Nef and Big Divisors on Toric Weak Fano 3-Folds

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Abstract

We show that a nef and big line bundle whose adjoint bundle has non-zero global sections on a nonsingular toric weak Fano 3-fold is normally generated. As a consequence, we see that any ample line bundle on a nonsingular toric weak Fano 3-fold is normally generated. As an application, we see that an ample line bundle on a Calabi-Yau hypersurface in a nonsingular toric Fano 4-fold is normally generated.

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Introduction

We call an invertible sheaf on an algebraic variety a line bundle. A line bundle L on an algebraic variety is called *normally generated* (by Mumford [19]) if the multiplication map of global sections $\Gamma(L)^{\otimes l} \rightarrow \Gamma(L^{\otimes l})$ is surjective for all $l \geq 1$. We are interested in normal generation of ample line bundles on a toric variety. If an ample line bundle L on a normal algebraic variety X is normally generated, then we see that it is very ample and that the graded ring $\bigoplus_{l \geq 0} \Gamma(X, L^{\otimes l})$ is generated by elements of degree one and is a normal ring. It is known that an ample line bundle on a nonsingular toric variety is always very ample (see [23, Corollary 2.15]). We may ask whether any ample line bundle be normally generated.

In general, for an ample line bundle L on a (possibly singular) toric variety of dimension n , we see that

$$\Gamma(L^{\otimes l}) \otimes \Gamma(L) \longrightarrow \Gamma(L^{\otimes(l+1)}) \quad (1)$$

is surjective for $l \geq n - 1$ (see [5], [21] or [22]). When $n \leq 2$, hence, we see that all ample line bundles are normally generated (see [14]). We also have examples of very ample but not normally generated line bundles for $n \geq 3$ (see [3], [4], [25] and [26]).

An algebraic variety X is called *Gorenstein* if its dualizing sheaf is invertible. A Gorenstein variety X is called *Fano* (or *weak Fano*) if its anti-canonical divisor $-K_X$ is ample (or nef and big), respectively. We know that the anti-canonical line bundle on a nonsingular toric Fano variety of dimension n is normally generated if $n \leq 7$ (see [10]). Ogata [24] shows that an ample line bundle L on a nonsingular toric 3-fold X is normally generated if $h^0(L + K_X) = 0$ or if $h^0(L + K_X) \neq 0$ and $L + K_X$ is not big.

In this paper we restrict X to be a nonsingular toric weak Fano 3-fold, but we assume that L is a certain nef and big line bundle on X .

Theorem 1 *Let X be a nonsingular toric weak Fano variety of dimension three. If a nef and big line bundle L on X satisfies the condition that $h^0(L + K_X) \neq 0$, then L is normally generated.*

Since a Gorenstein toric weak Fano 3-fold admits a crepant resolution, Theorem 1 implies the following corollary.

Corollary 1 *Let Y be a Gorenstein toric weak Fano variety of dimension three. If an ample line bundle L on Y satisfies the condition that $h^0(L + K_Y) \neq 0$, then L is normally generated.*

By combining the result of [24] we obtain the following theorem.

Theorem 2 *Let X be a nonsingular toric weak Fano variety of dimension three. Then any ample line bundle on X is normally generated.*

Ogata already shows that all ample line bundles on a nonsingular toric 3-fold with non-trivial morphism onto the projective line are normally generated in [27]. Thus we have two classes of nonsingular toric 3-folds such that all ample line bundles are normally generated.

We also obtain the projective normality of ample line bundles on a toric Calabi-Yau hypersurface of dimension three. Since an irreducible torus invariant divisor on a nonsingular toric Fano variety is a weak Fano variety, Theorem 2 implies the following theorem.

Theorem 3 *Let Z be a nonsingular toric Fano 4-fold and X be a general member of the anti-canonical linear system on Z . Then an ample line bundle on X is always normally generated.*

For the proof of Theorem 1, we need to prove the weak version.

Theorem 4 *Let X be a nonsingular toric weak Fano variety of dimension three. If a nef and big line bundle L on X satisfies the condition that $h^0(L + K_X) \neq 0$ and that $h^0(L + 2K_X) = 0$, then L is normally generated.*

We will give a proof of Theorem 4 in Sections 3 and 5 by dividing into two cases.

In our proof we do not use full classifications of Fano polytopes but use a classification of minimal Fano polytopes of Kasprzyk[12]. There are 4,319 Gorenstein toric Fano 3-folds (cf. [16]).

We note that there is an ample but not normally generated line bundle L on a Gorenstein toric Fano 3-fold Y with $h^0(L + K_Y) = 0$. See Remark in Section 6.

1 Line bundles on toric varieties

In this section we recall the fact about toric varieties and line bundles on them from Oda's book [23] or Fulton's book [8].

