Sturm's Theorems for Canonical Equation of the Second Order with Non-monotonous Coefficients

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Abstract. For canonical equation $y'' + a(x) \cdot y = 0$, if $a(x)$ is strictly monotonous function, we have determined number of zeros and exact locations of zero solutions which are main weak points of classic Sturm's theorems. However, if in equation, $y'' = -a(x) \cdot y$, $a(x)$ is non-monotonous function, then function $F(x, y) = -a(x) \cdot y$ is not monotonous, as well and its inverse function is the multi-valued one. This naturally influences both oscillarity and zero oscillations. Therefore, this case has been considered in this work. When the coefficient $a(x)$ is non-monotonous function, we have first determined all intervals of monotony of coefficient. Then in each interval, we have determined number of zeros and very precise locations of zeros, linearly particular solutions y_1 and y_2 and thus improved Sturm's theorems, which are almost 180 years old.

Keywords: Differential equations, iterations, oscillating solutions, locations of zero oscillations, non-monotonous coefficient, amplitudes, frequencies

1. Introduction

Let us have canonical differential equation

$$y'' + a(x) \cdot y = 0 \quad (1.1)$$

for which the following is valid:

- 1) $a(x) > 0$,
- 2) $\int_0^{+\infty} a(x) dx = +\infty$

Then on the semi-axis $[0, +\infty)$ both particular solutions of the equation are oscillating and have infinite number of zeros. This is in accordance with the physical meaning of the coefficient $a(x)$ which represent difference between forces which trigger oscillations and resistance. Since $a(x) \neq 0$, i.e. $a(x) > 0$, from the equation $y'' = -a(x) \cdot y$ it follows that $y'' = 0$ if and only if $y(x) = 0$, which means that zero solutions correspond to curving points of the solution. The curving points are simple curves with steep tangent and they cannot be contacts, i.e. curving points with horizontal tangent, which derives from the derivative of the given equation:

$$y''' = -a'(x) \cdot y - a(x) \cdot y' \quad (1.2)$$

If ξ is zero solution of $y(x)$, i.e. $y(\xi) = 0$ and since $a(\xi) > 0$, then we have that $y'''(\xi) \neq 0$, because $y'(\xi) \neq 0$, which follows based on a classic theorem according to which the solutions of the linear differential equation of the second order cannot have double zeros. Based on this, it follows that for the quality of the curve, including quality of zero solutions, derivative $a'(x)$ is important, as well as its fore-sign. Furthermore, it is very important whether $a(x)$ is monotonous or not, which Sturm could not see due to his algebraic approach to the problem. If coefficient $a(x)$ is non-monotonous function, then the number of zero solutions of the equation $y'' + a(x) \cdot y = 0$ is determined by firstly determining intervals of monotony. Then in each interval separately the number of zeros and locations of zeros are determined. Sturm also sensed this to some point when he defined his famous theorem on minimum number of zeros (case when $a(x)$ is strictly monotonous in $[0, X]$), but he did not have better mathematical apparatus to establish and introduce a definite proof for it.

However, despite of this, each basic literature on oscillating solutions of differential equation contains also Sturm's theorems on minimum number of zero oscillations in canonical interval which goes as follows:

Theorem 1.1 Let us have the following equation:

$$(\varphi \cdot y')' + g \cdot y = 0 \quad (1.3)$$

with two more equations with constant coefficients:

$$ky'' + k\left(\frac{m\pi}{b-a}\right)^2 \cdot y = 0, \quad (1.4)$$

$$Ky'' + K\left(\frac{m\pi}{b-a}\right)^2 \cdot y = 0 \quad (1.5)$$

In this case the following is valid:

1. each non-trivial solution of the equation (1.3) on canonical interval $[a, b]$ has maximum one zero if $g \leq 0$ or if $\varphi \geq k > 0$ and $g < k\left(\frac{m\pi}{b-a}\right)^2$
2. each non-trivial solution of the equation (1.3) on segment $[a, b]$ has no less than m zeros if $0 < \varphi \leq K$ and $g > K\left(\frac{m\pi}{b-a}\right)^2$, where m is a whole number and $m \geq 1$.

Sturm proved Theorem 1.1 by method of comparing coefficients of differential equation (1.3) by framing them between two constants k and K , which are coefficients of equations of harmonic oscillations. In this way he applied avant-garde approach in using method of differential inequalities and non-equations for evaluation of squarely insolvable equations. However, we cannot find this theorem satisfying, because it does not determine better and more precise zero locations. Furthermore, it does not determine the precise number of zeros either, but it only specifies the minimum number of zeros.

