Asymptotic Behavior of Solutions of Nonautonomous Ordinary Differential Equations of n-th Order

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Abstract

The asymptotic representations for solutions of some classes of nonautonomous n-th order ordinary differential equations that are close, in a certain sense, to linear equations are established.

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1 Statement of the Problem and Main Results

Consider the differential equation

\[ y^{(n)} = \alpha_0 p(t) y \ln |y|^{\sigma}, \quad (1.1) \]

where \( \alpha_0 \in \{-1, 1\}, \ \sigma \in \mathbb{R}, \ p : [a, \omega \rightarrow 0, +\infty] \rightarrow \mathbb{R} \) is a continuous function, \(-\infty < a < \omega \leq +\infty\).

Here, the function \( y \ln |y|^{\sigma} \) is regularly varying function of order one as \( y \rightarrow 0 \), even as \( y \rightarrow \pm \infty \) [10], so the equation (1.1) is not covered by the

\[ \text{We assume that } a > 1 \text{ for } \omega = +\infty, \text{ and } \omega - a < 1 \text{ for } \omega < +\infty. \]
results of work of V.M. Evtuhova, A.M. Samojlenko [3], in which investigated the asymptotic behavior of solutions of the differential equation

\[ y^{(n)} = \alpha_0 p(t) \varphi(y), \]

in which, \( \varphi : \Delta_{Y_0} \rightarrow [0, +\infty[ \) is regularly varying function of order \( \gamma \neq 1 \) for \( y \rightarrow Y_0 \), where \( Y_0 \) is either zero or \( \pm \infty \), and \( \Delta_{Y_0} \) - one-sided neighborhood of \( Y_0 \). In case of \( \gamma = 1 \), the last differential equation is no longer essentially nonlinear, and is asymptotically close to the linear equation

\[ y^{(n)} = \alpha_0 p(t)y, \]  

(1.2)

that requires the use of other approaches to its study. 

A solution \( y \) of Eq. (1.1) defined on the interval \([t_y, \omega[ \subset [a, \omega[\) is called a \( P_\omega(\lambda_0) \)- solution if it satisfies the following conditions:

\[
\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} 
\text{either 0,} & (k = 0, n-1), \\
\text{or } +\infty & (k = 0, n-1), \\
\end{cases}
\lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0. 
\]

(1.3)

For \( n = 2 \) the asymptotic \( P_\omega(\lambda_0) \)-solution of equation (1.1) were studied by [2] and [6-9].

The aim of the present work is to establish for \( n \geq 2 \) the necessary and the sufficient conditions for the existence of \( P_\omega(\lambda_0) \)- solution of Eq. (1.1), for which \( \lambda_0 \in R \setminus \{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1\} \), and to find the asymptotic representations for all these solutions and their derivatives up to order \( n-1 \) inclusive as \( t \uparrow \omega \).

We introduce the following auxiliary notation:

\[
a_{0k} = (n - k)\lambda_0 - (n - k - 1) \quad (k = 1, n) \quad \text{for} \quad \lambda_0 \in R, \quad (1.4)
\]

\[
\pi_\omega(t) = \begin{cases} 
t, & \text{if } \omega = +\infty, \\
t - \omega, & \text{if } \omega < +\infty,
\end{cases}
J_B(t) = \int_B \frac{1}{p^n} d\tau, \quad (1.5)
\]

where

\[
B = \begin{cases} 
a, & \text{if } \int_a^\omega \frac{1}{p^n} d\tau = +\infty, \\
\omega, & \text{if } \int_0^\omega \frac{1}{p^n} d\tau < +\infty.
\end{cases}
\]

For equation (1.1) the following theorem holds.