Let N be a free $\mathbb{Z}$ -module of rank n and $M := \text{Hom}(N, \mathbb{Z})$ its dual with the pairing $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$. By scalar extension to $\mathbb{R}$, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We also have the pairing of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ by scalar extension, which is denoted by the same symbol $\langle \cdot, \cdot \rangle$.

The group ring $\mathbb{C}[M]$ defines an algebraic torus $T_N := \text{Spec } \mathbb{C}[M] \cong (\mathbb{C}^*)^n$ of dimension n . Then the character group $\text{Hom}_{\text{gr}}(T_N, \mathbb{C}^*)$ of the algebraic torus T_N coincides with M . For $m \in M$ we denote the corresponding character by $e(m) : T_N \rightarrow \mathbb{C}^*$.

Let Δ be a finite complete fan of N . A convex cone $\sigma \in \Delta$ defines an affine variety $U_{\sigma} := \text{Spec } \mathbb{C}[M \cap \sigma^{\vee}]$. Here $\sigma^{\vee} := \{y \in M_{\mathbb{R}}; \langle y, x \rangle \geq 0 \text{ for all } x \in \sigma\}$ is the dual cone of σ . Then we obtain a normal algebraic variety $X(\Delta) := \bigcup_{\sigma \in \Delta} U_{\sigma}$, which is called a *toric* variety. We note that $U_{\{0\}} \cong T_N$ is a unique dense T_N -orbit in $X(\Delta)$. Set $\Delta(i) := \{\sigma \in \Delta; \dim \sigma = i\}$. Then an element $\sigma \in \Delta(i)$ corresponds to a T_N -invariant subvariety $V(\sigma)$ of dimension $n - i$. In particular, $\Delta(1)$ corresponds to the set of all irreducible T_N -invariant divisors on $X(\Delta)$.

Let $\Delta(1) = \{\rho_1, \dots, \rho_s\}$ and v_i the generator of the semi-group $\rho_i \cap N$. We simply write as $X = X(\Delta)$ and $D_i := V(\rho_i)$ for $i = 1, \dots, s$. For a T_N -invariant line bundle L there exists a T_N -invariant divisor $D = \sum_i a_i D_i$

satisfying $L \cong \mathcal{O}_X(D)$. For a T_N -invariant Cartier divisor D we define a rational convex polytope $P_D \subset M_{\mathbb{R}}$ as

$$P_D := \{y \in M_{\mathbb{R}}; \langle y, v_i \rangle \geq -a_i \text{ for } i = 1, \dots, s\}.$$

By definition we note that $P_{lD} = lP_D$ for any positive integer l . Moreover, for another T_N -invariant Cartier divisor E we have $P_{D+E} = P_D + P_E$. Here $P_D + P_E := \{x + y \in M_{\mathbb{R}}; x \in P_D \text{ and } y \in P_E\}$ is the Minkowski sum of P_D and P_E . By using this polytope, we can describe the space of global sections (see [23, Section 2.2], or [8, Section 3.5])

$$\Gamma(X, \mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{C}e(m).$$

Hence, we see that the surjectivity of the multiplication map of global sections

$$\Gamma(X, \mathcal{O}_X(D)) \otimes \Gamma(X, \mathcal{O}_X(E)) \longrightarrow \Gamma(X, \mathcal{O}_X(D + E))$$

is equivalent to the equality

$$P_D \cap M + P_E \cap M = (P_D + P_E) \cap M.$$

If $\mathcal{O}_X(D)$ is generated by global sections, then all vertices of P_D are lattice points, that is, P_D is the convex hull of finite subset of M . Conversely, if for all $\sigma \in \Delta$ there exist $u(\sigma) \in M$ with

$$\langle u(\sigma), v_i \rangle = -a_i \text{ for } v_i \in \sigma$$

and if P_D is the convex hull of $\{u(\sigma); \sigma \in \Delta\}$, then $\mathcal{O}_X(D)$ is generated by global sections (see [23, Theorem 2.7], or [8, Section 3.4]). We also know [17] that if $\mathcal{O}_X(D)$ is generated by global sections, then there exists an equivariant surjective morphism $\pi : X \rightarrow Y$ to a toric variety Y and an ample line bundle A on Y with $\mathcal{O}_X(D) \cong \pi^*A$. Thus we see that $\mathcal{O}_X(D)$ is generated by global sections if and only if D is *nef* (see also [20, Theorem 3.1]).

If X is Gorenstein, then $-K_X = \sum_i D_i$ is a Cartier divisor. By definition P_{-K_X} is an integral polytope of dimension n since the polytope is the intersection of half-spaces containing the origin as their interiors. This implies that $-K_X$ is *big*.

