Geodesic Curvatures and Natural Lifts of Fixed Pole Curve Belong to Timelike Mannheim Pair Curves

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Abstract: In this paper, we consider that Mannheim curve is timelike curve and Mannheim partner curve is timelike curve. Then, we investigate the relation between arc lengths and geodesic curvature of fixed pole curve \( C^* \) generated by Darboux vector \( C^* \) with respect to Lorentz space \( IL^3 \), Lorentz sphere \( S^2 \) or Hyperbolic sphere \( H^2 \). In addition, we get, if the natural lifts geodesic spray of spherical indicator curvatures of Mannheim partner curve is an integral curve, then how Mannheim Curve is found to be.

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1. Introduction

There are lots of studies about curve in differential geometry in Euclidean space. Especially, many theories were obtained by making connections two curves corresponding points and between Frenet frames. Bertrand curves and Involute-Evolute curves are two of them [7], [4], [8], [19]. Those curves were studied in
different spaces, and there are a lot of results were gained. Frenet frames of Bertrand curves and Frenet frames of Involute-Evolute curves create spherical indicator curves on unit sphere surface. Natural lifts and geodesic sprays of spherical indicator curves are defined in [5], [17], [3], [6].

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if \( \kappa = \lambda (\kappa^2 + \tau^2) \), where \( \lambda \) is a nonzero constant \( \kappa \) is the curvature and \( \tau \) is the torsion. Mannheim curve was redefined by Liu and Wang. According to this new definition, if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve [22], [10]. There are a lot of studies about Mannheim curve according to the new definition, [12], [14], [15], [18].

2. Preliminaries

Let \( \alpha : I \rightarrow E^3 \), \( \alpha(t) = (\alpha_1(s), \alpha_2(s), \alpha_3(s)) \) be a unit speed differentiable curve. If we denote \( \alpha : I \rightarrow E^3 \) curve’s Frenet as \( \{T, N, B\} \), curvature as \( \kappa \), and torsion as \( \tau \) then, there are some equations among them:

\[
\begin{align*}
T'(s) &= \kappa(s)N(s), \\
N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\
B'(s) &= -\tau(s)N(s).
\end{align*}
\]

(see [9]). By using (2.1), we can get \( W \) Darboux vector as:

\[
W = \tau T + \kappa B
\]

(2.2)

If \( \varphi \) is the angle between \( W \) and \( B \), then the unit Darboux vector is:

\[
C = \sin \varphi T + \cos \varphi B
\]

(2.3)

Let \( X \) be a differentiable vector space on \( M \). (Note that \( M \) is any vector space)

\[
\frac{d}{ds}(\alpha(s)) = X(\alpha(s))
\]

(2.4)
The curve $\alpha$ is an integral curve of $X$ if and only if 
\[
\frac{d}{ds}(\alpha(s)) = X(\alpha(s)).
\]
Suppose that $TM = \bigcup_{P \in M} T_P(P)$ then,
\[
\overline{\alpha} : I \rightarrow TM, \quad \overline{\alpha}(s) = (\alpha(s), \alpha'(s)) \tag{2.5}
\]
$\overline{\alpha} : I \rightarrow TM$ curve is natural lift of $\alpha : I \rightarrow M$ and for $v \in TM$
\[
X(v) = -\langle v, S(v) \rangle N_p
\]
Vector space $X$ is called geodesicspray \[5,19\].
\[
\overline{D}_XY = D_XY + \langle S(X), Y \rangle N. \tag{2.6}
\]
The Equation (2.6) is called Gauss equation on $M$. Equation of pole curve $(C)$ is
\[
\alpha_c(s) = C(s).
\]
The arc lengths and geodesic curvatures of those curves with respect to $E^3$ are given, respectively as;
\[
s_c = \int_0^s \phi' ds \tag{2.7}
\]
\[
k_c = \sqrt{1 + \left(\frac{W}{\phi'}\right)^2} \tag{2.8}
\]
The geodesic curvatures with respect to $S^2$, are given by;
\[
\gamma_c = \frac{W}{\phi'} \tag{2.9}
\]
Let $\tilde{\alpha} : I \to TM$ be natural lift of $\alpha : I \to M$. X geodesic spray is an integral curve if and only if there is a geodesic curve on $M$ \cite{5}. Let $\alpha : I \to E^3$ and $\alpha^* : I \to E^3$ be two differentiable curves. Suppose that Frenet frames on the points of $\alpha(s)$ and $\alpha^*(s)$ are respectively given as $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$. If $\alpha$ curve’s principal normal vector and $\alpha^*$ curve’s binormal vector are linearly dependent, $\alpha$ curve is named Mannheim curve, $\alpha^*$ curve is named Mannheim partner curve, and it is shown as $(\alpha, \alpha^*)$ \cite{22}. Mannheim curve’s equation is given as:

$$\alpha^*(s^*) = \alpha(s) - \lambda N(s) \text{ or } \alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*)$$

