International Mathematical Forum, Vol. 9, 2014, no. 13, 579 - 592
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.4236

Coextending Modules

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Abstract

Throughout this paper we introduce the notion of coextending module as a dual of the class of extending modules. Various properties of this class of modules are given, and some relationships between these modules and other related modules are introduced.

Keywords: Coextending modules, Extending modules, Essential submodules, Coessential submodules, Closed submodules, Coclosed submodules

1 Introduction

Let R be a commutative ring with unity and let M be a unitary left R-module. A submodule N of M is said to be essential in M (denoted by $N \leq_e M$), if for any submodule K of M, $N \cap K = 0$ implies that $K = 0$ [11]. A submodule N of M is said to be closed in M (denoted by $N <_c M$)if N has no proper essential extension in M; that is if $N \leq_e W < M$ then N=W [11].A submodule

N of M is said to be small in M (denoted by $N \ll M$), if whenever $K \leq M$, $N + M = M$, then $N = M$. A submodule N of M is said to be coclosed in M (denoted by $N <_{cc} M$),if whenever $\frac{N}{K} \ll \frac{M}{K}$ then $K = N$ for each submodule K of N [10]. An R-module M is called extending (or CS-module), if every submodule of M is essential in a direct summand [6]. Equivalently, M is extending if and only if every closed submodule of M is a direct summand [6].

In this paper we introduce a new concept (up to our knowledge)namely coextending module as a dual of the class of extending modules, where M is called coextending (briefly CCS-module),if every coclosed submodule of M is a direct summand of M.

This research consists of three sections. In $S2$ some basic properties and examples of coextending modules are given. In $S3$ we show by example that the direct sum of coextending module may not be coextending (see Ex(3.1). However, we give certain conditions under which the direct sum of coextending modules be coextending module (see Th (3.4) and Th (3.5)). In $S4$ we investigate some relationships between coextending modules and other related modules such as lifting, semisimple modules,discrete, quasi-discrete and UCC-modules.

2 Some Basic Properties

In this section we introduce the concept of Coextending modules. We investigate the basic properties of this type of modules.

Definition 2.1. *An R-module M is called coextending (or CCS-module), if every coclosed submodule of M is a direct summand of M.*

Remarks and Examples 2.2.

1. *It is clear that every hollow module is CCS-module, where an R-module M is called hollow if every proper submodule of M is small [7].*

   proof. *Let M be a hollow module and let N be a coclosed submodule of M, then $N \ll M$, and so for each submodule K of N, $\frac{N}{K} \ll \frac{M}{K}$. This implies that $\frac{N}{0} \ll \frac{M}{0}$. But N is a coclosed submodule of M, thus N=(0), which is a direct summand of M.*

2. *coextending module may not be hollow module. For example, Z as Z-module is coextending module, but not hollow module.*

3. *It is clear that every semisimple module is CCS-module, but the converse is not true for examples: Z as Z module is CCS-module but not semisimple, also Z_{12} is CCS-module but not semisimple.*

4. *Every local module (module which has only maximal submodule), is a CCS-module.*

5. *Every uniserial module is a CCS-module. In particular Z_{P^∞} as Z-module is CCS-module.*

6. *By a direct computation, we can see that $M = Z_2 \oplus Z_4$ is a CCS-module.*

7. *If a module M is isomorphic to a module $\acute{M}$, then M is a CCS-module if and only if $\acute{M}$ is a CCS-module.*

8. *Every couniform module is CCS-module, where an R-module M is called couniform , if every proper submodule N of M is either (0) or there exists a proper submodule N_1 of N such that $\frac{N}{N_1} \ll \frac{M}{N_1}$ [12].*

   proof. *Let N be a submodule of M. If $N = (0)$ then it is clear that N is a coclosed direct summand of M and we are done. If $N \neq (0)$. Since M is a couniform module so there exists a proper N_1 of N such that $\frac{N}{N_1} \ll \frac{M}{N_1}$. Hence N is not coclosed in M. Thus (0) is the only (proper) coclosed submodule of M, and so M is CCS-module.*

   However, a CCS-module need not be a couniform module. In fact the Z-module Z_6 is a CCS-module because it is a semisimple module. On the other hand Z_6 is not a couniform module, see ([12], Rem (1.2)(2)).