Now we introduce a criterion of nef-ness on nonsingular toric surfaces.

Proposition 1 *Let X be a nonsingular complete toric surface and let D a T_N -invariant divisor with $|D| \neq \emptyset$. If $|D|$ has no fixed components, then it is free from base points.*

Proof. Since $\Delta(1) = \{\rho_1, \dots, \rho_s\}$ consists of half-lines from the origin in the plane $N_{\mathbb{R}}$, we may assume that ρ_i and ρ_{i+1} sit next to each other (as usual we consider as $\rho_{s+1} = \rho_0$). Set $\sigma_i = \rho_i + \rho_{i+1} \in \Delta(2)$ for $i = 1, \dots, s$. Take $D = \sum_i a_i D_i$ with $|D| \neq \emptyset$. We may assume that $a_i \geq 0$ for all i .

First we consider the case that P_D is an integral convex polytope, that is, it is the convex hull of a finite subset of M . Set $H^+(a_i) := \{y \in M_{\mathbb{R}}; \langle y, v_i \rangle \geq -a_i\}$ the half-plane and its boundary line $H(a_i)$. By definition (1) we see that P_D is the intersection of all half-planes $H^+(a_i)$'s. Let u_0 be a vertex of P_D . If $\dim P_D = 2$, then a 1-dimensional face of P_D containing u_0 is contained in some line $H(a_i)$. If $\dim P_D \leq 1$, then P_D itself is contained in some $H(a_i)$. We may set $i = 1$.

Since P_D is the intersection of $H^+(a_i)$'s, we take another line $H(a_j)$ ($j \neq 1$) meeting with $H(a_1)$ at u_0 . We may assume that all σ_i with $i = 1, \dots, j-1$ does not contain $-v_1$. We claim that the line $H(a_i)$ contains u_0 for $i = 2, \dots, j$.

For $\sigma_i = \rho_i + \rho_{i+1} \in \Delta(2)$, since $\{v_i, v_{i+1}\}$ is a $\mathbb{Z}$ -basis of N , there exists $u(\sigma_i) \in M$ satisfying the condition (1). Then we have

$$u_0 \in H^+(a_1) \cap H^+(a_j) \subset u(\sigma_i) + \sigma_i^\vee$$

for $i = 1, \dots, j-1$. If $u(\sigma_1) \neq u_0$, then the half-plane $H^+(a_2 - 1)$ would contain P_D . This implies that D_2 is a fixed component of $|D|$. Then we see that $u(\sigma_1) = u_0$. Considering $v_3, \dots, v_j$ successively, we see that $u(\sigma_i) = u_0$ for $i = 1, \dots, j-1$.

When $\dim P_D = 2$, since we can take $H(a_j)$ so that it contains a 1-dimensional face of P_D , we see that the opposite vertex on the edge $H(a_j) \cap P_D$ coincides with $u(\sigma_j)$.

When $\dim P_D \leq 1$, the vector $-v_1$ coincides with some v_k ($j < k$). By the same argument, we see that $u(\sigma_i) = u_0$ for $i = j, \dots, k-1$. And we see that $u(\sigma_k)$ is also a vertex of P_D . Hence, $\mathcal{O}_X(D)$ is generated by global sections.

Next we assume only that P_D is a rational convex polytope. We can choose a positive integer l so large that lP_D is an integral polytope. Since $lP_D = P_{lD}$, the line bundle $\mathcal{O}_X(lD)$ is generated by global sections, hence it is nef. Then D is nef. On a toric variety, if D is nef, then $\mathcal{O}_X(D)$ is generated by global sections. $\square$

Remark. If $\dim X \geq 3$, then the same statement of Proposition 1 does not hold. We can easily construct counterexamples, as Professor Payne points out.

2 Adjoint line bundles

Let ω_X be the dualizing sheaf on a toric variety X . If a T_N -invariant Cartier divisor D is ample, then we have (see [23, Proposition 2.24])

$$\Gamma(X, \mathcal{O}_X(D) \otimes \omega_X) \cong \bigoplus_{m \in (\text{Int}(P_D)) \cap M} \mathbb{C}e(m).$$

If we take a resolution $\pi : \tilde{X} \rightarrow X$ of singularities by a subdivision of Δ , then $L = \pi^*\mathcal{O}_X(D)$ is nef and big, and we have

$$\Gamma(\tilde{X}, L + K_{\tilde{X}}) \cong \Gamma(X, \mathcal{O}_X(D) \otimes \omega_X).$$

In [24] we show that an ample line bundle A on a nonsingular toric 3-fold X is normally generated if A satisfies the condition that $h^0(X, A + K_X) \neq 0$ and that $A + K_X$ is not big. In order to treat more general case, we have to know the adjoint bundle $L + K_X$ with $h^0(L + K_X) \neq 0$ for a nef and big line bundle L .

