Global Attractor for a 3D Reaction-Diffusion Equation

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Abstract

In this paper, we consider the long time behavior of a 3D reaction-diffusion equation and obtain the existence of global attractor for a autonomous reaction-diffusion system by establishing the asymptotical smoothness for the semigroup by decomposition method which introduced by Zelik (CPAA 2004).

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1 Introduction

Let $\Omega \subset R^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, we consider the non-autonomous 3D reaction-diffusion equation:

$$u_t - \Delta u + \lambda u + u^3 = f(x), \quad x \in \Omega, \quad t \in R^+, \quad (1)$$

$$u(t, x)|_{\partial\Omega} = 0, \quad (2)$$

$$u(0, x) = u_0(x), \quad \tau \geq 0. \quad (3)$$

Here $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity, the length scale α is a characterizing parameter of the elasticity of the fluid.

Since the global wellposedness and long time behavior of reaction-diffusion equation pay an important role in explain the stability of reaction diffusion phenomena in physics, biology, chemistry and engineering science, the reaction-diffusion equation has become a hot topic in pure and applied mathematical research. For the classical result of reaction-diffusion equation, such as the global existence of solutions in $L^2(\Omega)$ and $H_0^1(\Omega)$, long-time behavior for autonomous system (such as global attractor, Hausdorff dimension of attractor), we can refer to [9], [4], [3]. Later, Chepyzhov and Vishik [1] gave a new method which is well suited to investigating equations arising in mathematical physics without unique solvability, the theory of trajectory attractor has developed and used to reaction-diffusion equation. Zhong, Yang and Sun [10] represented a general theory of norm-to-weak semigroup and applied to reaction-diffusion equation which obtained the existence of global attractor. Song and Zhong [7] investigate the case of external force function with less regularity (such as the external force is translation bounded but not translation compact or belong to norm functions class) and derived the uniform attractor in $L^p(\Omega)$ for the reaction-diffusion equation.

In this paper, we consider the long time behavior of a 3D reaction-diffusion equation and obtain the existence of global attractor for a autonomous reaction-diffusion system by establishing the asymptotical smoothness for the semigroup by decomposition method which introduced by Zelik [11] (CPAA 2004).

2 Global existence of solutions and attractors for the reactive-diffusion equation

In the sequel, we set $H = L^2(\Omega)$, $V = \{u \in H^1(\Omega) \cap H, u|_{\partial\Omega} = 0\}$, $W = H^2(\Omega) \cap V$. then we state the existence of global solution to problem (1)-(3).

We denote the family of Hilbert spaces $D(A^{\frac{s}{2}})$ ($s \in R$) with the usual product and norm as $\langle \cdot, \cdot \rangle_{D(A^{\frac{s}{2}})} = \langle A^{\frac{s}{2}} \cdot, A^{\frac{s}{2}} \cdot \rangle$ and $\| \cdot \|_{D(A^{\frac{s}{2}})} = \| A^{\frac{s}{2}} \cdot \|$ respectively. Obviously, we have $D(A^{\frac{s}{2}}) \hookrightarrow D(A^{\frac{r}{2}})$ for any $s > r$. Especially we denote $\mathcal{H}_s = D(A^{\frac{s+1}{2}})$ with the norm $\| \cdot \|_{\mathcal{H}_s} = \| \cdot \|_{1+s}$.

By the standard Faedo-Galerkin approximation method, we can prove the existence of strong solutions to (1)-(3) as the following theorem.

Theorem 2.1 (1) Let $f \in V'$ and $u_0 \in H$, then there exists a unique strong solution of problem (1)-(3) such that

$$u \in H^1(0, T; H) \cap L^2(0, T; W), \quad (4)$$

for all $T > 0$.