**Theorem 1.1.** Let \( \sigma \neq n \) and \( \lambda_0 \in R \setminus \{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1\} \). Then for the existence of \( P_\omega(\lambda_0) \)- solution of equation (1.1) it is necessary, and if the next inequality is hold

\[
\sigma \neq a_{01} \left( 1 + \sum_{k=1}^{n-1} \frac{1}{a_{0k}} \right), \quad (1.6)
\]
and the next algebraic equation with respect to $\rho$, has no roots with zero real part

$$
\prod_{j=1}^{n-1} (a_{0j} + \rho) + \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} (a_{0j} + \rho) \prod_{j=k+1}^{n-1} a_{0j} = 0 \quad (1.7)
$$

then it is sufficient, for the next inequality to take place

$$
\alpha_0 \left( \prod_{k=1}^{n-1} a_{0k} \right) [(\lambda_0 - 1)\pi_\omega(t)]^n > 0 \quad \text{for} \quad t \in [a, \omega[,
$$

and the following condition is hold

$$
\lim_{t \uparrow \omega} \frac{1}{n} p_n(t) \left| \frac{n - \sigma}{n} J_B(t) \right| \frac{n}{n-\sigma} = \frac{|a_{01}|}{|\lambda_0 - 1|} \left( \frac{1}{n} \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right) \frac{n}{n-\sigma} [1 + o(1)], \quad (1.9)
$$

moreover, each of these solutions admits the following asymptotic representations as $t \uparrow \omega$:

$$
\ln |y(t)| = \nu \left( \frac{1}{n} \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right) \left| n - \sigma \right| J_B(t) \frac{n}{n-\sigma} [1 + o(1)], \quad (1.10)
$$

$$
\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)] \quad (k = 1, n - 1), \quad (1.11)
$$

where

$$
\nu = \text{sign} [a_{01}(\lambda_0 - 1)(n - \sigma)\pi_\omega(t)J_B(t)].
$$

In addition to these conditions, if the algebraic equation (1.7) has the $m$-roots (including multiples), the real parts of which have a sign opposite to the sign of the function $(\lambda_0 - 1)\pi_\omega(t)$ in the interval $[a, \omega[$, and the following inequality is satisfied

$$
\left( \frac{\sigma}{a_{01}} - 1 - \sum_{k=1}^{n-1} \frac{1}{a_{0k}} \right) \left( 1 + \sum_{k=1}^{n-1} \frac{1}{a_{0k}} \right) > 0 \quad (1.12)
$$

then the Eq. (1.1) has $m-$parameter family of solutions with representations (1.10) and (1.11), and when the opposite inequality is hold, it has $m + 1$-parameter family of such solutions.
Remark 1.1. Algebraic equation (1.7) has no roots with zero real part if the next inequality is hold:

\[ \sum_{k=1}^{n-1} \frac{1}{|a_{0k}|} < 1. \]

From this theorem for \( \sigma = 0 \), directly implies the following statement for the linear differential equation (1.2)

Corollary 1.1. For the existence of \( P_\omega(\lambda_0) \) solution of equation (1.2), where \( \lambda_0 \in R \setminus \{0, \frac{1}{2}, \ldots, \frac{1}{n-1}, 1\} \), necessary, and if the algebraic equation with respect to \( \rho \) of equation (1.7) has no roots with zero real part, then sufficient the inequality (1.8) and the next condition are hold

\[ \lim_{t \to \omega} p^n(t)J_B(t) = \frac{\prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}}}{|\lambda_0 - 1|}, \] (1.13)

and for each of these solutions take place as \( t \to \omega \) the next asymptotic representations

\[ \ln |y(t)| = \frac{\nu |a_{01}|}{\prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}}} |J_B(t)| [1 + o(1)], \] (1.14)

\[ \frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)] \quad (k = 1, n-1), \] (1.15)

where

\[ \nu = \text{sign} [a_{01}(\lambda_0 - 1)\pi_\omega(t)J_B(t)] . \]

Moreover, if in addition to these conditions, the algebraic equation (1.7) has the \( m \)-roots (including multiples), the real parts of which are a sign of opposite sign of the function \( (\lambda_0 - 1)\pi_\omega(t) \) in the interval \([a, \omega]\), then for the equation (1.2) exists \( m + 1 \)-parameter family of solutions with representations (1.14) and (1.15).

