Another Generalization of the m -Topology

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Abstract

It is well known that the m_I -topology is a generalization of the m -topology on $C(X)$, see [1]. Given two subsets $A, B \subseteq X$ such that $A \cup B = X$, we are going to define a topology on $C(X)$ namely the $m_{(A,B)}$ -topology, finer than the m -topology and $C(X)$ with this topology becomes a topological ring. Connectedness in this space is studied and it is shown that if A, B are closed realcompact subsets of X , then the component of the zero function in $C(X)$ with $m_{(A,B)}$ -topology is the ideal $C_K(X)$.

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1 Introduction

Throughout this paper we denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real valued continuous functions on a completely regular Hausdorff space X . The m -topology on $C(X)$ is defined by taking the set of the form

$$B(f, u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X\}$$

as a base for a neighborhood system at f , for each $f \in C(X)$ and $u \in U^+$, where U^+ is the set of all positive elements of $C(X)$. $C(X)$ endowed with the m -topology is denoted by $C_m(X)$ which is a Hausdorff topological ring. The m -topology is first introduced in the late 40s in [2] and later the research in this area became active over the last 20 years, for example, the works in [1], [3] and [4].

In this paper, for two subsets A, B in X , we define a topology namely the $m_{(A,B)}$ -topology on $C(X)$ in which the component of 0 is exactly the ideal $C_K(X)$, whenever A and B be closed realcompact subsets of X . This topology is finer than the m -topology and makes $C(X)$ a topological ring. We denote the space $C(X)$ with the $m_{(A,B)}$ -topology by $C_{m(A,B)}(X)$.

An ideal I in $C(X)$ is called a z -ideal if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. We recall that $C_\psi(X)$ ($C_K(X)$) is a z -ideal in $C(X)$ consisting of functions with pseudocompact (compact) support and it is well-known that $f \in C_\psi(X)$ if and only if $X \setminus Z(f)$ is relatively pseudocompact, i.e., every function in $C(X)$ is bounded on $X \setminus Z(f)$, see Theorem 2.1 in [5]. For terminology and notations, the reader is referred to [6] and [7].

At first we need the following proposition that is well-known.

Proposition 1.1. For any closed subset A of a topological space X , if $f \in C(X)$, then

$$\text{cl}_X(A \setminus Z(f|_A)) = \text{cl}_X(A \setminus Z(f|_A)) \cap A$$

2. $m_{(A,B)}$ -topology on $C(X)$

Let $A \subseteq X$, we denote by U_A^+ the set of all $f \in C(A)$ such that $f(x) > 0$ for all $x \in A$.

Now suppose that $A, B \subseteq X$ that $A \cup B = X$. For each $f \in C(X)$, $u \in U_A^+$ and $v \in U_B^+$, we define the subset $B(f, u, v)$ of $C(X)$ as follow:

$$B(f, u, v) = \{g \in C(X) : |f - g| < u \text{ on } A \text{ and } |f - g| < v \text{ on } B\}.$$

We define the $m_{(A,B)}$ -topology on $C(X)$ by taking the family $\{B(f, u, v) : u \in U_A^+ \text{ and } v \in U_B^+\}$ as a base for a neighborhood system at f for each $f \in C(X)$. The set $C(X)$ endowed with the $m_{(A,B)}$ -topology is denoted by $C_{m(A,B)}(X)$. To see that $\{B(f, u, v) : u \in U_A^+ \text{ and } v \in U_B^+\}$ is a base at f , it is evident that $f \in B(f, u, v)$, $B(f, u_1 \wedge u_2, v_1 \wedge v_2) \subseteq B(f, u_1, v_1) \cap B(f, u_2, v_2)$, for all $u_1, u_2 \in U_A^+$ and $v_1, v_2 \in U_B^+$, and whenever $g \in B(f, u, v)$ for some $u \in U_A^+$ and $v \in U_B^+$, then $B(g, w, z) \subseteq B(f, u, v)$, where $w = u - |f - g| \in U_A^+$ and $z = v - |f - g| \in U_B^+$. It is clear that $m_{(A,B)}$ -topology is finer than the m -topology, for $B(f, u) = B(f, u|_A, u|_B)$, for each $u \in U^+$. Also, if A and B are

closed and open subsets of X , then $m_{(A,B)}$ -topology and m -topology coincide, because for each $u \in U_A^+$ and $v \in U_B^+$, clearly $B(f, t) \subseteq B(f, u, v)$, where

$$t(x) = \begin{cases} u(x) & x \in A \setminus B \\ u(x) \wedge v(x) & x \in A \cap B \\ v(x) & x \in B \setminus A \end{cases}$$

Proposition 2.1. The space $C_{m(A,B)}(X)$ is a topological ring.

