Exact Two Waves Solutions with Variable Amplitude to the KdV Equation¹

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Abstract

By means of the Darboux transformation method, a series of explicit variable amplitude-two-wave solutions to the Korteweg-de Vries (KdV) equation are generated from a particular solution with respect to the independent variable x . These new solutions include single soliton solutions, two-soliton solutions, single periodic solutions, doubly periodic solutions.

Mathematics Subjection Classification: 35Q51; 35A

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1. Introduction

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1)$$

is a typical soliton equation, its exact expressions of traveling wave solutions have been studied extensively in many papers [1,2,3,4,5,6]. Especially, the line-two-soliton solutions obtained by the Inverse scattering method in [7] have been well known.

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We know that, Darboux transformation method is one of the powerful and direct method for finding exact multiple line-soliton solutions, it is scarcely used for solving other type of multiple waves solutions because of the matter of calculation [8,9,10,11,12]. Generating new solutions from the non-trivial solution amounts to solving the variable coefficient partial differential system by this method, a major difficult is that for a given partial differential equations there is to date no completely systematic method of solving it, work has usually relied on some special techniques. In the paper, the author employ a exploratory method to solve partial differential system, further construct some two-wave solutions which amplitude depends on the independent variable x .

2. Solution to the lax pair

As described in [13], the Lax pair for equation (1) is given by

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \Phi, \\ \Phi_t = \begin{pmatrix} u_x & -(4\lambda + 2u) \\ -(4\lambda + 2u)(\lambda - u) + u_{xx} & -u_x \end{pmatrix} \Phi, \end{cases} \quad (2)$$

with the Darboux matrix

$$D(x, t, \lambda) = \begin{pmatrix} -\sigma_0 & 1 \\ \lambda - \lambda_0 + \sigma_0^2 & -\sigma_0 \end{pmatrix}, \quad (3)$$

where λ, λ_0 are the spectral parameters, in particular, when $\Phi(x, t, \lambda) = (a_{ij}(x, t, \lambda))_{2 \times 2}$ is the fundamental solution matrix to the system (2) with $u = u_0$, σ_0 is defined as

$$\sigma_0 = \frac{a_{21}(x, t, \lambda_0)\mu_0 + a_{22}(x, t, \lambda_0)\gamma_0}{a_{11}(x, t, \lambda_0)\mu_0 + a_{12}(x, t, \lambda_0)\gamma_0}, \quad (4)$$

here, μ_0 and γ_0 are arbitrary constants, but $\mu_0^2 + \gamma_0^2 \neq 0$. Furthermore, if u_0 is a given solution to equation (1), then

$$u_1 = 2\lambda_0 - u_0 - 2\sigma_0^2 \quad (5)$$

becomes new solution for equation (1), which further concludes that

$$u_2 = 2\lambda_1 - u_1 - 2\sigma_1^2 \quad (6)$$

is another new solution based on u_1 , where

$$\sigma_1 = \frac{\tilde{a}_{21}(x, t, \lambda_1)\mu_1 + \tilde{a}_{22}(x, t, \lambda_1)\gamma_1}{\tilde{a}_{11}(x, t, \lambda_1)\mu_1 + \tilde{a}_{12}(x, t, \lambda_1)\gamma_1}, \quad (7)$$

and $\tilde{\Phi}(x, t, \lambda) = D(x, t, \lambda)\Phi(x, t, \lambda) = (\tilde{a}_{ij}(x, t, \lambda))_{2 \times 2}$ is the fundamental solution matrix of the Lax pair on u_1, μ_1, γ_1 are similar to μ_0, γ_0 .

The essential procedure to construct new solutions is to solve the fundamental solution matrix of the lax pair on initial solution u_0 . Substituting $u_0 = -2x^{-2}$ into the system (2) yields

$$\begin{cases} \Phi_x = \begin{pmatrix} 0 & 1 \\ \lambda + 2x^{-2} & 0 \end{pmatrix} \Phi, \end{cases} \quad (8a)$$

$$\begin{cases} \Phi_t = \begin{pmatrix} 4x^{-3} & -4\lambda + 4x^{-2} \\ -4\lambda^2 - 4\lambda x^{-2} - 4x^{-4} & -4x^{-3} \end{pmatrix} \Phi, \end{cases} \quad (8b)$$