Remark 1.2. Corollary 1.1 belongs to the case, where the differential equation (1.2) is asymptotically close to the Euler equation. If

\[ \lim_{t \to \omega} p(t)\pi_\omega^n(t) = c_0 \neq 0 \]
and the next algebraic equation with respect to $\lambda_0$

$$c_0(\lambda_0 - 1)^n = \alpha_0 \prod_{k=1}^{n-1}[(n - k)\lambda_0 - (n - k - 1)],$$

which we obtain from (1.9), has $n$ distinct real roots $\lambda_{0j}$ $(j = \overline{1, n})$, then the fundamental system of solutions $y_j$ $(j = \overline{1, n})$ of differential equation (1.2) admits as $t \uparrow \omega$ the next asymptotic representations

$$\ln |y_j(t)| = \frac{\alpha_0(\lambda_{0j} - 1)^{n-1}J_B(t)}{\prod_{k=2}^{n-1}[(n - k)\lambda_{0j} - (n - k - 1)]}[1 + o(1)],$$

$$\frac{y_j^{(k)}(t)}{y_j^{(k-1)}(t)} = \frac{(n - k)\lambda_{0j} - (n - k - 1)}{(\lambda_{0j} - 1)\pi_{\omega}(t)}[1+o(1)] \quad (k = \overline{1, n-1}, \quad j = \overline{1, n}).$$

Remark 1.3. Theorem 1.1 in the particular case $n = 2$ complements the results from [2] and [6 - 9], and Corollary 1.1 - results from [8].

2 Auxiliary Statements

From the results obtained in [1] immediately implies the following statement on a priori asymptotic properties of $P_\omega(\lambda_0)$ - solutions of the differential Eq. (1.1).

Lemma 2.1. Let $\lambda_0 \in R \setminus \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{2}{n-1}, 1\}$ and $y : [t_0, \omega[\longrightarrow R \setminus \{0\}$-arbitrary $P_\omega(\lambda_0)$-solution of the equation (1.1). Then we have the asymptotic relations

$$\frac{y_j^{(k)}(t)}{y_j^{(k-1)}(t)} \sim \frac{a_{0k}}{(\lambda_0 - 1)\pi_{\omega}(t)} \quad (k = \overline{1, n}) \quad \text{as} \quad t \uparrow \omega, (2.1)$$

where $a_{0k}$ $(k = \overline{1, n})$ determined by the formulas (1.4) and $\pi_{\omega}$ - by (1.5).

Along with this lemma, to proof the Theorem 1.1 we will use a test for the existence of vanishing at infinity the solutions for the system of quasi-linear differential equations.
\[
\begin{align*}
\begin{cases}
  v'_k = \beta_0 \left[ f_k(\tau, v_1, \ldots, v_n) + \sum_{i=1}^{n} c_{ki} v_i + V_k(v_1, \ldots, v_n) \right] & (k = 1, n-1), \\
  v'_n = H(\tau) \left[ f_n(\tau, v_1, \ldots, v_n) + \sum_{i=1}^{n} c_{ni} v_i + V_n(v_1, \ldots, v_n) \right],
\end{cases}
\end{align*}
\]

(2.2)

in which \(\beta_0 \in R \setminus \{0\}\), \(c_{ik} \in R\) \((i, k = 1, n)\), \(H : [\tau_0, +\infty[ \longrightarrow R \setminus \{0\}\) - continuous function, \(f_k : [\tau_0, +\infty[ \times R^n_{1/2} (k = 1, n)\)-continuous functions satisfying the conditions:

\[
\lim_{t \uparrow \omega} f_k(\tau, v_1, \ldots, v_n) = 0 \quad \text{uniformly in} \quad (v_1, \ldots, v_n) \in R^n_{1/2},
\]

(2.3)

where

\[
R^n_{1/2} = \left\{ (v_1, \ldots, v_n) \in R^n : |v_i| \leq \frac{1}{2} (i = 1, n) \right\},
\]

and \(V_k : R^n_{1/2} \longrightarrow R (k = 1, n)\)- continuously differentiable function such that

\[
V_k(0, \ldots, 0) = 0 \quad (k = 1, n), \quad \frac{\partial V_k(0, \ldots, 0)}{\partial v_i} = 0 \quad (i, k = 1, n).
\]

(2.4)

By the Theorem 2.6 and 2.8 from ref. [4] for the system of differential equations (2.2), we have the following statement.

