On Geometric Hyper-Structures¹

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Abstract

The aim of this paper is to introduce the notion of hyper-semifields, hyper-spread sets and hyper-spreads on one hand and to investigate the relation among these structures.

1. INTRODUCTION

We recall some hyper-structures theory. A hyper-groupoid $(H; \otimes)$ is the set H endowed with a binary multi-valued operation (hyper-operation) i.e a function $\otimes$ from $H \times H$ to $\rho^*(H)$ the non empty set of subsets of H . A quasi-hyper-group is a hyper-groupoid such that $x \otimes H = H \otimes x = H \quad \forall x \in H$, (the reproduction axiom), where $H \otimes x = \bigcup_{h \in H} h \otimes x$. A semi-hyper-group is a hyper-groupoid $(H, \otimes)$ such that $(x \otimes y) \otimes z = x \otimes (y \otimes z) \quad \forall (x, y, z) \in H^3$. A semi-hyper-group $(H, \otimes)$ is a hyper-group (or also multi-group) if $H \otimes x = x \otimes H = H \quad \forall x \in H$, or equivalently if for all $(a, b) \in H^2$, there exists $(c, d) \in H^2$ such that $b \in c \otimes a$, $b \in a \otimes d$.

The condition $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ can be rephrased as : $\bigcup_{u \in x \otimes y} u \otimes z =$

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$\bigcup_{v \in y \otimes z} x \otimes v$. For more information about hyper-structures one may refer to [4], [2], [1], [3] and [5].

2. PRELIMINARIES AND EARLIER STUDIES

Definition 1. If $(H, \otimes)$ and $(H', *)$ are two hyper-groupoids, the function $\Phi : H \rightarrow H'$ is called a good homomorphism if and only if $\Phi(x \otimes y) = \Phi(x) * \Phi(y)$ for all $(x, y) \in H^2$.

Definition 2. Let $(G, \circ)$ be a hyper-groupoid. The action of $(G, \circ)$ on a non empty set A is a map $\bullet : G \times A \rightarrow \rho^*(A)$ such that for all $(g_1, g_2) \in G^2$, $a \in A$:

- (i) $\bigcup_{t \in g_1 \circ g_2} t \bullet a = \bigcup_{s \in g_2 \bullet a} g_1 \bullet s$
- (ii) There exists $e \in G$ such that $a \in e \bullet a$.

Remark 1. Let $A^{\rho(A)}$ be the set of all functions from A to $\rho(A)$, endowed with the composition operation $\circ$, then $\Phi : G \rightarrow A^{\rho(A)}$ defined by $\Phi(g)(a) = g \bullet a$ is a homomorphism.

Proof. $(\Phi(g_1 \circ g_2))(a) = \bigcup_{t \in g_1 \circ g_2} t \bullet a$. From the above definition one obtains

$$\bigcup_{t \in g_1 \circ g_2} t \bullet a = \bigcup_{s \in g_2 \bullet a} g_1 \bullet s = g_1 \bullet (g_2 \bullet a) = \Phi(g_1)(\Phi(g_2))(a) = (\Phi(g_1) \circ \Phi(g_2))(a).$$

□

The homomorphism $\Phi : G \rightarrow A^{\rho(A)}$ is called a hyper-representation associated with the hyper-groupoid action. This process is reversible in the sense that if $\Phi : G \rightarrow A^{\rho(A)}$ is any good homomorphism, then the map from $G \times A \rightarrow \rho(A)$ defined by $g \bullet a = \Phi(g)(a)$ satisfies the properties of hyper-groupoid action of G on A .