Lemma 1 *Let X be a nonsingular complete toric variety of dimension three. If a nef and big line bundle L on X satisfies that $h^0(X, L + K_X) \neq 0$, then the fixed part of $L + K_X$ is a reduced divisor $\sum_i E_i$ with $(E_i, L_{E_i}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $\mathcal{O}_X(E_i)|_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, and $L + K_X - \sum_i E_i$ is nef.*

Proof. Since a nef and big line bundle on a toric variety is the pull back of an ample line bundle on a toric variety of the same dimension, $K_X + tL$ is generated by global sections, that is, nef for a sufficiently large $t > 0$. If $K_X + tL$ is not nef, then Mori's theory [18] says that there exists an extremal curve R with $(K_X + tL)R < 0$ and $-K_X R \leq 4$. Thus we have $1 \leq t \leq 3$.

By the Mori-Kawamata theory (cf. [13], [18]) if $K_X + tL$ is not nef, then we have a contraction morphism $\varphi : X \rightarrow Y$. Following the same argument of Fujita [7, Theorem 11.8], our assumption $h^0(X, L + K_X) \neq 0$ implies that $t = 1$ and φ is birational and moreover that φ is a blowing-ups of a point and there exists a nef and big line bundle $\bar{L}$ on Y such that $L + E \cong \varphi^*\bar{L}$, where E is the exceptional divisor $E \cong \mathbb{P}^2$, $L_E \cong \mathcal{O}(1)$ and $\mathcal{O}_X(E)|_E \cong \mathcal{O}(-1)$. Since $K_X = \varphi^*K_Y + 2E$, we have $L + K_X = \varphi^*(\bar{L} + K_Y) + E$. $\square$

3 Proof of Theorem 4 (Part I)

Let $B := \sum_i D_i$ be the boundary divisor of T_N in X . We assume that B is nef. Let $L = \mathcal{O}_X(D)$ be a nef and big line bundle satisfying the condition in Theorem 4. Then P_D is an integral polytope of dimension three. The

assumption $h^0(X, L + K_X) \neq 0$ of Theorem 4 implies that $(\text{Int}(P_D)) \cap M \neq \emptyset$. Let F be the fixed components of $|D + K_X|$ and $A := (D + K_X) - F$. From Lemma 1 we see that $|A|$ is free from base points. Since $\Gamma(X, L + K_X) = \Gamma(X, \mathcal{O}_X(A))$, we see that P_A coincides with the convex full of $(\text{Int}(P_D)) \cap M$. We note that if $-K_X = B$ is nef, then $D - F = A + B$ is also nef.

Before treating nef divisors on 3-folds, we need to know more about nef divisors on toric surfaces. Fakhruddin [6] proves it when $A + B$ is ample. Haase, Nill, Paffenholz and Santos [9] and Kondo and Ogata [15] prove in general.

Lemma 2 *Let A and B be nef divisors on a nonsingular complete toric surface Y . Then the multiplication map of global sections*

$$\Gamma(Y, \mathcal{O}_Y(A)) \otimes \Gamma(Y, \mathcal{O}_Y(A + B)) \longrightarrow \Gamma(Y, \mathcal{O}_Y(2A + B))$$

is surjective.

We return to the case that $B = -K_X$ and $A = D + K_X - F$ in dimension three. First we prove Theorem 4 in a special case.

Proposition 2 *When $\dim P_A \leq 2$ the bundle $\mathcal{O}_X(A + B)$ is normally generated.*

Proof. Set $L = \mathcal{O}_X(A + B)$. Since $\mathcal{O}_X(A) = L + K_X$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(A) \rightarrow L \rightarrow L_B \rightarrow 0. \quad (2)$$

Since A is nef, we have $H^i(X, \mathcal{O}_X(A)) = 0$ for $i \geq 1$. Thus the global sections of (2) is exact. Take the tensor product with $\Gamma(X, \mathcal{O}_X(A))$. When $\dim P_A \leq 2$, we see that $\mathcal{O}_X(A)$ is normally generated (see (1)).

On the other hand, $\Gamma(B, (2L + K_X)_B)$ has a basis $\{e(m); m \in (\partial(2P_A + P_B)) \cap M\}$ as vector spaces. One $e(m)$ is contained in $\Gamma(D_i, (2L + K_X)_{D_i})$ for some D_i . Since the restriction map $\Gamma(X, G) \rightarrow \Gamma(D_i, G_{D_i})$ is surjective for any nef line bundle G on a toric variety X , from Lemma 2 we see that the multiplication map

$$\Gamma(B, L_B) \otimes \Gamma(B, (L + K_X)_B) \longrightarrow \Gamma(B, (2L + K_X)_B)$$

is surjective. Thus we obtain the surjectivity of $\Gamma(L) \otimes \Gamma(L + K_X) \rightarrow \Gamma(2L + K_X)$.