**Lemma 2.2.** Suppose the function \(H : [\tau_0, +\infty[ \longrightarrow R \setminus \{0\}\) - continuously differentiable and satisfies

\[
\lim_{\tau \to +\infty} H(\tau) = 0, \quad \lim_{\tau \to +\infty} \frac{H'(\tau)}{H(\tau)} = 0, \quad \int_{\tau_0}^{+\infty} H(\tau) \, d\tau = \pm \infty,
\]

(2.5)

and the matrices \(C_n = (c_{ki})_{k,i=1}^{n}\) and \(C_{n-1} = (c_{ki})_{k,i=1}^{n-1}\) such that \(\det C_n \neq 0\), and \(C_{n-1}\) has no eigenvalues with zero real part. Then the system of differential equations (2.2) has at least one solution \((v_k)_{k=1}^{n} : [\tau_1, +\infty[ \longrightarrow R^n_{1/2} (\tau_1 \geq \tau_0)\), tends to zero as \(t \to +\infty\). Moreover, if among the eigenvalues \(C_{n-1}\) has eigenvalues \(m\) (including multiple), which the real parts have opposite sign for \(\beta_0\), then as the inequality \(H(\tau) (\det C_n) (\det C_{n-1}) > 0\) is hold, there exist \(m\)-parameter family solutions of the system (2.2), and there exist \(m + 1\) - parameter family when the opposite inequality.
3 The proof of the main result

Proof of Theorem 1.1. Necessary. Let \( y : [t_y, \omega] \rightarrow R \setminus \{0, 1\} \) arbitrary \( P_\omega(\lambda_0) \)-solution of the Eq. (1.1), where \( \lambda_0 \in R \setminus \left\{ \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1 \right\} \). Then by Lemma 2.1 for this solution we have the asymptotic representation (1.11). Moreover, from (2.1) it is clear that

\[
\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}y'(t)}{a_{01}y(t)}[1 + o(1)] \quad (k = 1, n) \quad \text{as} \quad t \uparrow \omega.
\]

By these relationships we have

\[
y^{(n)}(t) = \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \cdot \frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} \cdots \frac{y'(t)}{y(t)} \sim \frac{\prod_{k=2}^{n-1} a_{0k}}{a_{01}^{n-1}} \left( \frac{y'(t)}{y(t)} \right)^n \quad \text{as} \quad t \uparrow \omega.
\]

and therefore from (1.1), it follows that

\[
\left( \frac{y'(t)}{y(t)} \right)^n = \frac{a_{00}a_{01}^n}{\prod_{k=1}^{n-1} a_{0k}} p(t) |\ln |y(t)||^\sigma [1 + o(1)] \quad \text{as} \quad t \uparrow \omega.
\]

Since, according to (2.1) we get

\[
\frac{y'(t)}{y(t)} \sim \frac{a_{01}}{\lambda_0 (\lambda_0 - 1) \pi_\omega(t)} \quad \text{as} \quad t \uparrow \omega, \quad (3.1)
\]

this implies inequality (1.8) and the next asymptotic representation

\[
y'(t) = \mu \left( \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right) ^\frac{1}{n} \frac{p_\omega(t) |\ln |y(t)||^\sigma}{|\ln |y(t)||^\frac{n-\sigma}{n} \cdot \text{sign} (\ln |y(t)|)} \left( n - \sigma \right) \frac{J_B(t)}{n} [1 + o(1)] \quad \text{as} \quad t \uparrow \omega, \quad (3.2)
\]

where

\[
\mu = \text{sign} [a_{01}(\lambda_0 - 1)\pi_\omega(t)].
\]

Dividing (3.2) by \( |\ln |y(t)||^\frac{n-\sigma}{n} \cdot \text{sign} (\ln |y(t)|) \) and integrating over the interval from \( t_y \) to \( t \), taking into account the conditions \( \sigma \neq n \) and the definition of \( P_\omega(\lambda_0) \)-solution, we obtain the relation of the form

\[
|\ln |y(t)||^\frac{n-\sigma}{n} \cdot \text{sign} (\ln |y(t)|) = \mu \left( \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right) \left( \frac{a_{01}}{n} \frac{J_B(t)}{n} \right) [1 + o(1)] \quad \text{as} \quad t \uparrow \omega,
\]
which implies the representation (1.10).