Proof. If $u \in U^+(A)$, $v \in U^+(B)$ and $B(f + g, u, v)$ is a neighborhood at $f + g$, then we consider the neighborhoods $B(f, \frac{u}{2}, \frac{v}{2})$ and $B(g, \frac{u}{2}, \frac{v}{2})$ at f and g respectively. Now suppose that $h \in B(f, \frac{u}{2}, \frac{v}{2})$ and $k \in B(g, \frac{u}{2}, \frac{v}{2})$, then we have $|h - f| < \frac{u}{2}$, $|k - g| < \frac{u}{2}$ on A and $|h - f| < \frac{v}{2}$, $|k - g| < \frac{v}{2}$ on B . Hence we have $|(h + k)(x) - (f + g)(x)| < u(x)$, for all $x \in A$ and $|(h + k)(x) - (f + g)(x)| < v$, for all $x \in B$, i.e., the function $+$ is continuous. For the continuity of $\times$, let $B(fg, u, v)$ be a neighborhood at fg and take $w_1 = \frac{u}{2}(\frac{1}{|g|+1})$, $w_2 = \frac{u}{2}(\frac{1}{|f|+w_1})$, $z_1 = \frac{v}{2}(\frac{1}{|g|+1})$ and $z_2 = \frac{v}{2}(\frac{1}{|f|+z_1})$. now if $h \in B(f, w_1, z_1)$ and $k \in B(g, w_2, z_2)$, then for each $x \in A$ we have $|h(x) - f(x)| < w_1(x)$ and $|k(x) - g(x)| < w_2(x)$, and also for each $x \in B$ we have $|h(x) - f(x)| < z_1(x)$ and $|k(x) - g(x)| < z_2(x)$. Hence whenever $x \in A$, we have $|(hg)(x) - (fg)(x)| < w_1(x)|g(x)| < \frac{u(x)}{2}$ and $|(hk)(x) - (hg)(x)| < w_2(x)|h(x)| < \frac{u(x)}{2}$, imply that $|(hk)(x) - (fg)(x)| < u(x)$, for all $x \in A$. Similarly, $|(hk)(x) - (fg)(x)| < v(x)$, for all $x \in B$ and hence we are through. $\square$

We need the following proposition in the sequel.

Proposition 2.2. The following statements hold.

- (a) $C^*(X)$ is a closed-open set in $C_{m(A,B)}(X)$.
- (b) Suppose that $\{x_n\} \subseteq A$ and $g \in C^+(A)$ such that $g(x_n) \rightarrow \infty$. Then the set of $L_{\{x_n, g\}}$ by

$$L_{\{x_n, g\}} = \{f \in C(X) : \exists u \in U_A^+ : u(x_n) \rightarrow 0, |(fg)|_A < u|_A\}$$

is closed-open set in $C_{m(A,B)}(X)$.

Proof. (a) It is evident, for $m_{(A,B)}$ -topology is finer than m -topology.

(b) Let $f \in L_{\{x_n, g\}}$. Then there exists $u \in U_A^+$ such that $u(x_n) \rightarrow 0$ and $|(fg)|_A < u|_A$. Now consider neighborhood base $B(f, \frac{u}{|g|+1}, 1)$. If $h \in B(f, \frac{u}{|g|+1}, 1)$, then $|(hg)(x)| < u(x) + |(fg)(x)| < 2u(x)$, for all $x \in A$ and therefore $h \in L_{\{x_n, g\}}$. this shows that $L_{\{x_n, g\}}$ is an open set in $C_{m(A,B)}(X)$.

In the other hand, if $f \in \text{cl}L_{\{x_n, g\}}$, then $B(f, \frac{1}{g^2+1}, 1) \cap L_{\{x_n, g\}} \neq \emptyset$. This means that there exists $h \in B(f, \frac{1}{g^2+1}, 1) \cap L_{\{x_n, g\}}$ and for all $x \in A$, we have $|(fg)(x)| < \frac{|g(x)|}{g^2(x)+1} + |(hg)(x)|$. Also there exists $u \in U_A^+$ such that $u(x_n) \rightarrow 0$ and $|(hg)(x)| < u(x)$, for all $x \in A$. Now take $v = \frac{|g|}{g^2+1} + u$. it is clear that $v(x_n) \rightarrow 0$ and $|(fg)(x)| < v(x)$, for all $x \in A$. Hence $f \in L_{\{x_n, g\}}$ and this implies that $L_{\{x_n, g\}}$ is closed. $\square$

3. Connectedness in $C_{m(A,B)}(X)$.

In this section we characterize the components of $C_{m(A,B)}(X)$ and investigate some connected subsets of $C_{m(A,B)}(X)$. At first, we generalize the set $C_\psi(X)$.

Definition 3.1. Let $A, B \subseteq X$ such that $A \cup B = X$. We define

$$C_{\psi(A,B)}(X) = \{f \in C(X) : f|_A \in C_\psi(A) \text{ and } f|_B \in C_\psi(B)\}$$

Lemma 3.2. If $f \in C_{\psi(A,B)}(X)$, then the function $\varphi_f : \mathbb{R} \rightarrow C_{m^*}(X)$ defined by $\varphi_f(r) = rf$, $\forall r \in \mathbb{R}$ is continuous.