There exist following equations among these curves:

\begin{align}
T &= \cos \theta T^* + \sin \theta N^* \\
N &= B^* \\
B &= -\sin \theta T^* + \cos \theta N^* \\
\cos \theta &= \frac{ds^*}{ds}, \quad \sin \theta = \lambda \tau^* \frac{ds^*}{ds} \\
T^* &= \cos \theta T - \sin \theta B \\
N^* &= \sin \theta T + \cos \theta B \\
B^* &= N
\end{align}

where $S (T, T^*) = \theta$, \cite{12}. Let $\kappa$ be curvature of $\alpha$, $\tau$ be torsion of $\alpha$, and let $\kappa^*$ be curvature of $\alpha^*$, $\tau^*$ be torsion of $\alpha^*$. Then, there are following equations:

\begin{align}
\kappa &= \tau^* \sin \theta \cdot \frac{ds^*}{ds} \\
\tau &= -\tau^* \cos \theta \cdot \frac{ds^*}{ds}
\end{align}
Geodesic curvatures and natural lifts

\[
\begin{align*}
\kappa^* &= \frac{d\theta}{ds^*} \\
\tau^* &= (\kappa \sin \theta - \tau \cos \theta) \frac{ds^*}{ds}, \quad (2.14) \\
\tau^* &= \frac{\kappa}{\lambda \tau} \quad (2.15)
\end{align*}
\]

(see [12]). Let \( I\mathbb{R}^3 = \{X = (x_1, x_2, x_3) : x_1, x_2, x_3 \in I\mathbb{R}\} \) be a 3-dimensional vector space, and let \( X = (x_1, x_2, x_3) \) and \( Y = (y_1, y_2, y_3) \) be two vectors in \( I\mathbb{R}^3 \). The lorentz scalar product of \( X \) and \( Y \) is defined by

\[
g : I\mathbb{R}^3 \times I\mathbb{R}^3 \to I\mathbb{R}, g(X, Y) = x_1y_1 + x_2y_2 - x_3y_3, \quad (2.16)
\]

This inner product space is defined as Lorentz space and symbolized as \( I\mathbb{L}^3 \). Norm of the vector \( X \in I\mathbb{L}^3 \) is \( \|X\|_{I\mathbb{L}} = \sqrt{|g(X, X)|} \).

For any \( X, Y \in I\mathbb{L}^3 \), vector product of \( X \) and \( Y \) is defined by

\[
X \times Y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2)
\]

(see [1]). Let \( T \) be tangent vector of \( \alpha : I \to I\mathbb{L}^3 \). \( \alpha : I \to I\mathbb{L}^3 \) is respectively defined as;

i) If \( g(T, T) > 0 \), then \( \alpha \) curve is a spacelike curve,

ii) If \( g(T, T) < 0 \), then \( \alpha \) curve is a timelike curve,

iii) If \( g(T, T) = 0 \), then \( \alpha \) curve is a lightlike or null curve, (see [11]).

Let \( \alpha : I \to I\mathbb{L}^3 \) be a differentiable timelike curve. In this case, \( T \) is timelike, \( N \) and \( B \) are spacelike, and Frenet formulae are given;

\[
\begin{align*}
T' &= \kappa N \\
N' &= \kappa T - \tau B \\
B' &= \tau N 
\end{align*}
\quad (2.17)
\]

(see [23]), where
\[ W = \tau T - \kappa B \] (2.18)

In this situation, there are two cases for \( W \) Darboux vector if \( W \) is spacelike, the Lorentzian timelike angle \( \phi \) between \( -B \) and \( W \), then;

\[
\kappa = \|W\| \cosh \phi , \quad \tau = \|W\| \sinh \phi
\] (2.19)

And unit Darboux vector is;

\[
C = \sinh \phi T - \cosh \phi B
\] (2.20)