Now, using (1.10) and (3.1), from (3.2) we find that

\[
\frac{a_{01}}{(\lambda_0 - 1)\pi_\omega(t)} = \mu \left( \frac{|a_{01}|}{\prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}}} \right)^{\frac{n}{n-\sigma}} p^n(t)^{\frac{n-\sigma}{n}} \left| \frac{n-\sigma}{n} J_B(t) \right|^{\frac{n}{n-\sigma}} [1 + o(1)] \text{ as } t \uparrow \omega.
\]

By virtue of this relations, the (1.9) is holds.

Sufficient. Suppose that for some \( \lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{3}{2}, \ldots, \frac{n-2}{n-1}, 1\} \) the conditions (1.6), (1.8), (1.9) are satisfied, and algebraic equation (1.7) has no roots with zero real part. We show that, in this case for \( \sigma \neq n \) differential equation (1.1) has a \( P_\omega(\lambda_0) \) solutions, admitting as \( t \uparrow \omega \) asymptotic representations (1.10), (1.11) and clarify the question about the number of such solutions.

Applying to equation (1.1), the transformation

\[
\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)} [1 + v_k(\tau)] \quad (k = 1, n-1),
\]

\[
\ln |y(t)| = \nu \left( \frac{|a_{01}|}{\prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}}} \right)^{\frac{n}{n-\sigma}} \left| \frac{n-\sigma}{n} J_B(t) \right|^{\frac{n}{n-\sigma}} [1 + v_n(\tau)],
\]

where

\[
\tau = \beta \ln |\pi_\omega(t)|, \quad \beta = \begin{cases} 1 & \text{for } \omega = +\infty, \\ -1 & \text{for } \omega < +\infty, \end{cases}
\]

we obtain a system of differential equations

\[
\begin{cases}
  v'_k = \frac{\beta(1 + v_k)}{\lambda_0 - 1} [\lambda_0 - 1 + a_{0k+1}(1 + v_{k+1}) - a_{0k}(1 + v_k)] \quad (k = 1, n-2), \\
  v'_{n-1} = \frac{\beta}{\lambda_0 - 1} \left[ \frac{h^n(\tau)|1 + v_n|^\sigma}{(1 + v_1) \ldots (1 + v_{n-2})} - a_{0n-1}(1 + v_{n-1})^2 + (\lambda_0 - 1)(1 + v_{n-1}) \right] \\
  v'_n = \frac{\beta n q(\tau)}{n-\sigma} \left[ \frac{1 + v_1}{h(\tau)} - 1 - v_n \right],
\end{cases}
\]

in which

\[
qu(\tau(t)) = \frac{\pi_\omega(t)p^n(t)}{J_B(t)},
\]
Asymptotic behavior of solutions

\[ h(\tau(t)) = \left| \frac{\lambda_0 - 1}{a_{01}} \right| \left( \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right)^{\frac{n}{n-\sigma}} |\pi_\omega(t)| \left( \frac{1}{p^{\frac{1}{n}}(t)} \right)^n \left| \frac{n-\sigma}{n} J_B(t) \right|^{\frac{\sigma}{n-\sigma}}. \]

Thus, taking into account the sign of the number \( \beta \), we note that

\[ \frac{\beta n q(\tau(t))}{n - \sigma} = \frac{|\pi_\omega(t)| p^{\frac{1}{n}}(t)}{n - \sigma} = M h(\tau(t)) \frac{\text{sign}[(n - \sigma) J_B(t)]}{\left| \frac{n-\sigma}{n} J_B(t) \right|^{\frac{n}{n-\sigma}}}, \quad (3.5) \]

where

\[ M = \left| \frac{a_{01}}{\lambda_0 - 1} \right| \left( \prod_{k=1}^{n-1} |a_{0k}|^{\frac{1}{n}} \right)^{\frac{n}{n-\sigma}}. \]