(2) Assume $f \in V'$ and $u_0 \in V$, then the system (1)-(3) possess a unique strong solution $u(x, t)$ satisfying

$$u \in C^0(R^+; V) \cap L^2(R^+; W), \quad u_t \in L^2(R^+; H). \quad (5)$$

Theorem 2.2 For any $f \in V'$, $u_0 \in H$, the semigroup $\{S(t)\}, t \geq \tau \in R^+$ generated by the system (1)-(3) has a global attractor $\mathcal{A}$ which is invariant, compact in H and attracts all bounded subsets of H in the topology of H .

The proof can divide into two steps, the first is to prove the existence of absorbing set, next we shall use decomposition method to prove the asymptotical smoothness for the semigroup.

Since the injection $H \hookrightarrow V'$ is dense, then for any $f \in V'$, we can find a function $f^\varepsilon \in H$ depending on f and ε such that

$$\|f - f^\varepsilon\| \leq \varepsilon. \quad (6)$$

Using the similar technique to Zelik [11], we decompose the semigroup into $S(t)u_0 = S_v(t)u_0 + S_w(t)u_0$ with $S_v(t)u_0 = v(t)$ and $S_w(t)u_0 = w(t)$ that satisfy the following equations respectively

$$v_t + Av_t + \lambda v = f - f^\varepsilon, \quad (7)$$

$$v(x, 0) = u_0(x) \quad (8)$$

and

$$w_t + Aw_t + \lambda w + u^3 = f^\varepsilon, \quad (9)$$

$$w(x, 0) = 0, \quad x \in \Omega. \quad (10)$$

3 Existence of global absorbing set

Lemma 3.1 Let the external force $f \in V'$, $u_\tau \in H$, then the semigroup has a bounded global absorbing set B_0 in H , where

$$B_0 = \{u \in H : \|u\|_H \leq \rho\}$$

is a bounded set in H .

Proof. Taking inner product of (1) with u , and integrating by parts over Ω , we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 + \int_{\Omega} (u^3 + \lambda u) \cdot u dx = \int_{\Omega} f(x) \cdot u dx, \quad (11)$$

where $\|\cdot\|$ denotes the H norm, $\|\nabla(\cdot)\|$ is equivalent to the V norm.

The Poincaré inequality implies $\|\nabla u\|^2 \geq \frac{1}{2} \|\nabla u\|^2 + \frac{\lambda}{2} \|u\|^2$. Hence, by the Cauchy inequality, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{\lambda}{2} \|u\|^2 + \lambda \|u\|^2 + \|u\|_{L^4}^4 \\ \leq \frac{1}{\lambda} \|f(x)\|_{V'}^2 + \lambda \|u\|^2, \end{aligned} \quad (12)$$

i.e.,

$$\frac{d}{dt} \|u\|^2 + \lambda \|u\|^2 \leq \frac{2}{\lambda} \|f(x)\|_{V'}^2. \quad (13)$$

By the Gronwall inequality, we have

$$\|u(x, t)\|^2 \leq e^{-\frac{\lambda t}{2}} \|u_0\|^2 + \frac{2}{\lambda^2} \|f(x)\|_{V'}^2, \quad (14)$$

Choosing $\rho^2 = e^{-\frac{\lambda t}{2}} \|u_0\|^2 + \frac{2}{\lambda^2} \|f(x)\|_{V'}^2$, such that $B_0 = \{u : \|u\|_H^2 \leq \rho^2\}$, i.e., B_0 is the global absorbing ball for the semigroup $\{S(t)\}_{t \geq 0}$ in H .

4 Asymptotic smoothness for the semigroup

Theorem 4.1 *For any $f \in V'$, the semigroup $\{S(t), t \geq 0\}$ generated by the system (1)-(3) is asymptotical smoothness in H .*

Proof. We shall use decomposition method to achieve the result in the following lemmas.

Lemma 4.2 *For the system (7)-(8), there exists a constant $\varepsilon = \varepsilon(f, \delta)$ such that the solution of (7)-(8) satisfies*

$$\|S_v(t)u_0\|_H^2 = \|v(x, t)\|_H^2 \leq Q(\|u_0\|_H) e^{-\frac{\lambda t}{2}} + \delta \quad (1)$$

for any $t \geq 0$, where $Q(\cdot)$ is a positive increasing function on $[0, +\infty)$.