If \( W \) timelike, \( \kappa \) and \( \tau \) are formulized;

\[
\kappa = \|W\| \sinh \phi , \quad \tau = \|W\| \cosh \phi
\] (2.21)

And unit Darboux vector is;

\[
C = \cosh \phi T - \sinh \phi B.
\] (2.22)

Let \( M \) be a Lorentz manifold, and \( \overline{M} \) be a hyper surface of \( M \). Suppose that \( S \) is a shape operator which is obtained from \( N \) normal of \( \overline{M} \), \( D \) is the connection on \( M \), \( \overline{D} \) is the connection on \( \overline{M} \), For \( X, Y \in \chi(\overline{M}) \), Gauss equation is;

\[
D_X Y = \overline{D}_X Y + \varepsilon g(S(X), Y)N
\] (2.23)

where \( S(X) = -D_X N \) and \( \varepsilon = g(N, N) \), \( (see[20]) \).

\[
S^2_i(r) = \left\{ X \in \mathbb{IR}^3_{\tau} \mid g(X, X) = r^2, \ r \in \mathbb{IR} \right\}
\]

is defined as Lorentz sphere,

\[
H^2_o(r) = \left\{ X \in \mathbb{IR}^3_{\tau} \mid g(X, X) = -r^2, \ r \in \mathbb{IR} \right\}
\]

is defined as hyperbolic sphere.

3. Timelike Mannheim Curve Pairs.

**Definition 3.1:** Let \( \alpha : I \rightarrow I \mathbb{R}^3 \) and \( \alpha' : I \rightarrow I \mathbb{R}^3 \) be a timelike curve. Suppose that Frenet frames of \( \alpha \) on the point of \( \alpha(s) \) is \( \{T(s), N(s), B(s)\} \) and Frenet frames of
\( \alpha^* \) on the point of \( \alpha^*(s) \) is \( \{T^*(s), N^*(s), B^*(s)\} \). If principal normal vector of \( \alpha^* \) and binormal vector of \( \alpha^* \) are linearly dependent, then the curve \( \alpha \) is called Mannheim curve and the curve \( \alpha^* \) is called Mannheim partner curve. This pair curve is symbolized as \( (\alpha, \alpha^*) \) and it is named timelike Mannheim curve pairs.

**Theorem 3.1:** The distance between \( (\alpha, \alpha^*) \) timelike Mannheim curve pair is constant.

**Proof:** It can be written that; \( \alpha(s) = \alpha^*(s^*) + \lambda(s^*)B^*(s^*) \). If this equation is derived with respect to \( s^* \) parameter, we can write;

\[
T \frac{ds}{ds^*} = T^* + \lambda \tau^* N^* + \lambda' B^*
\]

If we get inner product of the last equation and \( B^* \), then \( \lambda' = 0 \). From the definition of Euclidean distance, we can write;

\[
d(\alpha^*(s^*), \alpha(s)) = \|\alpha(s) - \alpha^*(s^*)\|
\]

\[
= |\lambda| = \text{constant}
\]

**Theorem 3.2:** Let \( (\alpha, \alpha^*) \) be a timelike Mannheim curve pair. Suppose that \( \alpha \) curve’s and \( \alpha^* \) curve’s Frenet frames of curves \( \alpha \) and \( \alpha^* \) are respectively \( \{T, N, B\} \) and \( \{T^*, N^*, B^*\} \). In this case, there are following equations;

\[
\begin{align*}
T^* &= -\cosh \theta T + \sinh \theta B \\
N^* &= \sinh \theta T - \cosh \theta B \\
B^* &= N
\end{align*}
\]

\[
-\cosh \theta = \frac{ds^*}{ds}, \quad \sinh \theta = \lambda \tau^* \frac{ds^*}{ds}
\]

\[
\begin{align*}
T &= -\cosh \theta T^* - \sinh \theta N^* \\
N &= B^* \\
B &= -\sinh \theta T^* - \cosh \theta N^*
\end{align*}
\]
Proof: If we derive $\alpha^*(s^*) = \alpha(s) - \lambda N(s)$ with respect to the parameter $s$, we can write:

$$T^* \frac{ds^*}{ds} = (1 - \lambda \kappa)T(s) + \lambda \tau B$$

(3.4)

If we take inner product of (3.4) and $T$, then;

$$-\cosh \theta \frac{ds^*}{ds} = 1 - \lambda \kappa,$$

(3.5)