3. HYPER-GEOMETRIC NOTATIONS

Definition 3. A hyper-vector space is a quadraplet $(V, +, \circ, K)$, where $(V, +)$ is abelian group, K is a field and $\circ$ is a mapping (product times a scalar in K). $\circ : K \times V \rightarrow \rho^*(V)$ where $\rho^*(V)$ is the non empty subsets of V , such that the following conditions hold:

- (i) For all $a \in K$ and for all $x, y \in V$, then $a \circ (x + y) \subseteq a \circ x + a \circ y$.
- (ii) For all $a, b \in K$ and for all $x \in V$, then $(a + b) \circ x \subseteq a \circ x + b \circ x$.
- (iii) For all $a, b \in K$ and for all $x \in V$, then $a \circ (b \circ x) = (a \cdot b) \circ x$.
- (iv) For all $a, b \in K$ and for all $x \in V$, then $a \circ (-x) = (-a) \circ x = -(a \circ x)$.

(v) For all $x \in V$, then $x \in 1 \circ x$.

In the right hand side of (i), the sum is meant in the sense of Frobenius, that is

$$a \circ x + a \circ y = \{w + u \mid w \in a \circ x, u \in a \circ x\}.$$

Definition 4. A hyper-algebra V over a field K is a triple $(V, \odot, K)$ where $(V, +, \circ, K)$ is a hyper-vector space over K , and $\odot$ is a hyper-operation $\odot : V \times V \rightarrow \rho^*(V)$ such that the following conditions hold:

- (i) There is no element $e \in V$, such that $x \in x \odot e \cap e \odot x$ for all $x \in V$, i.e e is a right unit element of V .
- (ii) $x \odot (y + z) = x \odot y + x \odot z$ for all $x, y, z \in V$ and $(x + y) \odot z = x \odot z + y \odot z$ for all $x, y, z \in V$, where the right hand side of (ii) are meant the Frobenius.
- (iii) If $t \in K$, then $(t \circ a) \odot b = t \circ (a \odot b) = a \odot (t \circ b)$ for all $t \in K$ and for all $a, b \in V$, where $(t \circ a) \odot b = \bigcup_{w \in t \circ a} w \odot b$, $t \circ (a \odot b) = \bigcup_{u \in a \odot b} t \circ u$ and

$$a \odot (t \circ b) = \bigcup_{s \in t \circ a} a \odot s.$$

Definition 5. A hyper-semifield over a field K , is a hyper-algebra $(V, \odot, K)$ such that V has no zero divisors, i.e if $0 \neq a \in V$ and $b \in V$ such that $0 \in a \odot b$, then $0 \in b \odot e$.

Definition 6. Let $(V, \odot, K)$ and $(W, \oplus, F)$ be two hyper-semifields over the fields K and F respectively, then a good hyper-homomorphism $\sigma : V \rightarrow W$ is a mapping such that $\sigma(x \odot y) = \sigma(x) \oplus \sigma(y)$ for all $x, y \in V$.

Definition 7. If $(V, +, \circ, K)$ is a hyper-vector space over K , then a hyper-linear transformation $\delta : V \rightarrow \rho^*(V)$ is defined by:

- (i) $\delta(v + w) = \delta(v) + \delta(w)$ for all $u, w \in V$.
- (ii) $\delta(t \circ a) = t \circ \delta(a)$ for all $t \in K$ and $a \in V$.

The set of all hyper-linear transformations of V is denoted by $HL(V)$. In particular if $(V, \odot, K)$ is a hyper-semifield over K , then a hyper-linear transformation δ of V is called a hyper-faithful-linear transformation if $0 \in \delta(b)$, $b \in V$ then $0 \in e \odot b$, i.e e is the unite element of $(V, \odot, K)$.

Proposition 1. Let $(V, \odot, K)$ be a hyper-semifield over a field K . For $a \in V$, let the hyper-mappings $\alpha_a : V \rightarrow \rho^*(V)$ and $\alpha_a^* : V \rightarrow \rho^*(V)$ defined by $\alpha_a(v) = a \odot v$ and $\alpha_a^*(v) = v \odot a$ respectively, where $v \in V$, are hyper-faithful-linear transformations.