By tracing the same argument after changing A with $L = A + B$, we obtain the normal generation of $\mathcal{O}_X(A + B)$. $\square$

4 Fano polytopes

For our proof of Theorem, we need some properties of Fano polytopes of dimension three. Batyrev [1] called *Fano polytope* a lattice polytope containing no lattice points except the origin in its interior. Kasprzyk [12] called a Fano polytope P of dimension n *minimal* if the convex hull $\text{Conv}\{P \cap (\mathbb{Z}^n) \setminus \{v\}\}$ is never Fano for any vertex v of P . He classifies minimal Fano polytopes of dimension three.

Let Y be a Gorenstein toric Fano 3-fold and $R \subset M_{\mathbb{R}}$ the lattice polytope corresponding to the ample anti-canonical divisor $-K_Y$. R contains only one lattice points in its interior. By an affine transformation of M , we may assume the origin is the interior point of R . Then R is a Fano polytope. Let F be a facet of R and G the minimal lattice triangle with vertices in $F \cap M$. By a suitable choice of the coordinates of $M_{\mathbb{R}}$, we may set $G = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. We fix the coordinates of $M_{\mathbb{R}}$ in this section.

Lemma 3 (Kasprzyk) *Let $S := \text{Conv}\{G, (-a, -b, -c)\}$ be a minimal Fano polytope with $1 \leq a \leq b \leq c$. Then the triplet (a, b, c) coincides with one of*

$$(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 2, 4) \\ (1, 3, 4), (1, 3, 5), (1, 4, 6), (2, 3, 5), (3, 4, 5).$$

We note that S contains all $(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0), (0, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})$ in $\frac{1}{2}M$. Here we call the point $v \in \frac{1}{2}M$ *perfect half-integral* if its any coordinates are not integers.

Lemma 4 *S contains a perfect half-integral point unless $(a, b, c) = (1, 2, 4)$.*

Set $S_0 := \text{Conv}\{G, (-1, -2, -4)\}$. We note $(0, 0, -1) \in S_0$.

Lemma 5 *A minimal Fano polytope of dimension three such that it contains G as a facet and has vertices more than four is one of*

- (1) $S_1 := \text{Conv}\{G, (0, 0, -1), (-1, -1, 0)\},$
- (2) $S_2 := \text{Conv}\{G, (0, 0, -1), (-2, -1, 0)\},$
- (3) $S_3 := \text{Conv}\{G, (1, 1, -1), (-1, -1, 0)\},$
- (4) $S_4 := \text{Conv}\{G, (-1, 0, -1), (-2, -1, 0)\},$
- (5) $S_5 := \text{Conv}\{G, (-1, 0, -2), (0, -1, -2)\},$
- (6) $S_6 := \text{Conv}\{G, (0, 0, -1), (0, -1, 0), (-1, 0, 0)\}.$

Lemma 6 *S_1, S_3, S_4 contain a perfect half-integral point.*

We note that S_2, S_5, S_6 contain the point $(0, 0, -1)$.

Proposition 3 *Let Y be a Gorenstein toric Fano 3-fold and R the lattice polytope corresponding $-K_Y$. If $Q \subset M_{\mathbb{R}}$ is a lattice polytope containing $\frac{m}{2} + \frac{3}{2}R$, then $\text{Int}(Q) \cap M \neq \emptyset$.*

Proof. If

$$\left\{ \frac{m}{2} + \text{Int} \left(\frac{3}{2}R \right) \right\} \cap M \neq \emptyset, \quad (3)$$

then Q contains lattice points in its interior.

If $\frac{m}{2} \in M$, then $R \subset \text{Int}(\frac{3}{2}R)$ contains the origin.

Set $\frac{m}{2} \notin M$. Then (3) is equivalent to the existence an element

$$\frac{u}{2} \in \left\{ \text{Int} \left(\frac{3}{2}R \right) \right\} \cap \left\{ \frac{1}{2}M \setminus M \right\} \quad \text{with} \quad \frac{m}{2} + \frac{u}{2} \in M. \quad (4)$$

We may replace R with a minimal Fano polytope S . We fix the coordinates of $M_{\mathbb{R}}$ as $G = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for a minimal lattice triangle G contained in a facet of R . From Lemmas 3, 4, 5 and 6, we see that (4) holds for a minimal Fano polytope S unless $\frac{m}{2}$ is a perfect half-integral point and S is one of S_0, S_2, S_5 and S_6 .