If we take inner product of (3.4) and $B$, then;

$$\sinh \theta \frac{ds^*}{ds} = \lambda \tau$$

(3.6)

If (3.5) and (3.6) are plugged into (3.4), we can write;

$$T^* = -\cosh \theta T + \sinh \theta B$$

From Frenet formulae, the following equations can be found;

$$N^* = \sinh \theta T - \cosh \theta B,$$

$$B^* = N.$$

Obviously, we have shown that equation (3.1). If we arrange this equation with respect to $T$ and $B$, then we can find equation of (3.3). If $\alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*)$ is derived with respect to parameter $s$, it can be found;

$$T = T^* \frac{ds^*}{ds} + \lambda \tau^* \frac{ds^*}{ds} N^*$$

If we consider the corresponding value of $T$ from (3.3), equation (3.2) is proven.

Theorem 3.3: Let $\left(\alpha, \alpha^*\right)$ be a timelike Mannheim curve pair. In this case, we get
Geodesic curvatures and natural lifts

\[ \kappa = \tau^* \sinh \theta \frac{ds^*}{ds}, \quad \tau = \tau^* \cosh \theta \frac{ds^*}{ds}, \] \hspace{1cm} (3.7)

\[ \kappa^* = -\frac{d\theta}{ds^*} = -\theta' \frac{ds}{ds^*}, \quad \tau^* = \kappa \sinh \theta \frac{ds}{ds^*} - \tau \cosh \theta \frac{ds}{ds^*}. \] \hspace{1cm} (3.8)

**Proof:** If we differentiate \( \langle T, B^* \rangle = 0 \), then \( \kappa = \tau^* \sinh \theta \frac{ds^*}{ds} \).

If we differentiate \( \langle B, B^* \rangle = 0 \), then \( \tau = \tau^* \cosh \theta \frac{ds^*}{ds} \).

If we differentiate \( \langle T, T^* \rangle = \sinh \theta \), then \( \kappa^* = -\frac{d\theta}{ds^*} = -\theta' \frac{ds}{ds^*} \).

If we differentiate \( \langle N, N^* \rangle = 0 \), then \( \tau^* = \kappa \sinh \theta \frac{ds}{ds^*} - \tau \cosh \theta \frac{ds}{ds^*} \).

**Theorem 3.4:** Let \( (\alpha, \alpha^*) \) be a timelike Mannheim curve pair. Let \( \kappa^* \) be curvature of \( \alpha^* \), \( \tau^* \) be torsion of \( \alpha^* \) and \( \tau \) be torsion of \( \alpha \). Then the following equation holds;

\[ \begin{cases} 
\tau^* = \frac{\kappa}{\lambda \tau} \\
\kappa^* = \theta' \frac{\kappa}{\lambda \tau \|W\|} 
\end{cases} \] \hspace{1cm} (3.9)

**Proof:** If we consider the equations (3.2), (3.5) and (3.6), then we have \( \tau^* = \frac{\kappa}{\lambda \tau} \). On the other hand, from equation (3.7), we can write that \( \frac{ds}{ds^*} = \frac{\tau^*}{\|W\|} \).

Then, substituting the value of \( \tau^* \) in \( \tau^* = \frac{\kappa}{\lambda \tau} \) in this equation, then we get \( \frac{ds}{ds^*} = \frac{\kappa}{\lambda \tau \|W\|} \). Then \( \kappa^* = \theta' \frac{ds}{ds^*} \), the following equation holds.
\[ \kappa^* = \theta' \frac{\kappa}{\lambda \tau \|W\|} \]

**Theorem 3.5:** Let \((\alpha, \alpha^*)\) be a timelike Mannheim curve pair. Then the following equation holds between \(W\) Darboux vector of the curve \(\alpha\) and \(T^*\) tangent vector of the curve \(\alpha^*\):

\[ W = -\tau^* \frac{ds^*}{ds} T^* \quad (3.10) \]

**Proof:** We know that \(W = \tau T - \kappa B\). If we take the corresponding values of \(T\) and \(B\) from (3.3), and then if we plug into those values in (3.10), then if we get the corresponding values of \(\kappa\) and \(\tau\) from (3.7), and then if we plug into those values in (3.10), the result is proven.