Today, there are works on zero solutions of differential oscillating equation, but they are based on theory of groups (Boruvka, Gregus, Biernacki, Suyama), while the thorough sources on these results can be found in famous monographs, those of Kamke [1] and Lamberto Cesari [2]. These works have not found their places in large books and manuals on differential equations (Eins, Hartman, Coddington-Levnison, Matveev, Pontryagin, etc). We also do not have impression that they are simple and adequate for engineering practice. According to our results, the right method for determination of number of zeros and very precise locations of zero oscillations is the method of iteration sequences, which is the most direct, but it has not been used, because it has not been worked out.

By applying this method we have solved not the equation (1.3), but its canonical equation:

$$y'' + a(x) \cdot y = 0 \quad (1.6)$$

for which it is

$$a(x) = \frac{\varphi}{g} - \frac{1}{2} \left(\frac{\varphi'}{\varphi} \right)' - \frac{1}{4} \left(\frac{\varphi'}{\varphi} \right)^2. \quad (1.7)$$

By series of works since 1993, we have shown that the equation (1.6), when the coefficient $a(x)$ fulfills the conditions 1) and 2), is solved most easily if integral form

$$y = C_1 x + C_2 - \int_0^x \int_0^x a(x) y(x) dx^2 \quad (1.8)$$

is submitted to iterations, regardless of general nature of the coefficient $a(x)$. Thus, the solution in the form of iteration sequences is easily obtained:

$$\begin{aligned} y = & C_1 \left(x - \int_0^x \int_0^x x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x x a(x) dx^2 - \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x x a(x) dx^2 + \dots \right) \\ & + C_2 \left(1 - \int_0^x \int_0^x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 - \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 + \dots \right) \end{aligned} \quad (1.9)$$

Even at first glance, it can be seen that for $a(x) > 0$, this solution has the structure of sine and cosine, and these are exactly obtained for $a = \text{const}$. If $a(x)$ is limited (due to continuity) on canonical interval $m \leq a(x) \leq M$, then the upper sequences made of successive integrals (iterations) are actually some non-elementary functions similar to simple Euclid's sine and cosine which depend on $a(x)$. Nevertheless, this $a(x)$ does not change drastically their usual main properties – schedule of zeros, change of sign, limitation, curving points in zeros, changes of curves, oscillating nature. Testing this, we have obtained the base for statement which says that solutions of the equation (1.6) given with (1.9) have the following form:

$$y_1 = \sin_{a(x)} x = x - \int_0^x \int_0^x x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x x a(x) dx^2 - \dots, \quad (1.10)$$

$$y_2 = \cos_{a(x)} x = 1 - \int_0^x \int_0^x a(x) dx^2 + \int_0^x \int_0^x a(x) dx^2 \int_0^x \int_0^x a(x) dx^2 - \dots \quad (1.11)$$

and that general solution is:

$$y = C_1 \cos_{a(x)} x + C_2 \sin_{a(x)} x \quad (1.12)$$

By using the theorem on mean value of integral and method of geometric means in some definite interval $[0, x]$, but which can also be big, we easily obtained very precise approximate formulae for the above generalized (general)

sine and cosine. However this is now complex argument which depends on $a(x)$. Namely, we obtain:

$$y_1 = \sin_{a(x)} x \approx \frac{1}{\sqrt{a(x)}} \cdot \sin(x\sqrt{a(x)}), \quad (1.13)$$

$$y_2 = \cos_{a(x)} x \approx \cos(x\sqrt{a(x)}), \quad (1.14)$$

which are some complete functions of the complex argument. Now, based on aforesaid, it has not been difficult to determine approximately much better zero locations, which Sturm completely missed.

The following theorem is valid:

Theorem 1.2 Zeros of oscillating solutions of the equation (1.6) are approximately located in solutions of plain (algebraic or transcendent) equations in respect to x :

$$\text{for } y_1: x\sqrt{a(x)} = n\pi, \quad n=0,1,2,\dots, \quad (1.15)$$

$$\text{for } y_2: x\sqrt{a(x)} = (2n-1) \cdot \frac{\pi}{2}, \quad n=1,2,3,\dots \quad (1.16)$$

Graphically, this becomes quite obvious, if based on given function $a(x)$, $\sqrt{a(x)}$ is found, and then the graph of the function $F(x) = x \cdot \sqrt{a(x)}$ is cut with horizontal lines

$$y = n\pi \quad \text{and} \quad y = (2n-1) \cdot \frac{\pi}{2}, \quad (1.17)$$

whose heights increasingly rise.