Proof. Let $a \in V$, $a \neq 0$, then

1. $\alpha(x + y) = a \odot (x + y) = a \odot x + a \odot y = \alpha_a(x) + \alpha_a(y)$. (Use (ii) of Definition 4)
2. $\alpha_a(t \circ x) = a \odot (t \circ x) = t \circ (a \odot x) = t \circ (\alpha_a(x))$ for all $t \in K$ and $x \in V$ (use (iii) of Definition 4). If $0 \in \alpha_a(b)$, this implies that $0 \in a \odot b$. As $a \neq 0$, then from the non-zero divisors property, it follows that $0 \in b \odot e$, which means that α_a is hyper-faithful linear transformation.

□

Definition 8. Let $(V, +, \odot, K)$ be a hyper-vector space over K , and let $\Sigma = \{\alpha_a \mid 0 \neq a \in V\} \subseteq HL^*(V)$ the set of all hyper-faithful-linear transformations of V over K such that:

- (i) $t \cdot \alpha_S \subseteq \Sigma$ where $(t \cdot \alpha_S)(x) = t \circ (\alpha_S(x))$ for all $S, x \in V$ and for all $t \in K$.
- (ii) If $\alpha_S, \alpha_T \in \Sigma$, then $\alpha_S + \alpha_T \in \Sigma$ where $(\alpha_S + \alpha_T)(x) = \alpha_S(x) + \alpha_T(x)$, for all $x \in V$.
- (iii) There exists $I \in \Sigma$ such that $x \in I(x)$, for all $x \in V$.
- (iv) There is a fixed element $e \in V$ and for all non-zero $x \in V$, there exists a unique $T \in \Sigma$ such that $x \in T(e)$.
- (v) For all $t \in K$ and for $a, b \in V$, the following condition must hold:

$$\bigcup_{z \in t \circ a} \alpha_z(b) = \bigcup_{w \in \alpha_a(b)} t \circ w = \bigcup_{u \in t \circ b} \alpha_a(u)$$

Then $(\Sigma, \cdot, +)$ is called a hyper-spread set over K .

Theorem 1. If $(V, \odot, K)$ is a hyper-semifield over K , then the hyper-regular representation $\xi : a \rightarrow \alpha_a, 0 \neq a \in V$ affords a hyper-spread set Σ over K .

Proof. Let $\Sigma = \{\alpha_a \mid 0 \neq a \in V\}$, where $\alpha_a : V \rightarrow \rho^*(V)$ defined by $\alpha_a(x) = a \odot x$ is a hyper-faithful-linear transformation, by Proposition 1

- (i) If $\alpha_a, \alpha_b \in \Sigma$, then $(\alpha_a + \alpha_b)(x) = \alpha_a(x) + \alpha_b(x) = a \odot x + b \odot x = (a + b) \odot x = \alpha_{a+b}(x)$. This implies that $\alpha_a + \alpha_b \in \Sigma$.
- (ii) $(t \cdot \alpha_a)(x) = t \circ (\alpha_a(x)) = t \circ (a \odot x) = (t \circ a) \odot x = \alpha_{t \circ a}(x) = \alpha_w(x)$, where $w \in t \circ a$, $t \in K$. This implies that $t \cdot \alpha_a \subseteq \Sigma$, for all $t \in K$.
- (iii) $I = \alpha_e \in \Sigma$ for: $\alpha_e(x) = e \odot x$ and as $x \in e \odot x$ for all $x \in V$, then it follows that $\alpha_e(x) = I(x)$.
- (iv) As $x \in x \odot e$ for all $x \in V$, it follows that there exists a unique α_x such that $x \in \alpha_x(e)$. Setting $\alpha_x = T$, thus, there is a unique $T \in \Sigma$ such that $x \in T(e)$, $x \in V$.

Hence the claim follows.

□

Theorem 2. Let Σ be a hyper-spread set over a field K , then Σ affords a hyper-semifield over K .