**Corollary 3.1:** Let \((\alpha, \alpha^*)\) be a timelike Mannheim curve pair. Let \(\varphi\) be the angle between Darboux vector of the curve \(\alpha\) with binormal vector of the \(\alpha\). Then the following equation occurs between \(\theta\) and \(\varphi\):

\[ \cosh \varphi = \cosh \theta \quad , \quad \sinh \varphi = \sinh \theta. \quad (3.11) \]

**Proof:** From (3.10) \(W\) is a timelike vector. From equation (2.22), we can write;

\[ C = \cosh \varphi T - \sin h\varphi B \]

If we consider \(C = -T^*\), we can write;

\[ \begin{cases} \cos h\varphi = \cosh \theta \\ \sin h\varphi = \sinh \theta. \end{cases} \]

Let \(k_c\) be geodesic curvature of fixed pole curve of Lorentzian sphere of generated by Darboux vector of \(\alpha : I \rightarrow IL^3\) timelike Mannheim curve. Suppose that \(T_c\) is unit tangent vector of \((C)\), then;

\[ k_c = \left\| D_{t_c} T_c \right\|. \]

If \(\alpha_c(s) = C(s)\) fixed pole curve is derived with respect to \(s_c\) parameter, we can write;
From the definition of geodesic curvature

\[
\begin{align*}
\left\{ \begin{array}{l}
k_c = \sqrt{\left(\frac{\|W\|}{\theta'}\right)^2 + 1} & , W \text{ spacelike} \\
k_c = \sqrt{\left(\frac{\|W\|}{\theta'}\right)^2 - 1} & , W \text{ timelike} 
\end{array} \right.
\end{align*}
\]

(3.14)

**Corollary 3.2:** Let \( (\alpha, \alpha^*) \) be a timelike Mannheim pair curve. Geodesic curvature of fixed pole curve of Lorentzian sphere of generated by Darboux vector of \( \alpha : I \to IL^3 \) timelike Mannheim curve \( (C) \)

\[
\begin{align*}
\left\{ \begin{array}{l}
k_c = \sqrt{\left(\frac{\|W\|}{\theta'}\right)^2 - 1} & , W \text{ spacelike} \\
k_c = \sqrt{\left(\frac{\|W\|}{\theta'}\right)^2 + 1} & , W \text{ timelike.}
\end{array} \right.
\end{align*}
\]
Let $D$ be $\alpha : I \rightarrow IL^3$ Mannheim curve’s connection on $IL^3$, let $\bar{D}$ be $\alpha : I \rightarrow IL^3$ Mannheim curve’s connection on $S^2_i$. Suppose that $\xi$ is unit normal vector space of $S^2_i$ then;

\[
D_\alpha Y = \bar{D}_\alpha Y + \varepsilon g(S(X),Y)\xi, \quad \varepsilon = g(\xi,\xi)
\]

where $S$ is shape operator of $S^2_i$ and corresponding matrix is;

\[
S = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\]

(see[16]). Let $\gamma_\tau$ be geodesic curvature of $(C)$ fixed pole curves on $S^2_i$, from Gauss equation, we can write;

\[
D_{\tau_c} T_c = \bar{D}_{\tau_c} T_c + \varepsilon g(S(T_c),T_c)C
\]

where

\[
\varepsilon = \begin{cases}
g(C,C) = +1, & C \text{ spacelike} \\
g(C,C) = -1, & C \text{ timelike}
\end{cases}, \quad \begin{cases}
g(S(T_c),T_c) = +1, & T_c \text{ timelike} \\
g(S(T_c),T_c) = -1, & T_c \text{ spacelike}
\end{cases}
\]

If we write those values in Gauss equation

\[
\begin{align*}
\bar{D}_{\tau_c} T_c &= \frac{\|W\|}{\theta'} N, & C \text{ timelike} \\
\bar{D}_{\tau_c} T_c &= -\frac{\|W\|}{\theta'} N, & C \text{ spacelike}
\end{align*}
\]

From the definition of geodesic curvature;

\[
\gamma_c = \frac{\|W\|}{\theta'} \tag{3.15}
\]
Corollary 3.3: Let \( (\alpha, \alpha^*) \) be a timelike Mannheim pair curve. Let \( \gamma_c \) be geodesic curvature of fixed pole curve \( C \) with respect to Lorentzian sphere \( S^2 \) of generated by Darboux vector of \( \alpha: I \to \mathbb{L}^3 \) timelike Mannheim curve. Then we get

\[
\gamma_c = \frac{\|W\|}{\theta'}.
\]

