Fuzzy Group Congruences on an E -inversive Semigroup¹

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Abstract

In this paper, we introduce the concepts of the fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on an E -inversive semigroup. Some of their properties are investigated. Finally, we show that there is a one-one correspondence between the fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on an E -inversive semigroup.

Keywords: Fuzzy subsemigroups; Fuzzy complete inner-unitary subsemigroups; Fuzzy group congruences; E -inversive semigroup

1 Introduction and preliminaries

Al-Thukair [4] introduced the concepts of fuzzy sets and fuzzy congruences on inverse semigroups and set up a one-one correspondence between the set of fuzzy congruences and the set of fuzzy congruence pairs on an inverse semigroup. Kuroki [3] introduced the notions of a quotient semigroup S/ρ induced by a fuzzy congruence ρ on S . Zhang [2] discussed the concepts of the fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on a semigroup and showed that there is a one-one correspondence between the

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fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on a regular semigroup. In this paper, we introduce the concepts of the fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on an E -inversive semigroup. Finally, we show that there is a one-one correspondence between the fuzzy complete inner-unitary subsemigroups and the fuzzy group congruences on an E -inversive semigroup.

A semigroup S is called E -inversive semigroup if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$. As usual, $E(S)$ is the set of all idempotents of S . An element $x \in S$ is called weak inverse of $a \in S$ if $xax = x$. Devote by $W(a)$ the set of all weak inverses of $a \in S$. A subsemigroup H of S is called inner-unitary, if $xay, xy \in H$ implies $a \in H$ for any a, x, y in S .

Lemma 1.1^[1] Let S be a semigroup and ρ be a congruence on S . If ρ is a group congruence, then $\ker \rho$ is a seminormal subsemigroup of S and $\ker \rho = \{a \in S | a\rho = 1_\rho\}$, where 1_ρ is the identity of S/ρ .

Let X be a non-empty, a mapping $\mu : X \rightarrow [0, 1]$ is called a fuzzy subset of X . Let S be a semigroup, a mapping $\rho : S \times S \rightarrow [0, 1]$ is called a fuzzy relation on S .

Definition 1.2^[3] Let α and β be two fuzzy relations on a semigroup S . Then the product $\alpha \circ \beta$ of α and β is defined by

$$\alpha \circ \beta(a, b) = \bigvee_{x \in S} \{\alpha(a, x) \wedge \beta(x, b)\}$$

for all a, b in S .

A fuzzy relation ρ is called a fuzzy equivalence relation on a semigroup S if

- (1) $\rho(a, a) = 1$ for all $a \in S$;
- (2) $\rho(a, b) = \rho(b, a)$ for all $a, b \in S$;
- (3) $\rho(a, c) \geq \rho(a, b) \wedge \rho(b, c)$ for all $a, b, c \in S$.

A fuzzy equivalence relation ρ on a semigroup S is called a fuzzy congruence if $\rho(ab, cd) \geq \rho(a, c) \wedge \rho(b, d)$ or $(\rho(ac, bc) \geq \rho(a, b) \text{ and } \rho(ca, cb) \geq \rho(a, b))$ for all $a, b, c, d \in S$.

Lemma 1.3^[2] Let S be a semigroup. Then a fuzzy relation ρ on S is a fuzzy congruence if and only if $\rho^t = \{(a, b) \in S \times S | \rho(a, b) \geq t\}$ is a congruence on S , for any $t \in [0, 1]$.

Kuroki [3] defined a fuzzy subset ρ_a of S as $\rho_a(x) = \rho(a, x)$ for all $a, x \in S$. Let ρ be a fuzzy equivalence relation on S , then $\rho_a = \rho_b$ if and only if $\rho(a, b) = 1$ for all $a, b \in S$. ρ_a is called a fuzzy equivalence class of ρ containing $a \in S$. Let ρ be a fuzzy congruence on S . Then $S/\rho = \{\rho_a | a \in S\}$.

Let S be a semigroup, a mapping $\mu : S \rightarrow [0, 1]$ is called a fuzzy subsemigroup of S if $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all x, y in S .

It is clear that μ is a fuzzy subsemigroup of S if and only if the t -level subset $\mu^t = \{x \in S | \mu(x) \geq t\}$ of μ is a subsemigroup of S , for any $t \in [0, 1]$.

By definition 2.2 and 2.4 in [2], we can easily obtain the following definition.