The consequences of this, obviously good result, are as follows:

1. Sturm's theorem on intricacy of zero solutions is immediately obtained
2. From the derivatives of solutions

$$y_1' = 1 - \int_0^x a(x) \sin_{a(x)} x dx, \quad (1.18)$$

$$y_2' = - \int_0^x a(x) \cos_{a(x)} x dx, \quad (1.19)$$

the same arrangement is obtained for zeros of derivatives.

3. Theorem (1.1) on number of zeros, which is not less than m , can now be formulated more precisely. If we put $\varphi = 1$ in (1.3), then the condition $0 < \varphi \leq K$ is satisfied for $K=1$. In this case the second condition:

$$g > K \left(\frac{m\pi}{b-a} \right)^2, \quad (1.20)$$

becomes $g > \left(\frac{m\pi}{x} \right)^2$, because we consider the interval $[0, X]$. Since $\varphi = 1$, then the function φ is not valid any more so we can put $g(x) = a(x)$, i.e. we have symbolism of the equation (1.1). Then, according to Sturm, the following is valid:

$$a(x) > \frac{m^2 \pi^2}{x^2}$$

i.e. our equation for finding zeros is obtained.

$$x^2 a(x) > m^2 \pi^2, \quad m = 1, 2, \dots, \text{ i.e. } x\sqrt{a(x)} > m\pi.$$

Now, everything seems much clearer and number of sections of the function $F(x) = x \cdot \sqrt{a(x)}$ with horizontal lines depends considerably on behavior of the coefficient $a(x)$. If $a(x)$ also has oscillations on its side, then the number of sections can be considerably higher, which leads to higher number of zeros and consequently to higher frequencies. Then the following theorem is valid:

Theorem 1.3. If $a(x)$ is monotonously rising function on $[0, X]$ and if the value $F(X)$ in the end point x is maximum of the function $F(X)$ and $m\pi \leq F(X) < (m+1)\pi$, then the solution y_1 has precisely $m+1$ zero on a closed interval $[0, X]$, while the other solution y_2 has exactly m zeros on the same interval.

The reason for solution y_1 to have one zero more than solution y_2 lies in the fact that sine always has one zero more than cosine in a closed interval which contains the whole number of full periods of these functions. This is obvious both from analytic sequences for $\sin x$ and $\cos x$ and from iteration sequences for the same functions.

2. Main results

If non-continuous function $a(x)$ is not monotonous and if it oscillates infinite number of times remaining positive, as, for example, in equation

$$y'' + [(x-1) \cdot (x-2) \cdot (x-3) \dots (x-n) + M] \cdot y = 0, \quad (2.1)$$

where M is high enough so that $a(x)$ can remain positive for each $x > 0$, then the number of zeros must be searched separately in every interval of monotony either of rising or falling.

Fig. 1.

If $a(x)$ is a positive function with definite number of zeros of derivative $a'(x) = 0$ and if $a(x) \rightarrow +\infty$ when $x \rightarrow +\infty$, then the function of the frequency $F(x) = x\sqrt{a(x)}$ seems somehow similar and the following is important for it: 1) it has compulsory zero in the point $x=0$; 2) the stationary points of the functions $a(x)$ and $F(x)$ do not correspond because:

$$F'(x) = \sqrt{a(x)} + \frac{x \cdot a'(x)}{2\sqrt{a(x)}} = 0 \quad (2.2)$$

for those values of x which satisfies the equation:

$$2 \cdot a(x) + x \cdot a'(x) = 0 \quad (2.3)$$

Since $a(x) > 0$ and $x > 0$, then the equation (2.3) can have solutions only in area in which $a'(x) < 0$, i.e. in which $a(x)$ is a falling function. Therefore, if η is an abscissa of maximum of the function $F(x)$ and ξ is an abscissa of the maximum of the function $a(x)$, then $\xi < \eta$, which is very important for these Sturm's theorems.

Fig. 2

Let now the non-monotonous function $a(x)$ fall and reach its minimum in point ζ . Then the function $F(x)$ is also changed during this process and the minimum is reached in some point δ , for which $a'(\delta) < 0$, thus making $\delta < \zeta$

Fig. 3

Therefore, for the arrangements of abscissas of the functions $a(x)$ and $F(x)$ the following inequality is valid:

$$0 < \xi_{\max a(x)} < \eta_{\max F(x)} < \delta_{\min F(x)} < \zeta_{\min a(x)} \quad (2.4)$$

This intricacy, originally, was characteristic for Sturm's theorems.