   Note that every Artinian couniform module is a hollow module [12], hence it is a CCS-module.

The following results give some important properties of the coextending modules.

Proposition 2.3. *A direct summand of coextending module is a coextending module.*

proof. Let M be an R-module, and let N be a direct summand of M, Let K be a coclosed submodule of N. Since N is a direct summand of M, so N is a coclosed submodule of M. It follows that K is a coclosed submodule of M, hence K is a direct summand of M, that is $M = K \oplus L$ for some submodule L of M. $N = M \cap N = (K \oplus L) \cap N = K \oplus (L \cap N)$ by modular law. Thus K is a direct summand of N, i.e N is a CCS-module.

Corollary 2.4. *If M is a CCS-module and N is a coclosed submodule of M, then $\frac{M}{N}$ is a CCS-module.*

proof. Since M is a CCS-module and N is a coclosed submodule of M, then N is a direct summand of M, i.e $M = N \oplus W$ for some submodule W of M. Hence $(\frac{M}{N}) \cong W$. But W is a direct summand of M, W is a CCS-module. Then $\frac{M}{N}$ is a CCS-module.

Corollary 2.5. *Let $f : M \to \acute{M}$ be an epimorphism from an R-module M to a projective R-module $\acute{M}$. If M is CCS-module, then $\acute{M}$ is CCS-module.*

proof. Consider the following short exact sequence:

$$(0) \to kerf \xrightarrow{i} M \xrightarrow{f} \acute{M} \to (0)$$

where i is the inclusion homomorphism. But M is a projective homomorphism, so the sequence splits. Therefore $M \cong (kerf \oplus \acute{M})$, that is M is an isomorphic to a direct summand of M. By Prop (2.3), $\acute{M}$ is a CCS-module.

Remark 2.6. *Let M be an R-module and let $\bar{R} = (\frac{R}{annM})$. Then M is a CCS-module if and only if M is a CCS-module as $\bar{R}$-module.*

proof. It is clear.

Recall that a submodule L of an R-module M is called coessential of N in M (denoted by $L \leq_{ce} N$ in M), if $\frac{N}{L} \ll \frac{M}{L}$ [8].

Remark 2.7. *If every submodule of M is a coessential in a direct summand of M,then M is a CCS-module.*

Proof. Let L be a coclosed submodule of M. By hypothesis there exists a direct summand N of M such that $L \leq_{ce} N$, such that $\frac{N}{L} \ll \frac{M}{L}$. But L is a coclosed in M, therefore $L = N$, and hence L is a direct summand of M.

The following theorem gives the hereditary property for the CCS-module.Before that, an R-module M is called multiplication if for each submodule N of M there exists an ideal I of R such that $N = IM$ [4].

Theorem 2.8. *Let M be a finitely generated faithful multiplication R-module. Then R is a CCS-ring if and only if M is a CCS-R-module.*

Proof. $\Rightarrow$) Let N be a coclosed submodule of M. Since M is a multiplication R-module, then $N = IM$ for some ideal I of R. It is easy to see that I is a coclosed in R. Hence I is a direct summand of R, and so $R = I \oplus J$ for some ideal J of R. It follows that $M = IM \oplus JM = N \oplus JM$. i.e N is a direct summand of M.

$\Leftarrow$) Let I be a coclosed ideal of R. Put $N = IM$, N is a coclosed submodule of M. But M is a CCS-module, so N is a direct summand of M, that is there exist a submodule W of M such that $N \oplus W = M$. But $W = JM$ for some ideal J of R. Now $IM \oplus JM = M$ implies that $(I + J)M = RM$. Since $M$is a finitely generated faithful multiplication module, then $I + J = R$, i.e I is a direct summand of R.