Proof. As $\Sigma \subseteq HL(V)$ is a hyper-spread set, then pick some vector $0 \neq e \in V$. By ((iv) of Definition 8), and for any $x \in V$, there exists a unique $T \in \Sigma$ (can be called T_x) such that $x \in T_x(e)$. Hence we can define a hyper-multiplication operation $\odot$ on V as follows: $x \odot y = T_x(y)$, where $x \in T_x(e)$ for all $x, y \in V$. This turns V into a hyper-semifield over K for: $x \odot (y + z) = T_x(y + z) = T_x(y) + T_x(z) = x \odot y + x \odot z$.

As $x + y \in V$, then there exists a unique $T_{x+y} \in \Sigma$ such that $(x + y) \in (x + y) \odot e = T_{(x+y)}(e)$.

Arguing similarly $x + y \in x \odot e + y \odot e = T_x(e) + T_y(e)$. From the uniqueness property of T_{x+y}, T_x and T_y , it follows that $T_x + T_y = T_{x+y}$ and hence

$$\begin{aligned} (T_{x+y})(z) &= (x + y) \odot z = (T_x + T_y)(z) \\ &= T_x(z) + T_y(z) = x \odot z + y \odot z, \quad \forall x, y, z \in V \end{aligned}$$

Similarly one can show that $x \odot (y + z) = x \odot y + x \odot z$ for all $x, y, z \in V$.

From ((iv) Definition 8), there is $e \in V$ such that there exists a unique $T \in \Sigma$ such that $x \in T(e) = T_x(e) = x \odot e$. This implies that this e is the unit element of V .

From ((v) Definition 8), it follows that:

$$\begin{aligned} (t \circ a) \odot x &= T_{t \circ a}(x) = \bigcup_{z \in t \circ a} T_z(x) = \bigcup_{w \in \alpha_a} t \cdot w \\ &= t \circ (\alpha_a(b)) = \bigcup_{u \in t \circ b} T_a(u) = T_a(t \circ b) \end{aligned}$$

for all $t \in K$ and for all $a, b \in V$.

The property of has no zero-divisors is an immediate consequence from the faithful condition of elements of Σ .

This completes the proof of the theorem. $\square$

Definition 9. Let $(V, +, \circ, K)$ be a hyper-vector space over K . Then $H \subseteq V$ is a hyper-subspace of V if $(H, +, \circ, K)$ is a hyper-vector space over K .

Definition 10. Let $(V, +, \circ, K)$ be a hyper-vector space, a hyper-spread over K is a collection $\mathfrak{K}$ of hyper-subspaces of V such that:

1. If $X, Y \in \mathfrak{K}$ and $X \neq Y$, then $0 \in X \cap Y$.
2. $\bigcup_{X \in \mathfrak{K}} X = V$. The elements of $\mathfrak{K}$ are called components.

Lemma 1. Let $(V, +, \circ, K)$ be a hyper-vector space, then $A = V \times V$ can be turned into a hyper-vector space as follows:

Define the hyper-operation $\oplus : V \times V \rightarrow \rho^*(V)$

$$(a, b) \oplus (c, d) = (a \oplus c, b + d), \quad \forall a, b, c, d \in V,$$

and define the hyper-operation $\hat{\odot} : K \times (V \times V) \rightarrow \rho^*(V)$

$$t\hat{\odot}(a, b) = (t \circ a, t \circ b), \quad \forall a, b \in V, \quad t \in K.$$

The proof of $(V \times V, \oplus, \hat{\odot}, K)$ is a hyper-vector space is easy and can be omitted.