We assume that $\frac{m}{2}$ is a perfect half-integral point and $S = S_0, S_2, S_5$, or S_6 . Since S contains the point $(0, 0, -1)$, two half-integer points $(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$ are contained in the boundary of $\frac{3}{2}S$. Set $\frac{m}{2} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $S_m := \frac{m}{2} + \frac{3}{2}S$ the rational convex polytope. Then S_m contains the lattice points $(1, 1, 1), (1, 1, 0)$. Let F_1, F_2 be the facets of S_m containing $(1, 1, 1)$ and $(1, 1, 0)$, respectively. Then the intersection $F_1 \cap F_2$ is the line segment ℓ connecting $(2, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 2, \frac{1}{2})$. Let Q be a lattice polytope containing $\frac{m}{2} + \frac{3}{2}S$. Assume that Q contains no lattice points in its interior. Since $(1, 1, 1), (1, 1, 0)$ are contained in the boundary of Q and since these lattice points are contained in the interior of the facets F_1 or F_2 , Q has two facets G_1, G_2 containing F_1, F_2 , respectively. In particular, the intersection $G_1 \cap G_2$ contains ℓ . Thus Q has a face E of dimension one containing ℓ . Since Q is a lattice polytope, the end points of E are lattice points. But it is impossible. Hence, we have $(\text{Int } Q) \cap M \neq \emptyset$. $\square$

5 Proof of Theorem 4 (Part II)

We recall our situation. Let X be a nonsingular toric weak Fano 3-fold and $B = \sum_i D_i$ the nef anti-canonical divisor. Let A be a nef and big divisor with $h^0(X, \mathcal{O}_X(A) + K_X) = 0$. We consider the normal generation of $\mathcal{O}_X(A + B)$.

Proposition 4 *If $\dim P_A = 3$ and if $(\text{Int}(P_A)) \cap M = \emptyset$, then $\mathcal{O}_X(A + B)$ is normally generated.*

Proof. Assumption $(\text{Int}(P_A)) \cap M = \emptyset$ implies that $h^0(X, \mathcal{O}_X(A + K_X)) = 0$.

First we consider the multiplication with $\Gamma(\mathcal{O}_X(B))$. Since $\mathcal{O}_X(B + K_X) \cong \mathcal{O}_X$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(B) \rightarrow \mathcal{O}_B(B|_B) \rightarrow 0.$$

Since the multiplication map $\Gamma(D_i, \mathcal{O}_{D_i}((A + B)|_{D_i})) \otimes \Gamma(D_i, \mathcal{O}_{D_i}(B|_{D_i})) \rightarrow \Gamma(D_i, \mathcal{O}_{D_i}((A + 2B)|_{D_i}))$ is surjective from Lemma 2, we have the surjectivity of $\Gamma(X, \mathcal{O}_X(A + B)) \otimes \Gamma(X, \mathcal{O}_X(B)) \rightarrow \Gamma(X, \mathcal{O}_X(A + 2B))$.

Set $L = \mathcal{O}_X(A + 2B)$. Next we consider the multiplication of $\Gamma(L)$ and $\Gamma(\mathcal{O}_X(A))$. In order that, we will show the vanishing of $H^i(X, L(-iA))$ for $i \geq 1$.

We have $H^1(X, L(-A)) = H^1(X, \mathcal{O}_X(2B)) = 0$ since B is nef.

From the Serre duality we have $h^3(X, L(-3A)) = h^3(X, \mathcal{O}_X(2B - 2A)) = h^0(X, \mathcal{O}_X(2A + 3K_X))$. If $h^0(X, \mathcal{O}_X(2A + 3K_X)) \neq 0$, we have an injective homomorphism $\mathcal{O}_X(3B) \rightarrow \mathcal{O}_X(2A)$. By taking global sections, we have an inclusion

$$m + P_{3B} \subset P_{2A}$$

for some $m \in M$. We may write it as

$$\frac{m}{2} + \frac{3}{2}P_B \subset P_A.$$

This implies that P_A has lattice points in its interior from Proposition 3. This contradicts to the assumption $(\text{Int}(P_A)) \cap M = \emptyset$. Hence we have $h^3(L(-3A)) = 0$.

Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(A + 2K_X) \rightarrow \mathcal{O}_X(A + K_X) \rightarrow \mathcal{O}_B((A + K_X)_B) \rightarrow 0.$$

Since $H^j(X, \mathcal{O}_X(A + K_X)) = 0$ for $j = 0, 1$, if $h^1(X, \mathcal{O}_X(A + 2K_X)) \neq 0$, then $h^0(B, \mathcal{O}_B((A + K_X)_B)) \neq 0$. A non-zero section defines an injective homomorphism $\Gamma(D_i, \mathcal{O}_{D_i}(B|_{D_i})) \rightarrow \Gamma(D_i, \mathcal{O}_{D_i}(A|_{D_i}))$ for each irreducible invariant divisor D_i . Since all facets of P_B are contained in P_A , $P_B \subset P_A$ and $(\text{Int}P_A) \cap M \neq \emptyset$. This contradicts the assumption on A . thus we have $h^1(X, \mathcal{O}_X(A + 2K_X)) = 0$.