The following proposition gives a necessary and sufficient condition on a free module to be a CCS-module.

Proposition 2.9. *Let R be a ring, then every free R-module is a CCS-module if and only if every free projective R-module is a CCS-module.*

Proof. $\Rightarrow$) Let M be a projective R-module. M is an epimorphic image of a free R-module say F [13]. By the hypothesis , F is a CCS-module, and by Cor (2.5), M is a CCS-module.

$\Leftarrow$) It is a clear.

Corollary 2.10. *Let R be a ring, then every finitely generated free R-module is a CCS-module if and only if every finitely generated projective R-module is a CCS-module.*

3 Direct Sum of Coextending Modules

In this section we study when the direct sum of coextending module is coextending.In fact this is not true in general as the following example shows.

Example 3.1. *We saw in Rem and Ex (2.2), that both of Z and Z_{p^∞} are CCS-module, but we can easily see that $Z \oplus Z_{p^\infty}$ is not.*

We study some cases in which the direct sum of CCS-module be a CCS-module. Before that we need the following lemmas.

Lemma 3.2. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 be two R-modules, and let $K = K_1 \oplus K_2$, where $K_1 \leq M_1$ and $K_2 \leq M_2$. If K is a coclosed submodule of M, then K_1 is a coclosed submodule of M_1 and K_2 is a coclosed submodule of M_2.*

Proof. Assume that $(\frac{K_1}{W_1}) \ll (\frac{M_1}{W_1})$

and

$(\frac{K_2}{W_2}) \ll (\frac{M_2}{W_2})$ where $W_1 \leq K_1$ and $W_2 \leq K_2$. Hence:

$$\frac{K1}{W_1} \oplus \frac{K2}{W_2} \ll \frac{M_1}{W_1} \oplus \frac{M_2}{W_2}$$

Therefore:

$$\frac{K_1 \oplus K_2}{W_1 \oplus W_2} \ll \frac{M_1 \oplus M_2}{W_1 \oplus W_2}$$

That is:

$$\frac{K}{W_1 \oplus W_2} \ll \frac{M}{W_1 \oplus W_2}$$

Since K is a coclosed submodule of M, then $K = W_1 \oplus W_2$; that is $K_1 \oplus K_2 = W_1 \oplus W_2$. Hence $K_1 = W_1$ and $K_2 = W_2$. Thus K_1 and K_2 are coclosed submodules in M_1 and M_2 respectively.

Lemma 3.3. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 be R-modules and let $ann_R M_1 + ann_R M_1 = R$. Then a submodule K is coclosed in M if and only if there exist coclosed submodules K_1 of M_1 and K_2 of M_2 such that $K = K_1 \oplus K_2$.*

Proof. $\Rightarrow$) Since $K \leq M$ and $M = M_1 \oplus M_2$, $ann_R M_2 + ann_R M_2$, so by ([1], Prop(4.2)) there exists submodules K_1 and K_2 of M_1 and M_2 respectively such that $K = K_1 \oplus K_2$, and by lemma(3.2) both of K_1 and K_2 are coclosed submodules in M_1 and M_2 respectively.
$\Leftarrow$) In order to prove that K is a coclosed submodule of M, assume that $\frac{K}{W} \ll \frac{M}{W}$ where W is a submodule of M. Since $ann_R M_1 + ann_R M_2 = R$, so by the same proof of ([1], Prop(4.2)), $W = W_1 \oplus W_2$ for some submodules W_1 and W_2 of M_1 and M_2 respectively. Thus:

$$(\frac{K}{W}) = (\frac{K_1 \oplus K_2}{W_1 \oplus W_2}) \ll \frac{M_1 \oplus M_2}{W_1 \oplus W_2}$$