Theorem 2.1 In one interval of monotony of the function $a(x)$, the abscissas of extremes of functions $a(x)$ and $F(x) = x \cdot \sqrt{a(x)}$ are interlaced successively.

If we now find solutions of the equation (1.15) and (1.16) in each interval of the monotony of $F(x)$, then the number of zeros is searched separately in every interval. First we take the first interval of monotony on which, depending on $a(x)$, the function $F(x)$ reaches its minimum and maximum. On graph 3, it is the interval $[0, \delta]$ on which we intersect non-monotonous graph of the function $F(x)$ with horizontal lines $y = m\pi$. Furthermore, on the interval $[0, \delta]$, i.e. up to the height of maximum of the function $F(x)$, let there be p_1 number of sections, while on interval $[\eta, \delta]$, i.e. up to the height of minimum of the function $F(x)$, let there be another p_2 number of sections. Then the total number of sections of the horizontal lines $y = m\pi$ and function $F(x)$ is $1 + p_1 + (p_1 - p_2)$ where the zero in the point $x=0$ is also obligatorily included. Since $p_1\pi \leq \max F(x) < (p_1 + 1)\pi$, then

$$p_1 = \left\lceil \frac{\max F(x)}{\pi} \right\rceil = M_1 \quad (2.5)$$

Completely analogically the minimum of the function $F(x)$ is between two whole multiples of the number π , i.e. $p_2\pi \leq \min F(x) < (p_2 + 1)\pi$, i.e.

$$p_2 = \left\lceil \frac{\min F(x)}{\pi} \right\rceil = m_2 \quad (2.6)$$

Fig.4

In this way we obtained that on the interval $[0, \delta]$, the total number of zeros is of the sine solution:

$$N(y_1) = 1 + M_1 + (M_1 - M_2) \quad (2.7)$$

The procedure is continued and in the next interval of monotony the number of zeros will be $M_3 - m_2$ in the rising part of the interval, while in the falling part of the interval there will be $M_3 - m_4$, where M_3 is the maximum and m_4 is the minimum of the function $F(x)$ and so on. If we now sum up the number of these sections, then it is the number of zero solutions of y_1 .

However, the coefficient $a(x)$ is considered on canonical interval on which it oscillates the canonical number of times, so this procedure should be finished after many steps in some point until which, according to our construction, there are $2n-1$ maximums $M_1, M_3, M_5, \dots, M_{2n-1}$ and $2n-2$ minimums $m_2, m_4, m_6, \dots, m_{2n-2}$. Furthermore, in the rest of canonical interval, the solution y_1 can have M zeros more, although the number M can be possibly equal to zero.

Theorem 2.2 The number of zeros of the oscillating solutions of the equation $y'' + a(x) \cdot y = 0$ on canonical interval on which $a(x)$ is positive continuous non-monotonous function, which has oscillations from its side, should be as follows:

$$1 + 2 \cdot (\sum M_i - \sum m_i) + M, \quad (2.8)$$

where M_i is the number of multiples of number π up to the maximum of the function $F(x)$ on i^{th} interval, while m_i is number of multiples of number π up to the minimum of the function $F(x)$ on the same interval and M is number of multiples of number π on the last interval within $[0, X]$ where $a'(x) > 0$ and from which $a(x)$ starts monotonously to rise.

Fig. 5

Therefore, the exact number of zeros of sine solution is:

$$N(y_1) = N(\sin_{a(x)} x) = 1 + 2 \cdot (\sum M_i - \sum m_i) + M \quad (2.9)$$

In M_i we marked the indices with odd numbers, while we used even number for indices in m_i in order to connect the adequate maximums with their corresponding minimums.

Analogically we obtain the number of zeros of cosine solution

$$N(y_2) = N(\cos_{a(x)} x) = 2 \cdot (\sum M_i - \sum m_i), \quad (2.10)$$

which is understandable, because sine always has one zero more in closed interval which contains at least one period.

In this way, we have considerably improved Sturm's theorems which are almost 180 years old. Namely, we have determined very precise locations of zeros, as well as number of zero solutions on a definite interval, when the trigger of oscillations is non-monotonous function.

As for the amplitudes of solutions of the equation (1.1), between each two zeros, the problem is more complex and unbalanced, because the amplitudes depend on local extremes of the function of the frequency $F(x) = x\sqrt{a(x)}$.