Considering the exact sequence

$$0 \rightarrow \mathcal{O}_X(A + 3K_X) \rightarrow \mathcal{O}_X(A + 2K_X) \rightarrow \mathcal{O}_B((A + 2K_X)_B) \rightarrow 0,$$

we can show $h^1(X, \mathcal{O}_X(A + 3K_X)) = 0$ in the same way. By Serre duality, we have $h^2(X, \mathcal{O}_X(-2K_X - A)) = h^2(X, L(-2A)) = 0$.

From vanishing of $H^i(X, L(-iA))$ for $i \geq 1$ we can apply [19, Theorem 2] to obtain the surjectivity of the multiplication map

$$\Gamma(X, \mathcal{O}_X(A)) \otimes \Gamma(X, L) \longrightarrow \Gamma(X, L(A)). \quad (5)$$

Since $L = \mathcal{O}_X(A + 2B)$, we obtain the normal generation of $\mathcal{O}_X(A + B)$ from the first step of the proof and the surjectivity of (5). $\square$

Finally we will complete the proof of Theorem 4.

Proof of Theorem 4. By assumption that L is nef and big with $h^0(X, L + K_X) \neq 0$ and $h^0(L + 2K_X) = 0$.

If $L + K_X$ has no fixed components, then we see the normal generation of L from Propositions 2 and 4. Let F be the fixed components of $L + K_X$. Then $F = \sum_i E_i, E_i \cong \mathbb{P}^2$ and E_i 's are disjoint from Lemma 1. And we have $L_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $L(-F)_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(2)$.

Consider the exact sequence

$$0 \rightarrow L(-F) \rightarrow L \rightarrow L_F \rightarrow 0. \quad (6)$$

Since $L(-F)$ is nef, we have $H^1(X, L(-F)) = 0$. Thus the sequence of global sections of (6) is exact. Taking the tensor product with $\Gamma(X, L(-F))$, we see the surjectivity of the map

$$\Gamma(X, L(-F)) \otimes \Gamma(X, L) \longrightarrow \Gamma(X, 2L(-F))$$

since $L(-F)$ is normally generated from Propositions 2 and 4. By changing the role of $\Gamma(X, L(-F))$ with $\Gamma(X, L)$ we see the normal generation of L . $\square$

6 Proof of Theorems

Proof of Theorem 1. Let L be a nef and big line bundle on a nonsingular toric weak Fano 3-fold X with $h^0(X, L + K_X) \neq 0$. Set F_1 the fixed part of $L + K_X$. Then $L + K_X - F_1$ is nef from Lemma 1. Set $L_1 := L + K_X - F_1$. Set $P_1 := P_{L_1}$ the integral convex polytope corresponding to the nef line bundle L_1 . If $\dim P_1 \leq 2$ or if $\dim P_1 = 3$ and $\text{Int}(P_1) = \emptyset$, then L is normally generated from Theorem 4.

Assume that $\dim P_1 = 3$ and $\text{Int}(P_1) \neq \emptyset$. Then L_1 is nef and big and $h^0(X, L_1 + K_X) \neq 0$. Set F_2 the fixed part of $L_1 + K_X$. Then $L_2 := L_1 + K_X - F_2$ is nef. If $h^0(X, L_2 + K_X) = 0$, then L_1 is normally generated from Theorem 4. Set $B = \sum_i D_i$ the boundary divisor. Since $L - F_1$ is a sum of two nef and big line bundles L_1 and $-K_X$, the fact that the short exact sequence

$$0 \rightarrow L_1 \rightarrow L_1 - K_X \rightarrow (L_1 - K_X)|_B \rightarrow 0$$

gives the short exact sequence of its global sections and the surjectivity of two multiplication maps $\Gamma(L_1) \otimes \Gamma(L_1) \rightarrow \Gamma(2L_1)$ and $\Gamma(B, (L_1 - K_X)|_B) \otimes \Gamma(B, L_1|_B) \rightarrow \Gamma(B, (2L_1 - K_X)|_B)$ implies the surjectivity of the spaces of global sections of $L_1 - K_X$ and L_1 . By the same argument of the proof of Proposition 4, we have the surjectivity of $2L_1 - K_X$ and $-K_X$. This implies the normal generation of $L - F_1 = L_1 - K_X$. Next we apply the same argument of "Proof of Theorem 4" in the previous section to obtain the normal generation of L .