Definition 1.4 Let S be an E -inversive semigroup and μ be a fuzzy subsemigroup of S . μ is called a fuzzy inner-unitary subsemigroups, if

$$\mu(a) \geq \mu(xay) \wedge \mu(xy)$$

for all a, x, y in S .

Theorem 1.5 Let S be an E -inversive semigroup and μ be a fuzzy subsemigroup of S . Then μ is a fuzzy inner-unitary subsemigroup if and only if

- (1) $\mu(xy) = \mu(yx)$ and
 - (2) $\mu(a) \geq \mu(ax) \wedge \mu(x)$ or $\mu(a) \geq \mu(xa) \wedge \mu(x)$
- for all a, x, y in S .

Proof. (1) If μ is a fuzzy inner-unitary subsemigroup, for all x, y in S , then

$$\mu(xy) \geq \mu(y \cdot xy \cdot x) \wedge \mu(yx) \geq \mu(yx).$$

Similarly, we have $\mu(yx) \geq \mu(xy)$. Hence $\mu(xy) = \mu(yx)$.

(2) For all a, x in S , we get

$$\mu(a) \geq \mu(x \cdot a \cdot x) \wedge \mu(xx) \geq \mu(x) \wedge \mu(ax) \wedge \mu(x) = \mu(ax) \wedge \mu(x).$$

Similarly, we have $\mu(a) \geq \mu(xa) \wedge \mu(x)$.

Conversely, for all a, x, y in S , we get

$$\mu(a) \geq \mu(a \cdot yx) \wedge \mu(yx) = \mu(xay) \wedge \mu(xy).$$

Thus, μ is a fuzzy inner-unitary subsemigroup of S .

By definition 2.9 in [2], we can easily obtain the following definition.

Definition 1.6 Let S be an E -inversive semigroup. A fuzzy inner-unitary subsemigroup μ of S is called fuzzy complete, if $\mu(e) = 1$ for all $e \in E(S)$.

2 Fuzzy group congruences

In this section, S always be an E -inversive semigroup. A fuzzy congruence ρ on S is called a fuzzy group congruence if ρ^t is a group congruence (i.e. S/ρ^t is a group), for any $t \in [0, 1]$.

We denote the fuzzy kernel of ρ by $K(\rho)$ is a fuzzy subset of S and is defined as $K(\rho)(x) = \bigvee_{e \in E(S)} \rho(x, e)$ for all $x \in S$.

Lemma 2.1 Let ρ be a fuzzy group congruence on S . Then $K(\rho) = \rho_e$ is a fuzzy complete inner-unitary subsemigroup of S for any $e \in E(S)$.

Proof. Let ρ be a fuzzy group congruence on S . Then there exists $e \in E(S)$ such that ρ_e is the identity in S/ρ . For all $x \in S$, then $\rho_e(x) = \rho(e, x)$. Since ρ_e is the identity of S/ρ , for all $f \in E(S)$, we have $\rho_e(x) = \rho_f(x)$. Hence $\rho_e(x) = \bigvee_{f \in E(S)} \rho(x, f) = K(\rho)(x)$. Therefore, $K(\rho) = \rho_e$.

We need only to prove that ρ_e is a fuzzy complete inner-unitary subsemigroup of S . For all x, y in S , we have

$$\rho_e(xy) = \rho(e, xy) \geq \rho(e, x) \wedge \rho(e, y) = \rho_e(x) \wedge \rho_e(y).$$

Thus, ρ_e is a fuzzy subsemigroup of S . By lemma 1.3, ρ^t is a group congruence on S for any $t \in [0, 1]$. By lemma 1.1, we know that $e\rho^t$ is a seminormal subsemigroup of S . For all $a, x, y \in S$, let $t = \rho_e(xay) \wedge \rho_e(xy)$, then $(e, xay), (e, xy) \in \rho^t$, i.e. $xay, xy \in e\rho^t$, thus, $a \in e\rho^t$, i.e. $(e, a) \in \rho^t$, hence $\rho(e, a) \geq t$ and

$$\rho_e(a) \geq \rho_e(xay) \wedge \rho_e(xy).$$

Therefore, ρ_e is a fuzzy inner-unitary subsemigroup of S . Since ρ_e is the identity in S/ρ , for any $f \in E(S)$, we have $\rho_e = \rho_f$, i.e. $\rho(e, f) = 1$. Hence $\rho_e(f) = 1$. Thus, ρ_e is a fuzzy complete inner-unitary subsemigroup of S .

Lemma 2.2 Let μ be a fuzzy complete inner-unitary subsemigroup of S .