Lemma 2. *Let $(V, \odot, K)$ be a hyper-semifield, then the subsets $L_\infty = \{(x \oplus e, 0) \mid x \in V\}$ and $R_\infty = \{(0, y \oplus e) \mid y \in V\}$ of $(V \times V, \oplus, \hat{\odot}, K)$ are hyper-semifields.*

Proof. Define the hyper-operation $\hat{\odot} : L_\infty \times L_\infty \rightarrow \rho^*(L_\infty)$ by

$$(x \odot e, 0) \hat{\odot} (y \odot e, 0) = ((x \odot y) \odot e, 0), \quad \forall x, y \in V,$$

where $\rho^*(L_\infty)$ is the set of non-empty subsets of L_∞ .

Also, the hyper-operations $\hat{\odot}, \oplus$ on $K \times L_\infty$ and $L_\infty \times L_\infty$ respectively by: $t\hat{\odot}(x \odot e, 0) = ((t \circ x) \odot e, 0)$ and $(x \odot e, 0) \oplus (y \odot e, 0) = ((x + y) \odot e, 0)$ for all $x, y \in V$ and for all $t \in K$.

(i) The hyper-operation $\hat{\odot}$ is distributive with respect to $\oplus$ in $L_\infty \times L_\infty$ for

$$\begin{aligned} (x \odot e, 0) \hat{\odot} ((y \odot e, 0) \oplus (z \odot e, 0)) &= (x \odot e, 0) \hat{\odot} ((y \oplus z) \odot e, 0) \\ &= ((x \odot (y \oplus z)) \odot e, 0) \\ &= ((x \odot y) \oplus (x \odot z)) \odot e, 0 \\ &= (x \odot y) \odot e, 0 \oplus (x \odot z) \odot e, 0 \\ &= (x \odot e, 0) \hat{\odot} (y \odot e, 0) \\ &\quad \oplus (x \odot e, 0) \hat{\odot} (z \odot e, 0) \end{aligned}$$

Also, one can show that:

$$(x \odot e, 0) \oplus ((y \odot e, 0) \hat{\odot} (z \odot e, 0)) = (x \odot e, 0) \hat{\odot} (z \odot e, 0) \oplus (y \odot e, 0) \hat{\odot} (z \odot e, 0)$$

(ii) If $t \in K$, then

$$\begin{aligned} t\hat{\odot}((x \odot e, 0) \hat{\odot} (y \odot e, 0)) &= t\hat{\odot}((x \odot y) \odot e, 0) \\ &= ((t \circ (x \odot y)) \odot e, 0) \\ &= ((x \odot (t \circ y)) \odot e, 0) \\ &= (x \odot e) \hat{\odot} ((t \circ y) \odot e, 0) \\ &= (x \odot e, 0) \hat{\odot} ((t \circ y) \odot e, 0) \\ &= (x \odot e, 0) \odot (t\hat{\odot}(y \odot e, 0)) \end{aligned}$$

Also, as

$$\begin{aligned} t\hat{\odot}((x \odot y) \odot e, 0) &= t \circ ((x \odot y) \odot e, 0) \\ &= (t \circ (x \odot y) \odot e, 0) \\ &= (((t \circ x) \odot y) \odot e, 0) \\ &= ((t \circ x) \odot e, 0) \hat{\odot} (y \odot e, 0) \\ &= (t\hat{\odot}(x \odot e, 0) \hat{\odot} (y \odot e, 0)) \end{aligned}$$

- (iii) $(e \odot e, 0)$ is the unit element of $L_\infty \times L_\infty$, for $(x \odot e, 0) \hat{\odot} (e \odot e, 0) = ((x \odot e) \odot e, 0)$ for all $(x \odot e, 0)$ in $L_\infty \times L_\infty$, and as $x \in x \odot e \cap e \odot x$ for all $x \in V$, then $x \odot e \subseteq \bigcup_{w \in x \odot e} w \odot e = (x \odot e) \odot e$. This implies that

$$(x \odot e, 0) \subseteq ((x \odot e) \odot e, 0) = (x \odot e, 0) \hat{\odot} (e \odot e, 0).$$