If $h^0(X, L_2 + K_X) \neq 0$, then we continue the same argument for L_2 . By induction we obtain the proof of Theorem 1. $\square$

If the anti-canonical divisor $-K_X$ of a Gorenstein toric variety X is nef, then it is nef and big, hence, there exists a polarized toric variety (Y, A) and a surjective morphism $\pi : X \rightarrow Y$ such that $-K_X \cong \pi^*A$. Since Y has only rational singularity, we see that $A = -K_Y$ and Y is a Gorenstein Fano 3-fold.

On the other hand, let Y be a Gorenstein toric weak Fano 3-fold. Then we have a resolution of singularities $\pi : X \rightarrow Y$ with $K_X \cong \pi^*K_Y$ (a crepant resolution). Thus we can apply Theorem 4 to a nef and big line bundle π^*A with ample A on Y . We obtain Corollary 1.

Proof of Theorem 3. Let Z be a nonsingular toric Fano variety of dimension n and X be a general member of the anti-canonical linear system $|-K_Z|$ on Z . Someone calls X a *toric Calabi-Yau hypersurface*. If $n \geq 4$, then the Lefschetz hyperplane theorem implies the isomorphism $\text{Pic}(Z) \rightarrow \text{Pic}(X)$ and, furthermore, Kollár [2, Appendix], Wiśniewski [28] and Hassett, Lin and Wang [11] show the isomorphism between the Mori cones for a Fano variety Z . Thus an ample line bundle L on X is the restriction of an ample line bundle $\mathcal{L}$ on Z , and L is very ample since an ample line bundle on a nonsingular toric variety is always very ample.

Set $n = 4$. Let $B := \sum_i D_i$ be the sum of all torus invariant irreducible divisors on Z . Let Q be the lattice polytope of dimension four corresponding to $\mathcal{L}$. Then a facet F_i of Q corresponds to the restriction $\mathcal{L}_{D_i}$. Furthermore, the set of lattice points contained in the boundary of kQ parametrizes the basis of the space of global sections of $\mathcal{L}_B^{\otimes k}$ as vector spaces. Theorem 2 implies that $\mathcal{L}_B$ is normally generated.

Consider the diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow \Gamma(k\mathcal{L} - B) \otimes \Gamma(\mathcal{L}) & \rightarrow & \Gamma(k\mathcal{L}) \otimes \Gamma(\mathcal{L}) & \rightarrow & \Gamma(k\mathcal{L}_B) \otimes \Gamma(\mathcal{L}) & \rightarrow & 0 \\ & \downarrow \alpha(B) & & \downarrow \beta & & \downarrow \gamma & \\ 0 \rightarrow \Gamma((k+1)\mathcal{L} - B) & \rightarrow & \Gamma((k+1)\mathcal{L}) & \rightarrow & \Gamma((k+1)\mathcal{L}_B) & \rightarrow & 0. \end{array} \quad (7)$$

Since the restriction map $\Gamma(\mathcal{L}) \rightarrow \Gamma(\mathcal{L}_B)$ is surjective, the down arrow γ is surjective, hence the homomorphism $\text{Coker}(\alpha(B)) \rightarrow \text{Coker}(\beta)$ is surjective.

Since X is linear equivalent to B , $\Gamma(k\mathcal{L} - B) \cong \Gamma(k\mathcal{L} - X)$. By replacing B with X in the diagram (7), set $\alpha(X)$ the corresponding down arrow in the left. Then we have a surjective homomorphism $\text{Coker}(\alpha(X)) \rightarrow \text{Coker}(\beta)$, hence, we see that $L = \mathcal{L}_X$ is normally generated. $\square$

Remark. In Theorem 1, 4 or Corollary 1 we cannot remove the condition $h^0(X, L + K_X) \neq 0$. We give an example of (X, L) such that $-K_X$ is nef but L is not normally generated and $h^0(X, L + K_X) = 0$.

Let $M = \mathbb{Z}^3$ and $P := \text{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$ in $M_{\mathbb{R}}$. Then there exists the polarized toric 3-fold $(Y, \mathcal{O}_Y(D))$ with $P_D = P$. This Y is Gorenstein toric Fano with $-K_Y = 2D$. Since P does not contain lattice points of the form $(a, b, 1)$, we can easily see that D is not very ample. We can make a toric crepant (partial) resolution $\pi : X \rightarrow Y$ of singularities with $K_X = \pi^*K_Y$. Then $-K_X$ is nef (and big) and $L := \pi^*\mathcal{O}_Y(D)$ is nef and big, and $h^0(X, L + K_X) = 0$. We note that a choice of (partial) resolution $\pi : X \rightarrow Y$ is not unique for this Gorenstein toric Fano 3-fold Y .

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