Proof. Multiplying (7) with $v(t)$, integrating over Ω , we deduce

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \frac{1}{2} \|\nabla v\|^2 + \frac{\lambda}{2} \|v\|^2 + \lambda \|v\|^2 \leq \frac{1}{\lambda} \|f - f^\varepsilon\|_{V'}^2 + \lambda \|v\|^2, \quad (2)$$

which means

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 \leq \frac{2}{\lambda}\|f - f^\varepsilon\|_{V'}^2, \quad (3)$$

applying the Gronwall lemma, we have

$$\|v\|_H^2 \leq Q(\|u_0\|_H)e^{-\frac{\lambda t}{2}} + \frac{2}{\lambda^2}\|f - f^\varepsilon\|_{V'}^2. \quad (4)$$

Choosing $\varepsilon^2 \leq 2\lambda^2\delta$, we complete the proof.

Lemma 4.3 *For any time $T > 0$ and constant $\varepsilon > 0$, there exists a non-negative constant $M > 0$ depending on the constants T , f^ε , $\|u_0\|_H$ such that the solution to the system (9)-(10) has the property*

$$\|S_w(t)u_0\|_{\mathcal{H}_s}^2 = \|w(t)\|_{\mathcal{H}_s}^2 \leq M \quad (5)$$

for all $0 < s \leq 1$.

Proof. Taking inner product of (9) with $A^s w(t)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{s}{2}} w|^2 dx + \int_{\Omega} |A^{\frac{s+1}{2}} w|^2 dx + \lambda \int_{\Omega} |A^{\frac{s}{2}} w|^2 dx + (u^3, A^s w) = \langle f^\varepsilon, w \rangle. \quad (6)$$

Since $0 < s \leq 1$, by the Cauchy inequality, we have

$$\langle f^\varepsilon, w \rangle \leq C\|f^\varepsilon\| \|A^{\frac{s+1}{2}} w\| \leq \frac{C_1}{2}\|f^\varepsilon\|^2 + \frac{C_2}{2} \int_{\Omega} |A^{\frac{s+1}{2}} w|^2 dx \quad (7)$$

and

$$|(u^3, A^s w)| \leq C\|u^3\| \|A^{\frac{s+1}{2}} w\| \leq \frac{C_3}{2}\|u\|_{L^6}^2 + \frac{C_4}{2} \|A^{\frac{s+1}{2}} w\|^2. \quad (8)$$

Since $H^1 \hookrightarrow L^6$, then we can choose $s = 1$ and appropriate constants C_i ($i = 1, 2, 3, 4$) such that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} w|^2 dx + \int_{\Omega} |Aw|^2 dx + \lambda \int_{\Omega} |A^{\frac{1}{2}} w|^2 dx \\ & \leq \frac{C_1}{2}\|f^\varepsilon\|^2 + \int_{\Omega} |Aw|^2 dx + C_0 \lambda \int_{\Omega} |A^{\frac{1}{2}} w|^2 dx, \end{aligned} \quad (9)$$

here $C_0 > 1$. This means

$$\frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} w|^2 dx \leq C_5 \left(\frac{C_1}{2}\|f^\varepsilon\|^2 + C(\lambda) \int_{\Omega} |A^{\frac{1}{2}} w|^2 dx \right), \quad (10)$$

Using the Gronwall lemma, we conclude that

$$\|A^{\frac{s}{2}}\|^2 \leq C\|A^{\frac{1}{2}}\| \leq C'e^{C_5 t} \quad (11)$$

for all $t \geq 0$.

From the compact embedding $\mathcal{H}_s \hookrightarrow V \hookrightarrow H$ is compact, we can deduce the asymptotical smoothness for the semigroup. This complete the proof of the lemma.

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