Hence by [11] :

$$((\frac{K_1}{W_1}) \oplus (\frac{K_2}{W_2})) \ll ((\frac{M_1}{W_1}) \oplus (\frac{M_2}{W_2}))$$

So by [3]:

$$(\frac{K_1}{W_1}) \ll (\frac{K_2}{W_2}) \quad \text{and} \quad (\frac{M_1}{W_1}) \ll (\frac{M_2}{W_2})$$

Since K_1 and K_2 are coclosed submodule of M_1 and M_2 respectively, thus $K_1 = W_1$ and $K_2 = W_2$, and hence $K = W_1 \oplus W_2 = W$.

In the following theorems we put certain conditions under which the direct sum of two CCS-modules is CCS-modules.

Theorem 3.4. *Let $M = M_1 \oplus M_2$ Where M_1 and M_2 be R-modules.If $ann_R M_1 + ann_R M_2 = R$, then M is a CCS-module if and only if both of M_1 and M_2 are CCS-modules.*

Proof. $\Rightarrow$) It follows from Prop(2.3).
$\Leftarrow$) Let K a coclosed submodule of M. By Lemma(3.3), $K = K_1 \oplus K_2$ for some coclosed submodules K_1 and K_2 of M_1 and M_2 respectively. But M_1 and M_2 are CCS-modules, so K_1 is direct summand of M_1 and K_2 is a direct summand of M_2; that is $K_1 \oplus W_1 = M_1$ and $K_2 \oplus W_2 = M_2$, for some $W_1 \leq M_1$ and $W_2 \leq M_2$. Hence:

$$K \oplus (W_1 \oplus W_2) = (K_1 \oplus K_2) \oplus (W_1 \oplus W_2)$$

$$= (K_1 \oplus W_1) \oplus (K_2 \oplus W_2)$$

$$M_1 \oplus M_2 = M$$

Therefore K is a is a direct summand of M, and hence M is a CCS-module.

Theorem 3.5. *Let $M = \bigoplus_{i=1}^{n} M_i$, where each of M_i is an R-module for each $i = 1, ..., n$. If every submodule of M is a fully invariant, then M is a CCS-module if and only if each M_i is a CCS-module for each $i = 1, ..., n$.*

Proof. $\Rightarrow$)It follows from Prop(2.3).
$\Leftarrow$) Let N be a coclosed submodule of M. By assumption N is a fully invariant submodule of M, so $N = \bigoplus_{i=1}^{n}(N \cap M_i)$. On the other hand N is a coclosed submodule of M, so by lemma(3.3), for each $i = 1, ..., n$, $N \cap M_i$ is a coclosed submodule of M_i. Hence $N \cap M_i$ is a direct summand of M_i for each $i = 1, ..., n$, since M_i is a CCS-module, for each $i = 1, ..., n$, thus $(N \cap Mi) \oplus B_i = M_i$ for some submodule B_i of M_i.Terefore:

$$\bigoplus_{i=1}^{n} M_i = \bigoplus_{i=1}^{n}\{(N \cap M_i) \oplus B_i\}$$

$$= \{\bigoplus_{i=1}^{n}(N \cap M_i)\} \oplus \{\bigoplus_{i=1}^{n} B_i\}$$

So $M = N \oplus B$, where $B = \bigoplus_{i=1}^{n} B_i$, that is M is a CCS-module.

4 Coextending Modules and other related concepts

In this section we give some relationships between CCS-modules and some other modules such as lifting, semisimple, quasi-discrete and UCC-modules.

Recall that an R-module M is called lifting, if for every submodule N of M there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$ [15]. Hence we have the following.

Proposition 4.1. *If M is a lifting R-module, then M is a CCS-module.*

Proof. Let N be a coclosed submodule of M. Since M is a lifting module, so there exists a direct summand K of M, $K \leq N$ such that $\frac{N}{K} \ll \frac{M}{K}$ [14]. But N is a coclosed submodule in M, then $N = K$. That is M is a CCS-module.