Note. Oscillations are possible even if $a(x)$ does not tend towards infinity, but $a(x) \rightarrow c > 0$, $x \rightarrow +\infty$, or even $a(x) \rightarrow +0$ when $x \rightarrow +\infty$. Then, in the similar way, the number of zeros can be determined on each definite interval $[0, X]$, but that is another problem, in which we come into conflict with the existing theorems from the theory on stability. It is above all Fatu's theorem which claims that if $a(x) \rightarrow c > 0$, $x \rightarrow +\infty$, then all solutions of the equation of oscillations are limited (analogy with plain sine and cosine which deceived Fatu and which actually is not correct). Fatu's theorem was wrong, but it contributed to the theory on stability of oscillations. Counter examples were given by Hukahara, Caciopoli, Peron, Wintner and others. Fatu's theorem was first corrected by the Japanese Hukahara whose theorem claims that if $a(x) \rightarrow c > 0$, $x \rightarrow +\infty$ and also if the integral is divergent, i.e.

$$\int_0^{+\infty} |a(x) - c| dx < +\infty,$$

it is only then that all solutions (all oscillations) are limited. However, this does not reduce significance of Fatu's endeavors. Next proofs for stability of oscillations were provided by Hukahara, Naguma, Caciopoli and many others. This opened an important chapter in testing stability of oscillations in general and not only those of the second order. However, this goes beyond scope of this work.

Example 1 Let us have differential equation $y'' + A(x-1) \cdot (x-2)y = 0$ with non-monotonous coefficient $a(x)$ which has oscillating solutions in the intervals $\{0,1\}$ and $[2,+\infty)$ for $A > 0$. Since $a(x) = A(x-1)(x-2)$, $F(x) = x\sqrt{a(x)}$, $a(0) = 2A$, $a(1) = 0$, $F(0) = 0$, $\max F(x) = 0,7 \cdot \sqrt{0,4A}$, then the number of zeros, i.e. the number of crossing points of the function of frequency $F(x)$ and horizontal lines $y = n\pi$ in the interval $[0,1]$ is $N = 1 + 2 \cdot \frac{0,7\sqrt{0,4A}}{\pi}$. The

number of zeros in the interval $[2, +\infty)$ is $M = \frac{\sqrt{98 \cdot 90} A}{\pi}$, so the total number of zeros of oscillations is $N(y_1) = \frac{\sqrt{A}}{\pi} \left(\sqrt{98 \cdot 90 + 1,4\sqrt{(0,4)}} \right) + 1$.

Example 2 As for differential equation $y'' + (x^2 \sin^2 x)y = 0$, which also has non-monotonous coefficient $a(x)$, it is $F(x) = x\sqrt{a(x)} = x^2 |\sin x|$, $\max F(x) \approx \left((2n-1) \frac{\pi}{2} \right)^2$. Therefore, the number of zeros, i.e. the number of frequencies is $N(y_1) = 1 + 2 \sum_{n=1}^m \frac{\left((2n-1) \frac{\pi}{2} \right)^2}{\pi}$.

3. Conclusion

The linear canonical differential equation (1.1) is equation of oscillations if conditions 1) and 2) are fulfilled or if:

1. $a(x) > 0$ on the segment $[0, x]$,
2. $a(x)y \in (C, Lip)D$, where canonical area is D:
$$\begin{aligned} 0 &\leq x \leq X, \\ a &\leq y \leq b. \end{aligned}$$

We have also reduced the famous Sturm's equation (1.3) to the equation (1.1). Sturm determined the conditions for this equation in the sense that, when its solutions are oscillations, they have infinite number of zeros, while when they are monotonous, they have maximum one Sturm's zero. According to classical qualitative Sturm's theorems, some ten of them, the zeros of oscillations y_1 and y_2 mutually intertwine, between every two zeros, there is one zero of the derivative of the other zero and the number of zeros cannot be less than some number m . However, the main deficiency of the Sturm's theorems is the fact that they determine neither the exact number of zeros not their locations. Furthermore, as the greatest algebraist of his time, Sturm noticed the importance of fore-sign of the coefficient of the equation for the nature of solution, but he did not realize the importance of monotonous i.e. non-monotonous nature of the initiator of oscillation. In this work, we have actually considered the case when the coefficient of the equation is non-monotonous function. We have split that interval into smaller intervals of monotony. For each of these, we have determined number of zeros and very precise locations of zeros as cross section of the function of frequency $F(x) = x\sqrt{a(x)}$ and horizontal lines $y = k\pi$ and $y = (2k-1)\frac{\pi}{2}$. In this way, we have improved classical Sturm's theories which are almost 150 years old.

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