(8a) is an ordinary differential system with variable coefficient, we have no effective method to deal with it, as for (8b), when x is looked as a parameter, it is an ordinary differential system with const coefficients. In the case of $\lambda > 0$, with the eigenvalue method, we solve the characteristic equation

$$\begin{vmatrix} \gamma - 4x^{-3} & 4\lambda - 4x^{-2} \\ 4\lambda^2 + 4\lambda x^{-2} + 4x^{-4} & \gamma + 4x^{-3} \end{vmatrix} = 0,$$

and obtain the characteristic roots $\gamma_1 = 4\lambda\sqrt{\lambda}$, $\gamma_2 = -4\lambda\sqrt{\lambda}$. After calculating the characteristic vectors associated with γ_1 and γ_2 , we obtain two basic solutions to (8b)

$$\begin{pmatrix} (\lambda - x^{-2})e^{4\lambda\sqrt{\lambda}t} \\ (x^{-3} - \lambda\sqrt{\lambda})e^{4\lambda\sqrt{\lambda}t} \end{pmatrix}, \begin{pmatrix} (\lambda - x^{-2})e^{-4\lambda\sqrt{\lambda}t} \\ (x^{-3} + \lambda\sqrt{\lambda})e^{-4\lambda\sqrt{\lambda}t} \end{pmatrix}.$$

Of course,

$$k_1(x) \begin{pmatrix} (\lambda - x^{-2})e^{4\lambda\sqrt{\lambda}t} \\ (x^{-3} - \lambda\sqrt{\lambda})e^{4\lambda\sqrt{\lambda}t} \end{pmatrix}$$

is also solution to (8b), where $k_1(x)$ is function to be determined later. In order to probe whether it is a solution of (8a), substituting it into (8a), we have

$$k_1'(x)(x^{-2} - \lambda) = k_1(x)(\lambda\sqrt{\lambda} + x^{-3}) \quad (9a)$$

$$k_1'(x)(\lambda\sqrt{\lambda} - x^{-3}) = -k_1(x)(\lambda^2 + \lambda x^{-2} + x^{-4}), \quad (9b)$$

more interesting, (9a) and (9b) have the same solution $k_1(x) = \frac{x}{1-\sqrt{\lambda}x} e^{-\sqrt{\lambda}x}$. In parallel, we know

$$k_2(x) \begin{pmatrix} (\lambda - x^{-2})e^{4\lambda\sqrt{\lambda}t} \\ (x^{-3} - \lambda\sqrt{\lambda})e^{4\lambda\sqrt{\lambda}t} \end{pmatrix}$$

is also solution to (8b), where $k_2(x) = \frac{x}{1+\sqrt{\lambda}x}e^{\sqrt{\lambda}x}$. In short, we obtain the fundamental solution matrix to the system (8)

$$\Phi(x, t, \lambda) = \begin{pmatrix} -(\sqrt{\lambda} + x^{-1})e^{\xi} & (\sqrt{\lambda} - x^{-1})e^{-\xi} \\ (\lambda + \sqrt{\lambda}x^{-1} + x^{-2})e^{\xi} & (\lambda - \sqrt{\lambda}x^{-1} + x^{-2})e^{-\xi} \end{pmatrix}, \quad (10)$$

where $\xi = \sqrt{\lambda}(4\lambda t - x)$.

When $\lambda < 0$, the fundamental solution matrix (10) is complex, that is to say, the system (8) has two linearly independent complex solutions. We know that the real and imaginary parts of a complex-valued solution are also solutions, thus taking

$$\Phi'(x, t, \lambda) = \begin{pmatrix} \sqrt{-\lambda} \sin \theta - x^{-1} \cos \theta & -(\sqrt{-\lambda} \cos \theta + x^{-1} \sin \theta) \\ I & J \end{pmatrix} \quad (11)$$

as the fundamental solution matrix to the system (8), where $I = \lambda \cos \theta - x^{-1}(\sqrt{-\lambda} \sin \theta - x^{-1} \cos \theta)$, $J = \lambda \sin \theta + x^{-1}(\sqrt{-\lambda} \cos \theta + x^{-1} \sin \theta)$, $\theta = \sqrt{-\lambda}(4\lambda t - x)$.