Let \( \alpha^*: I \to \mathbb{L}^3 \) be timelike Mannheim curve and let \( s_{C^*} \) be \( \alpha^*: I \to \mathbb{L}^3 \) Mannheim partner curve’s \( (C^*) \) fixed pole curves are length, then;

\[
s_{C^*} = \int_0^s (\varphi^*)' ds
\]

and

\[
\begin{cases}
(\varphi^*)' = \frac{\left(\tau^*\right)'}{1 - \left(\kappa^* \tau^*\right)^2}, & \text{\( W^* \) spacelike} \\
(\varphi^*)' = \frac{\left(\kappa^* \tau^*\right)'}{1 - \left(\kappa^* \tau^*\right)^2}, & \text{\( W^* \) timelike}
\end{cases}
\]

If the values of \( \kappa^* \) and \( \tau^* \) are written in the equations, we can write

\[
\begin{cases}
(\varphi^*)' = \frac{\left|\theta\right|}{\left|\theta^*\right|} \left(\frac{\left|W\right|}{\left|\theta^*\right|}\right)'}, & \text{\( W^* \) spacelike} \\
(\varphi^*)' = \frac{\left|\theta\right|}{\left|\theta^*\right|} \left(\frac{\left|W\right|}{\left|\theta^*\right|}\right)'}, & \text{\( W^* \) timelike}
\end{cases}
\]

(3.17)
or from equation (3.15)

\[
(\varphi')' = \frac{\gamma_c'}{1-\gamma_c^2}
\]  

(3.18)

If (3.17) and (3.18) values are written in (3.16) rather than

\[
\begin{aligned}
    s_{c'} &= \int_0^s \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1-\left(\frac{\|W\|}{\theta'}\right)^2} \, ds, & W^* \text{ spacelike} \\
    s_{c'} &= \int_0^s \frac{\left(\frac{\theta'}{\|W\|}\right)'}{1-\left(\frac{\theta'}{\|W\|}\right)^2} \, ds, & W^* \text{ timelike}
\end{aligned}
\]

(3.19)

or

\[
s_{c'} = \int_0^s \frac{\gamma_c'}{1-\gamma_c^2} \, ds
\]

(3.20)

**Corollary 3.4:** Let \((\alpha, \alpha^*)\) be a timelike Mannheim curve pair. On the point of \(\alpha^* (s), \alpha^*\) curve’s \((C^*)\) fixed pole curve which is drawn by the unit Darboux vector. The arc lengths of fixed pole curve \((C^*)\) in terms of \(IL^3\),

\[
\begin{aligned}
    s_{c'} &= \int_0^s \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1-\left(\frac{\|W\|}{\theta'}\right)^2} \, ds, & W^* \text{ spacelike} \\
    s_{c'} &= \int_0^s \frac{\left(\frac{\theta'}{\|W\|}\right)'}{1-\left(\frac{\theta'}{\|W\|}\right)^2} \, ds, & W^* \text{ timelike}
\end{aligned}
\]
Corollary 3.5: Let \((\alpha, \alpha^*)\) be a timelike Mannheim curve pair, and let \(k_C\) be \(\alpha\) timelike Mannheim curve's geodesic curvature. In this case, the arc length of \((C^*)\) fixed pole curve is:

\[
    s_C = \int_0^s \frac{\gamma_C^*}{1 - \gamma_C^2} ds.
\]

In terms of \(IL^3\), Let \(k_C^*\) be \(\alpha^*\) (s) = \(C^*\) (s) fixed pol curve’s geodesic curvature, let \(s_C\) be arc parameter, and let \(T_C^*\) be unit tangent vector. Then, we can say:

\[
    \begin{align*}
    T_C^* &= \cosh \varphi^* T^* - \sinh \varphi^* B^* , \quad W^* \text{ spacelike} \\
    T_C^* &= \sinh \varphi^* T^* - \cosh \varphi^* B^* , \quad W^* \text{ timelike} ,
    \end{align*}
\]

From the definition of geodesic curvature;

\[
    \begin{align*}
    k_C^* &= \sqrt{\left(\frac{\|W^*\|}{(\varphi^*)'} \right)^2 + 1}, \quad W^* \text{ spacelike} \\
    k_C^* &= \sqrt{\left(\frac{\|W^*\|}{(\varphi^*)'} \right)^2 - 1}, \quad W^* \text{ timelike}.
    \end{align*}
\]