By (1.9) and the form of the functions \( J_B(t), \tau(t) \) we have

\[ \lim_{\tau \to +\infty} h(\tau) = \lim_{t \uparrow \omega} h(\tau(t)) = 1, \quad (3.6) \]

\[ +\infty \int_{\tau_0}^{\omega} \beta q(\tau) \, d\tau = \int_{a_0}^{\omega} \frac{J_B'(t)}{J_B(t)} \, dt = \ln \left| J_B(t) \right|_{a_0}^{\omega} = \begin{cases} +\infty, & \text{if } B = a, \\ -\infty, & \text{if } B = \omega, \end{cases} \quad (3.7) \]

where \( \tau_0 = \beta \ln |\pi_\omega(a_0)| \) and \( a_0 \)- arbitrarily chosen number from the interval \([a, \omega] \). Hence, taking into account that \( q \) is different from zero on the interval \([\tau_0, +\infty] \), we have

\[ +\infty \int_{a_0}^{+\infty} \frac{q(\tau)}{h(\tau)} \, d\tau = \pm \infty. \quad (3.8) \]

In addition, by condition (1.9) we get

\[ +\infty \int_{a_0}^{\omega} p^{\frac{1}{n}}(t) \left| \frac{n-\sigma}{n} J_B(t) \right|^{\frac{\sigma}{n-\sigma}} \, dt = +\infty \]

and therefore

\[ \lim_{t \uparrow \omega} \left| J_B(t) \right|^{\frac{\sigma}{n-\sigma}} = +\infty. \quad (3.9) \]

Separation the linear parts on the right hand-sides of system (3.4), we
rewrite it according to (1.4) and (3.5) in the following form

\[
\begin{aligned}
    v_k' &= \frac{\beta}{\lambda_0 - 1} \left[ -a_{0k} v_k + a_{0k+1} v_{k+1} - a_{0k} v_k^2 + a_{0k+1} v_k v_{k+1} \right] \quad (k = \overline{1, n-2}), \\
    v_{n-1}' &= \frac{\beta}{\lambda_0 - 1} \left[ r_1(\tau, v_1, \ldots, v_n) - \sum_{i=1}^{n-2} v_i - (\lambda_0 + 1)v_{n-1} + \sigma v_n + V(v_1, \ldots, v_n) \right] \\
    v_n' &= \frac{\beta \eta(\tau)}{(n-\sigma)h(\tau)} \left[ r_2(\tau, v_n) + v_1 - v_n \right],
\end{aligned}
\]

(3.10)

where

\[
\begin{aligned}
    r_1(\tau, v_1, \ldots, v_n) &= \frac{[h^\alpha(\tau) - 1][1 + v_n]^{\sigma}}{[1 + v_1][1 + v_2] \ldots [1 + v_{n-2}]}, \\
    r_2(\tau, v_1) &= [h(\tau) - 1](1 + v_n), \\
    V(v_1, \ldots, v_n) &= \frac{|1 + v_n|^{\sigma}}{|1 + v_1| \ldots |1 + v_{n-2}|} - \sigma v_n + \sum_{i=1}^{n-2} v_i - \lambda_0 v_{n-1}.
\end{aligned}
\]

Here

\[
V(0, \ldots, 0) = 0, \quad \frac{\partial V(0, \ldots, 0)}{\partial v_i} = 0 \quad (i = \overline{1, n})
\]

and by the condition (3.6) we have

\[
\lim_{\tau \to +\infty} r_1(\tau, v_1, \ldots, v_n) = 0, \quad \lim_{\tau \to +\infty} r_2(\tau, v_1) = 0 \quad \text{uniformly in} \quad (v_1, \ldots, v_n) \in R^n_{\overline{1, n}},
\]

where

\[
R^n_{\overline{1, n}} = \{(v_1, \ldots, v_n) \in R^n : |v_i| \leq \frac{1}{2} \quad (i = \overline{1, n})\}.
\]