Following a similar argument one can show that $(x \odot e, 0) \subseteq (e \odot e) \hat{\odot} (x \odot e, 0)$

- (iv) Assume that $0 \in (x \odot e, 0) \hat{\odot} (y \odot e, 0)$, where $0 \notin (x \odot e, 0)$, then $0 \in (x \odot y) \odot e, 0)$. This implies that $0 \in (x \odot y) \odot e$. Since $0 \notin e \odot e$, $0 \in (x \odot y)$. But $0 \notin x \cdot e$, this implies that $0 \in y \odot e$ and thus $0 \in (y \odot e, 0)$. This completes the proof that L_∞ is a hyper-semifield. A similar argument can be followed to prove that R_∞ is a hyper-semifield.

□

Lemma 3. Let $(V, \odot, \hat{\odot}, \oplus, K)$ be a hyper-semifield, then $(L_\infty \hat{\times} R_\infty, \hat{\oplus}, K)$ is a hyper-vector space, where $L_\infty \hat{\times} R_\infty = \{(\overline{x \odot e}, \overline{y \odot e}) \mid x, y \in V\}$,

$$(\overline{x \odot e}, \overline{y \odot e}) \hat{\oplus} (\overline{a \odot e}, \overline{b \odot e}) = (((x \odot e), 0) \oplus (a \odot e, 0), (0, (y \odot e) \oplus (0, b \odot e)))$$

and

$$t\hat{\odot}(\overline{x \odot e}, \overline{y \odot b}) = ((t\hat{\odot}(x \odot e), 0), t\hat{\odot}(0, (y \odot e)))$$

for all $x, y, a, b \in V$, $\overline{x \odot e} = (x \odot e, 0)$.

Proof. The proof is an immediate consequence of Lemma 1 and Lemma 2. □

Theorem 3. Let $(V, \odot, K)$ be a hyper-semifield, then $(L_\infty \hat{\times} R_\infty, \wedge, K)$ is a hyper-semifield, where

$$(\overline{x \odot e}, \overline{y \odot e}) \wedge (\overline{a \odot e}, \overline{b \odot e}) = (((x \odot e), 0) \hat{\odot} (a \odot e, 0), (0, y \odot e) \hat{\odot} (0, b \odot e))$$

, where $\overline{x \odot e} = (x \odot e, 0) \forall x \in V$.

$$\begin{aligned} \text{Proof. } 1. & (\overline{x \odot y}, \overline{y \odot e}) \wedge ((\overline{a \odot e}, \overline{b \odot e}) \hat{\oplus} (\overline{c \odot e}, \overline{d \odot e})) = \\ & (x \odot e, y \odot e) \wedge ((a \odot e, 0) \oplus (c \odot e, 0), (0, b \odot e) \oplus (0, d \odot e)) = \\ & (x \odot e, y \odot e) \wedge (a + c) \odot e, (b + d) \odot e) = \\ & (((x \odot e, 0) \hat{\odot} (a \odot e, 0) \oplus (x \odot e, 0) \hat{\odot} (c \odot e, 0), (0, y \odot e) \hat{\odot} (0, b \odot e) \\ & \oplus (0, y \odot e) \hat{\odot} (0, d \odot e))) \\ & = (\overline{x \odot e}, \overline{y \odot e}) \wedge (\overline{a \odot e}, c \odot e) \hat{\oplus} (\overline{x \odot e}, \overline{y \odot e}) \wedge (\overline{c \odot e}, \overline{d \odot e}) \end{aligned}$$

Arguing in a similar manner one can show that

$$\begin{aligned} & ((\overline{x \odot e}, \overline{y \odot e}) \hat{\oplus} (\overline{a \odot e}, \overline{b \odot e})) \wedge (\overline{c \odot e}, \overline{d \odot e}) = \\ & ((\overline{x \odot e}, \overline{y \odot e}) \wedge (\overline{c \odot e}, \overline{d \odot e}) \hat{\oplus} (\overline{a \odot e}, \overline{b \odot e}) \wedge (\overline{c \odot e}, \overline{d \odot e})) \end{aligned}$$

From this it follows that $\wedge$ is distributive with respect to $\hat{\oplus}$.