Similarly, we obtain the fundamental solution matrix to the system (8) for the case of $\lambda = 0$

$$\Phi''(x, t, 0) = \begin{pmatrix} x^2 + 12tx^{-1} & x^{-1} \\ 2x - 12tx^{-2} & -x^{-2} \end{pmatrix}. \quad (12)$$

3. Solutions to the KdV equation

Substituting the entries of (10) into (4) shows

$$\sigma_0 = -x^{-1} - \frac{\lambda_0 x(e^{\xi_0} \mu_0 + e^{-\xi_0} \gamma_0)}{e^{\xi_0} \mu_0 + e^{-\xi_0} \gamma_0 + \sqrt{\lambda_0} x(e^{\xi_0} \mu_0 - e^{-\xi_0} \gamma_0)} \quad (13)$$

with $\xi_0 = \xi(\lambda_0)$. Then substituting (11) into (5) yields

$$u_1 = \frac{-2\lambda_0[(e^{\xi_0} \mu_0 + e^{-\xi_0} \gamma_0)^2 + 4\lambda_0 \mu_0 \gamma_0 x^2]}{[e^{\xi_0} \mu_0 + e^{-\xi_0} \gamma_0 + \sqrt{\lambda_0} x(e^{\xi_0} \mu_0 - e^{-\xi_0} \gamma_0)]^2} \quad (14)$$

in (12), choosing $\mu_0 = e^{c_0}$, $\gamma_0 = e^{-c_0}$ and $\mu_0 = e^{c_0}$, $\gamma_0 = -e^{-c_0}$ respectively, we have new soliton solutions

$$u_{1-1} = \frac{-2\lambda_0(\cosh^2 \eta_0 + \lambda_0 x^2)}{(\cosh \eta_0 + \sqrt{\lambda_0} x \sinh \eta_0)^2}$$

and

$$u_{1-2} = \frac{-2\lambda_0(\sinh^2 \eta_0 - \lambda_0 x^2)}{(\sinh \eta_0 + \sqrt{\lambda_0} x \cosh \eta_0)^2},$$

respectively, where c_0 is an arbitrary constant, $\eta_0 = \xi_0 + c_0$.

Now we construct the variable amplitude-two-soliton solutions to the KdV equation generated from u_1 . For convenience, we first give the new solution which is expressed in terms of σ_0 rather than u_1 , then consider μ_0 and γ_0 in σ_0 . According to the Darboux matrix (3) and (10), we can obtain the fundamental matrix to the lax pair associated with the known solution u_1 in the following manner

$$\tilde{\Phi}(x, t, \lambda) = \begin{pmatrix} -\sigma_0 & 1 \\ \lambda - \lambda_0 + \sigma_0^2 & -\sigma_0 \end{pmatrix} \Phi(x, t, \lambda) = \begin{pmatrix} P & Q \\ S & T \end{pmatrix},$$

where $P = [\lambda + (\sigma_0 + x^{-1})(\sqrt{\lambda} + x^{-1})]e^\xi$, $Q = [\lambda - (\sigma_0 + x^{-1})(\sqrt{\lambda} - x^{-1})]e^{-\xi}$, $S = -[\lambda\sigma_0 + (\lambda - \lambda_0 + \sigma_0^2 + \sigma_0x^{-1})(\sqrt{\lambda} + x^{-1})]e^\xi$, $T = [-\lambda\sigma_0 + (\lambda - \lambda_0 + \sigma_0^2 + \sigma_0x^{-1})(\sqrt{\lambda} - x^{-1})]e^{-\xi}$, from (7), this implies

$$\sigma_1 = -\sigma_0 -$$

$$\frac{(\lambda_1 - \lambda_0)[(e^{\xi_1}\mu_1 + e^{-\xi_1}\gamma_1) + \sqrt{\lambda_1}x(e^{\xi_1}\mu_1 - e^{-\xi_1}\gamma_1)]}{\lambda_1x(e^{\xi_1}\mu_1 + e^{-\xi_1}\gamma_1) + (\sigma_0 + x^{-1})[(e^{\xi_1}\mu_1 + e^{-\xi_1}\gamma_1) + \sqrt{\lambda_1}x(e^{\xi_1}\mu_1 - e^{-\xi_1}\gamma_1)]},$$

where $\xi_1 = \xi(\lambda_1)$. Choosing $\mu_1 = e^{c_1}$, $\gamma_1 = e^{-c_1}$ and $\mu_1 = e^{c_1}$, $\gamma_1 = -e^{-c_1}$ in the above formulation, respectively, we obtain

$$\sigma_{11} = -\sigma_0 - \frac{(\lambda_1 - \lambda_0)(\cosh \xi_1 + \sqrt{\lambda_1}x \sinh \xi_1)}{\lambda_1x \cosh \xi_1 + (\sigma_0 + x^{-1})(\cosh \xi_1 + \sqrt{\lambda_1}x \sinh \xi_1)}, \quad (15)$$

and

$$\sigma_{12} = -\sigma_0 - \frac{(\lambda_1 - \lambda_0)(\sinh \xi_1 + \sqrt{\lambda_1}x \cosh \xi_1)}{\lambda_1x \sinh \xi_1 + (\sigma_0 + x^{-1})(\sinh \xi_1 + \sqrt{\lambda_1}x \cosh \xi_1)}, \quad (16)$$