The converse of Prop(4.1) is not true in general, for example Z as Z-module is a CCS-module, but it is not lifting. However by using Th.(2.2.3) in [3], we get the following theorem, but first recall that an R-module M is called amply supplemented module, if for any two submodule U and V of M with $U + V = M$, then V contains a supplement of U in M [17].

Theorem 4.2. *Let M be an R-module, then M is a lifting module if and only if M is a CCS-module and amply supplemented.*

In the following result we put condition under which the CCS-module can be lifting module.

Proposition 4.3. *Let M be an R-module. If every submodule of M is a coclosed, then the following statements are equivalent.*

1. *M is a lifting module.*
2. *M is a CCS-module.*
3. *M is a semisimple module.*

Proof.
(1)⇒(2): It follows from Prop (4.1).
(2)⇒(3): It is clear.
(3)⇒(1): It is clear.

Recall that an R-module M is called discret, if M is lifting module and for any submodule $N \leq M$ with $\frac{M}{N}$ is isomorphic to a direct summand of M, implies that N is a summand of M. And M is called a quasi-discret if M is a lifting module and whenever M_1 and M_2 are summand of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M [15].

Corollary 4.4. *Let M be an R-module such that every submodule of M is coclosed, then the following statements are equivalent.*

1. *M is a lifting module.*
2. *M is a discret module.*
3. *M is a quasi-discret module.*

4. M is a CCS-module.

5. M is a semisimple module.

Proof.
(1)⇔(4)⇔(5): It follows from Prop (4.1).

(1)⇔(2)⇔(3): By ([3], Prop(2.3.8)).

Recall that a submodule N of an R-module M has coclosure or (S-closure), if there exists a coclosed submodule H of N such that $\frac{N}{H} \ll \frac{M}{H}$ [14]. So we can give the following.

Proposition 4.5. *If an R-module M is a CCS-module such that every submodule N of M has a coclosure, then M is a lifting module.*

Proof. Let N be a submodule of M. By assumption, there exists a coclosed submodule H of N such that $\frac{N}{H} \ll \frac{M}{H}$. But M is a CCS-module, therefore H is a direct summand of M. Thus M is a lifting module [14].

Recall that an R-module M is called UCC-module, if every submodule of M has a unique coclosure in M ([5], P.261). Hence we have the following.

"If an R-module M is a CCS-module and UCC-module, then M is a lifting module".

J.Abuhlial in [2] introduced the class of the completely distributive module, where an R-module M is called completely distributive, if for each a collection N_i of submodules of M and for each submodule N of M:

$$N + \bigcap_{i\in\Lambda} N_i = \bigcap_{i\in\Lambda}(N + N_i)$$

In order to give the main result in this section we need the following lemmas.

Lemma 4.6. *Let M be a completely distributive module, and let N be a submodule of M. If A_i be a collection of submodules of M such that $A_i \leq_{ce} N$ of M, then $\bigcap_{i\in\Lambda} A_i \leq_{ce} N$ of M.*

Proof. We want to prove that $\bigcap_{i\in\Lambda} A_i \leq_{ce} N$ of M, that is:

$$\frac{N}{\bigcap_{i\in\Lambda} A_i} \ll \frac{M}{\bigcap_{i\in\Lambda} A_i}$$

Suppose that:

$$\frac{N}{\bigcap_{i\in\Lambda} A_i} + \frac{B}{\bigcap_{i\in\Lambda} A_i} = \frac{M}{\bigcap_{i\in\Lambda} A_i}$$