From equations (3.9)
\[
\left\lfloor \|W\|^* = \frac{\kappa}{\lambda \tau} \sqrt{\left(\frac{\|W\|}{(\varphi')^2}\right)^2 - 1} \right. , W^* \text{ spacelike} \\
\left\lfloor \|W\|^* = \frac{\kappa}{\lambda \tau} \sqrt{1 - \left(\frac{\|W\|}{(\varphi')^2}\right)^2} \right. , W^* \text{ timelike.}
\]

From equations (3.15) and (3.18)

\[
\frac{\|W\|^*}{(\varphi')^2} = \frac{\kappa}{\lambda \tau C} \left(1 - \gamma_c^2\right) \frac{2}{3}
\]

(3.24)

If those values are written in equation (3.23)

\[
\left\lfloor k_c^* = \sqrt{\left(\frac{\kappa}{\lambda \tau c'}\right)^2 \left(1 - \gamma_c^2\right)^3 + 1} \right. , W \text{ spacelike} \\
\left\lfloor k_c^* = \sqrt{\left(\frac{\kappa}{\lambda \tau c'}\right)^2 \left(1 - \gamma_c^2\right)^3 - 1} \right. , W \text{ timelike}
\]

(3.25)

**Corollary 3.6:** Let \((\alpha, \alpha^*)\) be a timelike Mannheim pair curve. The geodesic curvatures of fixed pol curve \((C^*)\) in terms of \(IL^3\)

\[
\left\lfloor k_c^* = \sqrt{\left(\frac{\kappa}{\lambda \tau c'}\right)^2 \left(1 - \gamma_c^2\right)^3 + 1} \right. , W \text{ spacelike} \\
\left\lfloor k_c^* = \sqrt{\left(\frac{\kappa}{\lambda \tau c'}\right)^2 \left(1 - \gamma_c^2\right)^3 - 1} \right. , W \text{ timelike}
\]
Let $\alpha^*: I \to \mathbb{I}^3$ timelike Mannheim partner curve and $\gamma_c$ be geodesic curve in $S^2_1$ for $\alpha^*_c(s) = C^*(s)$. From Gauss equation, we get,

$$D_t \kappa^*_c T_c = \bar{D}_t \kappa^*_c T_c + \varepsilon g(S(T_c), T_c) C^*$$

where

$$\varepsilon = \begin{cases} 
    g(C^*, C^*) = +1, & C^* \text{ spacelike} \\
    g(S(T_c^*), T_c^*) = 1, & W^* \text{ timelike} \\
    g(C^*, C^*) = -1, & C^* \text{ timelike} \\
    g(S(T_c^*), T_c^*) = -1, & W^* \text{ spacelike}
  \end{cases}$$

If those values are written in Gauss equation, then, $C^* = -\sin \phi^* T^* + \cos \phi^* B^*$ and if equation (3.22) is considered, we can say;

$$\begin{cases} 
    \bar{D}_t \kappa^*_c T_c = \left\| \frac{W^*}{(\phi')} \right\| N^*, & C \text{ timelike} \\
    \bar{D}_t \kappa^*_c T_c = -\left\| \frac{W^*}{(\phi')} \right\| N^*, & C \text{ spacelike}
  \end{cases} \tag{3.26}$$

If equation (3.24) is considered, geodesic curve is,

$$\gamma_c = \frac{\kappa}{\lambda \tau'_c} \left( 1 - \gamma_c^2 \right)^{\frac{3}{2}}$$

$\bar{D}_t \kappa^*_c T_c = 0$ if and only if $(C^*)$ curve geodesic spray is an integral curve. In this case, from equation (3.26), we can write $\kappa^* = \tau^* = 0$ and from equation (3.9) $\kappa = 0$.

**Corollary 3.7:** Let $(\alpha, \alpha^*)$ be a timelike Mannheim curve pair. Since Mannheim curve $\alpha$ is a straight line, there is no Mannheim partner curve.

**Corollary 3.8:** Let $(\alpha, \alpha^*)$ be a timelike Mannheim curve pair. In terms of $S^2_1$ or $H^2_0 (C^*)$ fixed pol curve’s geodesic curvatures is
\[ \gamma_C^t = \frac{\kappa}{\lambda r_C^t} \left( 1 - r_C^t \right)^{\frac{3}{2}} \]

REFERENCES


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