d(v,u)|$$
$$|d(u,v) - d(v,u)| \leq |\alpha||d(u,v) - d(v,u)|$$

which is possible if

$$d(u,v) - d(v,u) = 0 \quad \Rightarrow \quad d(u,v) = d(v,u) \tag{5}$$

using (5) in (3) and (4) we have,

$$d(u,v) = d(v,u) = 0 \Rightarrow u = v$$

Hence fixed point of T is unique. □

Corollary 3.1. *Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self mapping satisfying,*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \mu[d(x,Ty) + d(y,Tx)]$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \mu \geq 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then T has a unique fixed point.

Proof. Taking the self mapping T continuous in theorem 3.2 we can easily prove the required result. □

Remark. Corollary 3.1 is the result of C. T. Aage and J. N. Salunke[2].

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