Then $N+B=M$, which implies that $\frac{N+B}{A_i}=\frac{M}{A_i}$ for each $i\in\Lambda$, hence

$$\frac{N}{A_i}+\frac{B+A_i}{A_i}=\frac{M}{A_i}$$

for each $i\in\Lambda$. Since $\frac{N}{A_i}\ll\frac{M}{A_i}$ for each $i\in\Lambda$, then $B+A_i=M$, hence

$$M=\bigcap_{i\in\Lambda}(B+A_i)$$

Since M is a completely distributive, therefore $M=B+(\bigcap_{i\in\Lambda}A_i)$. But $\bigcap_{i\in\Lambda}A_i\subseteq B$, thus $M=B$, hence

$$\frac{B}{\bigcap_{i\in\Lambda}A_i}=\frac{M}{\bigcap_{i\in\Lambda}A_i}$$

That is:

$$\frac{N}{\bigcap_{i\in\Lambda}A_i}\ll\frac{M}{\bigcap_{i\in\Lambda}A_i}$$

i.e $\bigcap_{i\in\Lambda}A_i\leq_{ce}N$.

Also we need the following lemma.

Lemma 4.7. *Let M be a completely distributive module , and let N be a submodule of M, then there exists a submodule coclosed submodule K of N such that $K\leq N of\ M$ (i.e there exists a coclosure submodule of N).*

Proof. Consider the following set:

$$C=\{H|H\leq_{ce}N of M\}$$

$C\neq\varphi$, since $N\leq_{ce}N$ of M. Let $K=\bigcap_{H_i\in C}H_i$. Then by lemma (4.6), $K\leq_{ce}N$ of M. We claim that K is a coclosed submodule of M. It is clear that K is a minimal element of C. Assume that $\frac{K}{T}\ll\frac{M}{T}$ for some submodule T of K. Hence $T\leq_{ce}N$ of M, thus $T\in C$. But $T\leq K$ and K is a minimal element of C, so $T=K$. Thus K is a coclosed submodule in M.

Now we can give the main result of this section.

Theorem 4.8. *Let M be a completely distributive module, then M is a CCS-module if and only if M is a lifting module.*

Proof.

$\Rightarrow$) Let N be a submodule of M. Since M is a completely distributive module, so by lemma (4.7), there exists a coclosed submodule K of

M, $K\leq N$ such that

$$\frac{N}{K}\ll\frac{M}{K}$$

. But M is a CCS-module, then K is a direct summand of M. Hence M is a lifting module.

$\Leftarrow$) It is a clear.

Recall that an R-module is called A-projective, where A is an R-module, if for each submodule X of A, every homomorphism $h : M \to \frac{A}{X}$ can be lifted to a homomorphism $g : M \to A$, i.e the following diagram :

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "h"] & \\ A \arrow[r, "\pi"'] & \frac{A}{X} \arrow[r] & 0 \end{tikzcd}$$

commutes, where π is the natural epimorphism, and M is called a projective module, if M is a A-projective for every modules A. And M is self projective if M is M-projective [6]. Also Iman in [3] introduced the concept of A-coprojective module, where an R-module M is called A-coprojective module, if for every homomorphism $f : M \to \frac{A}{K}$, where K is a coclosed submodule of A, can be lifted to a homomorphism $g : M \to A$ i.e the following diagram:

$$\begin{tikzcd} & M \arrow[dl, dotted, "g"'] \arrow[d, "f"] & \\ A \arrow[r, "\pi"'] & \frac{A}{K} \arrow[r] & 0 \end{tikzcd}$$

commutes, where π is the natural epimorphism.

By using Th (2.2.2) in [3], we get the following proposition.

Proposition 4.9. *For an R-module M the following statements are equivalent.*

1. *M is a CCS-module.*

2. *Every R-module is an M-coprojective module.*

3. *For every coclosed submodule K of M, $\frac{M}{K}$ is an M-coprojective module.*

Corollary 4.10. *Let M be a an R-module. If M is a CCS-module, then M is a self coprojective module.*

Proof. Since M is a CCS-module, then by Prop (4.9), every R-module is an M-coprojective; that is M is a self coprojective module.