Proof. Let $u \in U^+(A)$ and $v \in U^+(B)$. Since $f \in C_{\psi(A,B)}(X)$, u is bounded away from zero on $A \setminus Z(f)$ for $\frac{1}{u}$ is bounded on $A \setminus Z(f)$. v is also bounded away from zero on $B \setminus Z(f)$. Hence there exist $\alpha \in \mathbb{R}^+$ such that $u(x) > \alpha > 0$ for all $x \in A \setminus Z(f)$ and $v(x) > \alpha > 0$ for all $x \in B \setminus Z(f)$. Now if $|f| < M$, then $(r - \frac{\alpha}{M}, r + \frac{\alpha}{M}) \subseteq \varphi_f^{-1}(B(rf, u, v))$. So whenever $|r - s| < \frac{\alpha}{M}$, then $|rf - sf| < \frac{\alpha}{M}|f| < \alpha$ and hence φ is continuous. $\square$

The following theorem shows that the component of 0 in $C(X)$ with the m -topology is $C_\psi(X)$, see also [4]. At first we recall that A is relatively pseudocompact in X if and only if every function $f \in C(X)$ is bounded on A .

Theorem 3.4. The component of 0 in $C_{m(A,B)}(X)$ is $C_{\psi(A,B)}(X)$.

Proof. For each $f \in C_{\psi(A,B)}(X)$, the function φ_f is continuous by Lemma 3.2. Hence $\varphi_f(\mathbb{R})$ is connected. But $C_{\psi(A,B)}(X) = \bigcup_{f \in C_{\psi(A,B)}(X)} \varphi_f(\mathbb{R})$ means that $C_{\psi(A,B)}(X)$ is also connected. Now suppose that I is a connected ideal in $C_{m(A,B)}(X)$ and $I \not\subseteq C_{\psi(A,B)}(X)$. Hence there exists $f \in I \setminus C_{\psi(A,B)}(X)$ and therefore $f|_A \notin C_\psi(A)$ (or $f|_B \notin C_\psi(B)$). This implies that $A \setminus Z(f)$ is not relatively pseudocompact in A , i.e., there exists $g \in U^+(A)$ such that $g|_{A \setminus Z(f)}$ is unbounded. Now by 1-20 in [7], there exists a sequence $\{x_n\} \subseteq A \setminus Z(f)$ which is a copy of $\mathbb{N}$ and it is C -embedded in A and $g(x_n) \rightarrow \infty$. So by considering

$\frac{1}{|f|} \in C(A \setminus Z(f))$, there exists $k \in C^+(A)$ which $k|_{\{x_n\}} = \frac{1}{|f|}$. But the set $L_{\{x_n, gk\}}$ is open and closed in $C_{m(A,B)}(X)$, by part (b) of proposition 2.2, and it is clear that $0 \in L_{\{x_n, gk\}}$ and $f \notin L_{\{x_n, gk\}}$. This implies $L_{\{x_n, gk\}} \cap I \neq \emptyset$ and $L_{\{x_n, gk\}} \cap I \neq I$. it is a contradiction. $\square$

It is well-known that $C_m(X)$ is connected if and only if X is pseudocompact. But the following corollary shows that the pseudocompactness is not sufficient for connectedness in $C_{m(A,B)}(X)$.

Corollary 3.6. $C_{m(A,B)}(X)$ is connected if and only if A and B are pseudocompact.

Corollary 3.7. If A, B are realcompact subspace of X , then

$$C_{\psi(A,B)}(X) = \{f \in C(X) : f|_A \in C_K(A) \text{ and } f|_B \in C_K(B)\}$$

Now, we could see a generalization of m -topology which its component of zero is $C_K(X)$.

Corollary 3.8. If A, B are closed realcompact subspace of X , then $C_{\psi(A,B)}(X) = C_K(X)$.

Proof. Let $f \in C_{\psi(A,B)}(X)$, then $\text{cl}_A(A \setminus Z(f|_A))$ and $\text{cl}_B(B \setminus Z(f|_B))$ are compact, by corollary 3.7. A, B are closed and therefore by proposition 1.1, we have

$$\text{cl}_X(A \setminus Z(f|_A)) = \text{cl}_X(A \setminus Z(f|_A)) \cap A = \text{cl}_A(A \setminus Z(f|_A))$$

$$\text{cl}_X(B \setminus Z(f|_B)) = \text{cl}_X(B \setminus Z(f|_B)) \cap B = \text{cl}_B(B \setminus Z(f|_B))$$

But $\text{cl}_X \text{coz}(f) = \text{cl}_X(A \setminus Z(f|_A)) \cup \text{cl}_X(B \setminus Z(f|_B))$. Hence $f \in C_K(X)$. Conversely, if $f \in C_K(X)$, then $\text{cl}_X \text{coz}(f)$ is compact and therefore $\text{cl}_A(A \setminus Z(f|_A))$ is compact too, for it is easily $\text{cl}_A(A \setminus Z(f|_A)) \subseteq \text{cl}_X \text{coz}(f)$. Similarly, $\text{cl}_B(B \setminus Z(f|_B))$ is compact. So $f \in C_{\psi(A,B)}(X)$. $\square$

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