- (1) If $\mu(mn) = 1$ for all $m, n \in S$, then $\mu(mnx) = \mu(x)$ for all $x \in S$;
- (2) If $\mu(km) = \mu(kn) = 1$ for all $k, m, n, x \in S$, then $\mu(mx) = \mu(nx)$;
- (3) If $\mu(ab) = \mu(cd) = 1$ for all $a, b, c, d \in S$, then $\mu(da) = \mu(bc)$.

Proof. (1) Let $\mu(mn) = 1$ and $m, n, x \in S$. Then $\mu(mnx) \geq \mu(mn) \wedge \mu(x) = 1 \wedge \mu(x) = \mu(x)$. Conversely, by theorem 1.5, hence $\mu(x) \geq \mu(mnx) \wedge \mu(mn) = \mu(mnx) \wedge 1 = \mu(mnx)$. Thus, $\mu(mnx) = \mu(x)$.

(2) Let $\mu(km) = \mu(kn) = 1$ and $k, m, n \in S$. By theorem 1.5, then $\mu(kn) = \mu(nk) = 1$. For all $x \in S$, by (1) of this lemma, we have $\mu(mx) = \mu(nk \cdot mx) = \mu(km \cdot xn) = \mu(nx)$.

(3) Let $\mu(ab) = \mu(cd) = 1$ and $a, b, c, d \in S$. By theorem 1.5, we have

$$\begin{aligned} \mu(da) &\geq \mu(da \cdot bc) \wedge \mu(bc) = \mu(cdab) \wedge \mu(bc) \\ &\geq \mu(cd) \wedge \mu(ab) \wedge \mu(bc) \geq 1 \wedge 1 \wedge \mu(bc) = \mu(bc). \end{aligned}$$

Similarly, we can prove $\mu(bc) \geq \mu(da)$. Thus, $\mu(da) = \mu(bc)$.

Lemma 2.3 Let μ be a fuzzy complete inner-unitary subsemigroup of S . Define a mapping $\rho_\mu : S \times S \rightarrow [0, 1]$ by

$$\rho_\mu(a, b) = \mu(a'b)$$

for all $a, b \in S$, where $a' \in W(a)$. Then ρ_μ is a fuzzy group congruence on S and $\mu = K(\rho_\mu)$.

Proof. If for all $a \in S$, $a' \in W(a)$, it is clear that $\rho_\mu(a, a) = \mu(a'a) = 1$. By (3) of lemma 2.2, for all $a, b \in S$, then $\rho_\mu(a, b) = \mu(a'b) = \mu(b'a) = \rho_\mu(b, a)$. For all $a, b, c \in S$, $b' \in W(b)$, $b'b \in E(S)$, we have $\mu(b'b) = 1$. Hence

$$\begin{aligned}\rho_\mu(a, c) &= \mu(a'c) = \mu(b'b \cdot a'c) = \mu(ba'cb') \\ &\geq \mu(ba') \wedge \mu(cb') = \mu(a'b) \wedge \mu(b'c) = \rho_\mu(a, b) \wedge \rho_\mu(b, c).\end{aligned}$$

Furthermore, for all $a, b, c \in S$, then

$$\begin{aligned}\rho_\mu(ac, bc) &= \mu((ac)'bc) = \mu((c'a' \cdot ac) \cdot (ac)'bc) = \mu(ac(ac)'bcc'a') \\ &= \mu(bcc'a') = \mu(a'bcc') \geq \mu(a'b) \wedge \mu(c'c) = \rho_\mu(a, b).\end{aligned}$$

Similarly, we have $\rho_\mu(ca, cb) \geq \rho_\mu(a, b)$. Thus, ρ_μ is a fuzzy congruence on S .

By lemma 1.3, for any $t \in [0, 1]$, then ρ_μ^t is a congruence on S . In order to show that ρ_μ is a fuzzy group congruence on S , we need only to show that ρ_μ^1 is a group congruence on S . Let $e\rho_\mu^1, f\rho_\mu^1 \in S/\rho_\mu^1$ be idempotent congruence classes of ρ_μ^1 . Hence $\mu(e) = \mu(f) = 1$. Since μ be a fuzzy complete inner-unitary subsemigroup of S , $e' \in W(e)$, then

$$\mu(e'f) \geq \mu(e \cdot e'f \cdot e) \wedge \mu(ee) \geq \mu(ee') \wedge \mu(f) \wedge \mu(e) = 1.$$