Thus (3.10) is a system of differential equations that has the form of (2.2), where

\[
\beta_0 = \frac{\beta}{\lambda_0 - 1}, \quad H(\tau(t)) = \frac{\beta \eta(\tau(t))}{(n-\sigma)h(\tau(t))} = \frac{M \text{sign} \left[ (n-\sigma)J_B(t) \right]}{1 - \frac{n-\sigma}{M} J_B(t)},
\]

\[
\begin{aligned}
    f_k(\tau, v_1, \ldots, v_n) &\equiv 0 \quad (k = \overline{1, n-2}), \\
    f_{n-1}(\tau, v_1, \ldots, v_n) &= r_1(\tau, v_1, \ldots, v_n), \\
    f_n(\tau, v_1, \ldots, v_n) &= r_2(\tau, v_n), \\
    V_k(v_1, \ldots, v_n) &= -a_{0k} v_k^2 + a_{0k+1} v_k v_{k+1} \quad (k = \overline{1, n-2}), \\
    V_{n-1}(v_1, \ldots, v_n) &= V(v_1, \ldots, v_n), \\
    V_{n}(v_1, \ldots, v_n) &\equiv 0,
\end{aligned}
\]

\[
\begin{aligned}
    c_{kk} &= -a_{0k}, \quad c_{kk+1} = a_{0k+1}, \quad c_{ki} = 0 \quad \text{for} \quad i \in \{1, \ldots, n\} \setminus \{k, k+1\} \quad (k = \overline{1, n-2}), \\
    c_{n-1i} &= -1 \quad (i = \overline{1, n-2}), \quad c_{n-1n-1} = -1 - \lambda_0, \quad c_{n-1n} = \sigma, \quad c_{n1} = 1, \quad c_{ni} = 0 \quad (i = \overline{2, n-1}), \quad c_{nn} = -1,
\end{aligned}
\]

and here the functions \(f_k\) and \(V_k\) \((k = \overline{1, n})\) satisfy the conditions (2.3) and (2.4).
We show that for this system fulfilled, along with (2.3) and (2.4), all the other conditions of Lemma 2.2.

By (3.5), (3.6), (3.8), (3.9) and (1.9) we have

\[
\lim_{\tau \to +\infty} H(\tau) = 0, \quad \lim_{\tau \to +\infty} \frac{H'(\tau)}{H(\tau)} = 0, \quad \int_{a_0}^{+\infty} H(\tau) \, d\tau = \pm \infty, \quad \text{sign} \, H(\tau) = 1,
\]

that is the conditions (2.5) are satisfied.

Further, in the obtained system (2.2), the matrix \(C_n = (c_{ki})_{k,i=1}^{n}\) and \(C_{n-1} = (c_{ki})_{k,i=1}^{n-1}\) are such that:

\[
\det C_n = (-1)^{n+1} \prod_{j=1}^{n-1} a_{0j} \left( \frac{\sigma}{a_{01}} - 1 - \sum_{k=1}^{n-1} \frac{1}{a_{0k}} \right),
\]

\[
\det C_{n-1} = (-1)^n \prod_{j=1}^{n-1} a_{0j} \left( 1 + \sum_{k=1}^{n-1} \frac{1}{a_{0k}} \right)
\]

and the characteristic equation of the matrix \(C_{n-1}\) has the form (1.7).

Since, the inequality (1.6) is hold and the algebraic equation (1.7) has no roots with zero real part, then the system of differential equations (3.10) satisfies all the conditions of Lemma 2.2. On the basis of this lemma, for the system of differential equations (3.10) there exists at least one solution \((v_k)_{k=1}^{n} : [\tau_1, +\infty] \to \mathbb{R}^n (\tau_1 \geq \tau_0)\) which tends to zero as \(\tau \to +\infty\). If among the roots of the algebraic equation (1.7) has \(m\) roots, which the real parts have a sign opposite to the sign of the number \(\beta(\lambda_0 - 1)\), where \(\beta = \text{sign} \, \pi_\omega(t)\), when the inequality (1.12) is hold, there exists an \(m\)-parameter family of solutions vanishing at infinity, and there exists \(m+1\) parameter family when the opposite inequality.

According to substitutions (3.3) for every such solutions there exist a corresponding solution \(y : [t_1, \omega] \to \mathbb{R} (t_1 \in [a, \omega])\) of differential equation (1.1), that admit as \(t \uparrow \omega\) asymptotic representations (1.10), (1.11). The theorem is proved.

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References


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