2. $t\bar{\odot}((\overline{x\odot e}, \overline{y\odot e}) \wedge (\overline{a\odot e}, \overline{c\odot e})) =$
 $(t\hat{\odot}((x\odot e, 0)\hat{\odot}(a\odot e, 0)), t\hat{\odot}((0, y\odot e)\hat{\odot}(0, b\odot e)))$.
 Then (2) follows from ((ii) of Lemma 2).
3. $(\overline{e\odot e}, \overline{e\odot e})$ is the unit element of $L_\infty \hat{\times} R_\infty$. The proof directly follows from ((iii) of Lemma 2).
4. Assume that $0 \in ((\overline{x\odot e}, \overline{y\odot e}) \wedge (\overline{a\odot e}, \overline{b\odot e}))$ such that $0 \notin ((x\odot e, 0)\hat{\odot}(a\odot e, 0), (0, y\odot e)\hat{\odot}(0, b\odot e))$. From ((iv) of Lemma 2), $0 \in (a\odot e, 0)$ and $0 \in (0, b\odot e)$ which means that $0 \in (a\odot e, 0)\hat{\odot}(0, b\odot e)$ and hence $0 \in (\overline{a\odot e}, \overline{b\odot e})$.

□

Remark 2. Let $(V, \odot, K)$ be a hyper-semifield, then the good hyper-homomorphism $\sigma : V \rightarrow \rho^*(V)$, keeps $L_\infty \hat{\times} R_\infty$ invariant if $e^\sigma = \{e\}$.

Proof. Let $(\overline{x\odot e}, \overline{y\odot e}) \in L_\infty \hat{\times} R_\infty$ then
 $(\overline{x\odot e}, \overline{y\odot e})^\sigma = ((\overline{x\odot e})^\sigma, (\overline{y\odot e})^\sigma) = (((x\odot e)^\sigma, 0), (0, (y\odot e)^\sigma))$
 $((x^\sigma \odot y^\sigma), 0), (0, (y^\sigma \odot e^\sigma)) = ((z\odot e, 0), (0, (w\odot e))) = (\overline{z\odot e}, \overline{w\odot e}) \in L_\infty \hat{\times} R_\infty$,
 where $z, w \in V$. □

Definition 11. Let $(V, \odot, K)$ be a hyper-semifield. A left hyper-nucleus L_V of V is defined by

$$L_V = \{a \in V \mid a \odot (x \odot y) = (a \odot x) \odot y, \quad \forall x, y \in V\}$$

Theorem 4. $K \circ e \subseteq L_V$ if and only if $e \in L_V$.

Proof. Let $e \in L_V$ and consider

$$(t \circ e) \odot (x \odot y) = t \circ (e \odot (x \odot y)) = t \circ ((e \odot x) \odot y) = (t \circ (e \odot x)) \odot y = ((t \circ e) \odot x) \odot y$$

(use (iii) of Definition 4), where $t \in K$, $x, y \in V$.

From this it follows that $K \circ e \subseteq L_V$.

Conversely: Let $K \circ e \subseteq L_V$, this implies that

$$(t \circ e) \odot (x \odot y) = ((t \circ e) \odot x) \odot y$$

for all $t \in K$ and all $x, y \in V$. From this one has:

$$t \circ (e \odot (x \odot y)) = (t \circ (e \odot x)) \odot y = t \circ (e \odot x) \odot y$$

which means that

$$\bigcup_{w \in e \odot (x \odot y)} t \circ w = \bigcup_{z \in (e \odot x) \odot y} t \circ z.$$

Hence $e \in L_V$. □

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