Recall that an R-module M is called D_2, if for every submodule N of M, for which $\frac{M}{N}$ is a direct summand of M, then N is a direct summand of M [16]. And M is called WD_2-module, if for every coclosed submodule N of M for which $\frac{M}{N}$ is isomorphic to a direct summand of M, then N is a direct summand of M [3].

Remark 4.11. *We have the following relations:*

$$CCS\text{-module} \Longrightarrow \text{Self coprojective module} \Longrightarrow WD_2\text{-module}$$

Theorem 4.12. *Let M be a an R-module. Consider the following statements:*

1. *R is a semisimple ring.*
2. *Every R-module is a lifting module.*
3. *Every R-module is a CCS-module.*
4. *Every R-module is an M-coprojective module.*

Then (1)$\Rightarrow$ (2)$\Rightarrow$ (3)$\Leftrightarrow$ (4)$\Rightarrow$. And if $J(R) = (0)$, then (2)$\Rightarrow$ (1).

Proof.
(1)$\Rightarrow$ (2): Since R is a semisimple ring, then every R-module is a semisimple module, and hence it is a lifting module.
(2)$\Rightarrow$ (3): It is a clear.
(3)$\Leftrightarrow$ (4): By Prop (4.9).
(2)$\Rightarrow$ (1): Since $J(R) = (0)$, so by Prop (2.1.16) in [9], R is a semisimple ring. Note that if every R-module is an amply supplemented, then (3)$\Leftrightarrow$ (2).

Next we have the following.

Theorem 4.13. *Let R be a ring such that $J(\frac{R}{K}) = (0)$, for each ideal K of R. Then R is a semisimple ring if and only if every R-module is a CCS-module.*

Proof. $\Rightarrow$) It is a clear.
$\Leftarrow$) Let I be an ideal of R. Since every R-module is a CCS-module, then R is a CCS-R-module. So if I is a coclosed ideal in R, then I is a direct summand of R. Assume that I is not coclosed ideal in R, then there exists a proper subideal K of I such that $\frac{I}{K} \ll \frac{R}{K}$. Thus $\frac{I}{K} \subseteq J(\frac{R}{K})$.But $J(\frac{R}{K}) = (0)$, then $\frac{I}{K} = (0)$, and hence $I = K$ which is a contradiction.Thus I must be a coclosed in R, and so I is a direct summand od R.

We end this section by the following Corollary.

Corollary 4.14. *Let R be a ring such that $J(\frac{R}{K}) = (0)$, for each ideal K of R. Consider the following statemens:*

1. *R is a semisimple ring.*
2. *Every R-module is a lifting module.*
3. *Every R-module is a CCS-module.*
4. *Every projective R-module is a CCS-module.*
5. *Every free R-module is a CCS-module.*
6. *Every finitely generated free R-module is a CCS-module.*
7. *Every finitely generated projective R-module is a CCS-module.*
8. *$R \oplus R$ is CCS-module.*

Then (1)$\Leftrightarrow$ (2)$\Leftrightarrow$ (3)$\Rightarrow$ (4)$\Leftrightarrow$ (5)$\Leftrightarrow$ (6)$\Leftrightarrow$ (7), and (1)$\Rightarrow$ (8).

Proof.
(1)$\Leftrightarrow$ (2): Since $J(\frac{R}{K}) = (0)$ for each ideal K of R, then $J(R) = (0)$. Hence the result follows from ([9],Prop(2.1.16)).
(2)$\Leftrightarrow$ (3): By Th (4.13).
(3)$\Rightarrow$ (4)$\Leftrightarrow$ (5)$\Leftrightarrow$ (6)$\Leftrightarrow$ (7): It is clear.
(1)$\Rightarrow$ (8): Since R is a semisimple ring, then $R \oplus R$ is a semisimple ring, and hence $R \oplus R$ is a CCS-module.

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Received: August 1, 2013