This forces that $\mu(e'f) = 1$, i.e. $\rho_\mu(e, f) = 1$. Hence $e\rho_\mu^1 = f\rho_\mu^1$. If for all $x\rho_\mu^1 \in S/\rho_\mu^1$, $x' \in W(x)$, then

$$\rho_\mu(ex, x) = \rho_\mu(x, ex) = \mu(x'ex) = \mu(xx'e) \geq \mu(xx') \wedge \mu(e) = 1.$$

i.e. $ex\rho_\mu^1 = x\rho_\mu^1$. Dually, $xex\rho_\mu^1 = x\rho_\mu^1$. So $e\rho_\mu^1$ is the identity of S/ρ_μ^1 . Since

$$\rho_\mu(x'x, x'xx'x) = \mu((x'x)'x'x \cdot x'x) = \mu(x'x) = 1,$$

then $x'x\rho_\mu^1 = (x'x)^2\rho_\mu^1 = e\rho_\mu^1$. Dually, $xx'\rho_\mu^1 = e\rho_\mu^1$. So $x'\rho_\mu^1$ is an inverse of $x\rho_\mu^1$. Hence, S/ρ_μ^1 is a group. That is, ρ_μ is a fuzzy group congruence on S .

By lemma 2.1, $K(\rho_\mu)$ is the identity of S/ρ_μ . For all $x \in S$ and $e \in E(S)$, we have $e \in W(e)$, $\mu(e) = 1$, and $\mu(x) = \mu(ex) = \rho_\mu(e, x)$. Hence $\mu(x) = \bigvee_{e \in E(S)} \rho_\mu(x, e) = K(\rho_\mu)(x)$. Therefore, $\mu = K(\rho_\mu)$.

Lemma 2.4 Let ρ is a fuzzy group congruence on S . Then $\rho = \rho_{K(\rho)}$.

Proof. Let ρ is a fuzzy group congruence on S . For all $e \in E(S)$, by lemma 2.1, we know that $K(\rho) = \rho_e$ is a fuzzy complete inner-unitary subsemigroup of S . Let $t = \rho(x, y)$ for all $x, y \in S$, $x' \in W(x)$, $t \in [0, 1]$. Then

$$\rho(x'x, x'y) = \rho_{x'x}(x'y) = \rho_e(x'y) = \rho(e, x'y) \geq \rho(x, y) = t.$$

Thus, $\rho_e(x'y) \geq \rho(x, y)$. Conversely, let $t = \rho_e(x'y)$ for all $x, y \in S$, $t \in [0, 1]$. Then

$$\begin{aligned}\rho(x, y) &= \rho(y, x) = \rho_y(x) = \rho_{xx'y}(x) = \rho_x(xx'y) \\ &= \rho_{xe}(xx'y) = \rho(xe, xx'y) \geq \rho(e, x'y) = \rho_e(x'y).\end{aligned}$$

Thus, $\rho(x, y) = \rho_e(x'y)$. By (2) of lemma 2.2, for any $x'' \in W(x)$, we have $\rho_e(x'y) = \rho_e(x''y)$. By definition of $\rho_{K(\rho)}$ in lemma 2.3, we have

$$\rho(x, y) = \rho_e(x'y) = K(\rho)(x'y) = (\rho_{K(\rho)})_e(x'y) = \rho_{K(\rho)}(x, y).$$

Thus, $\rho = \rho_{K(\rho)}$.

Theorem 2.5 Let ρ be a fuzzy congruence on S . If ρ is a fuzzy group congruence on S , then $K(\rho)$ is a fuzzy complete inner-unitary subsemigroup of S , and $\rho = \rho_{K(\rho)}$.

Conversely, if μ is a fuzzy complete inner-unitary subsemigroup of S , then ρ_μ is a fuzzy group congruence on S .

Proof. By lemma 2.1, lemma 2.4 and lemma 2.3, we get this theorem.

Corollary 2.6 Let ρ, θ be a fuzzy group congruence on S . Then $\rho = \theta$ if and only if $K(\rho) = K(\theta)$.

Theorem 2.7 Let $FGC(S)$ be the set of all fuzzy group congruences on S and $FCI(S)$ be the set of all fuzzy complete inner-unitary subsemigroups of S . Define a mapping $\xi : FGC(S) \rightarrow FCI(S)$ by $\xi(\rho) = K(\rho)$ for all $\rho \in FGC(S)$. Then ξ is a one to one mapping from $FGC(S)$ onto $FCI(S)$.

Proof. By lemma 2.3 and corollary 2.6, we know that ξ is bijective.

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