Substituting (5) and (15) into (6), we get

$$u_2 = -2x^{-2} + 2(\lambda_1 - \lambda_0) \times \frac{\lambda_1(\cosh^2 \eta_1 + \lambda_1x^2) + (\lambda_0 - \sigma_0^2 + x^{-2})(\cosh \eta_1 + \sqrt{\lambda_1}x \sinh \eta_1)^2}{[\lambda_1x \cosh \eta_1 + (\sigma_0 + x^{-1})(\cosh \eta_1 + \sqrt{\lambda_1}x \sinh \eta_1)]^2}. \quad (17)$$

where $\eta_1 = \xi(\lambda_1) + c_1$, c_1 is an arbitrary constant. Substituting (13) into (17), at the same time, choosing $\mu_0 = e^{c_0}$, $\gamma_0 = e^{-c_0}$ and $\mu_0 = e^{c_0}$, $\gamma_0 = -e^{-c_0}$, respectively, we obtain the two-soliton solutions

$$u_{2-1} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2A_1^2 - \lambda_0^2B_1^2) - \lambda_0\lambda_1(\cosh \eta_1A_1 - \cosh \eta_0B_1)^2}{x^2(\lambda_1 \cosh \eta_1A_1 - \lambda_0 \cosh \eta_0B_1)^2}$$

and

$$u_{2-2} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2A_2^2 + \lambda_0^2B_1^2) - \lambda_0\lambda_1(\cosh \eta_1A_2 - \sinh \eta_0B_1)^2}{x^2(\lambda_1 \cosh \eta_1A_2 - \lambda_0 \sinh \eta_0B_1)^2},$$

respectively, where $A_1 = \cosh \eta_0 + \sqrt{\lambda_0}x \sinh \eta_0$, $B_1 = \cosh \eta_1 + \sqrt{\lambda_1}x \sinh \eta_1$, $A_2 = \sinh \eta_0 + \sqrt{\lambda_0}x \cosh \eta_0$.

In the same way, compound (5), (6), (13) and (16), we obtain new solutions

$$u_{2-3} = 2 \frac{(\lambda_0 - \lambda_1)x^2(\lambda_1^2 A_1^2 + \lambda_0^2 B_2^2) - \lambda_0 \lambda_1 (\sinh \eta_1 A_1 - \cosh \eta_0 B_2)^2}{x^2(\lambda_1 \sinh \eta_1 A_1 - \lambda_0 \cosh \eta_0 B_2)^2}$$

and

$$u_{2-4} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_0^2 B_2^2 - \lambda_1^2 A_2^2) - \lambda_0 \lambda_1 (\sinh \eta_1 A_2 - \sinh \eta_0 B_2)^2}{x^2(\lambda_1 \sinh \eta_1 A_2 - \lambda_0 \sinh \eta_0 B_2)^2},$$

with $B_2 = \sinh \eta_1 + \sqrt{\lambda_1}x \cosh \eta_1$.

Next, the same calculation as the case $\lambda > 0$ can be done in $\lambda < 0$, combining (11) with (4), we obtain

$$u'_{1-1} = \frac{-2\lambda_0(\cos^2 \theta_0 + \lambda_0 x^2)}{(\cos \theta_0 - \sqrt{-\lambda_0}x \sin \theta_0)^2} \quad (18)$$

and

$$u'_{1-2} = \frac{-2\lambda_0(\sin^2 \theta_0 + \lambda_0 x^2)}{(\sin \theta_0 + \sqrt{-\lambda_0}x \cos \theta_0)^2} \quad (19)$$

with $\theta_0 = \theta(\lambda_0)$. In exactly the same manner, we further obtain new doubly periodic solutions

$$u'_{2-1} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2 A_1'^2 - \lambda_0^2 B_1'^2) - \lambda_0 \lambda_1 (\cos \theta_1 A_1' - \cos \theta_0 B_1')^2}{x^2(\lambda_1 \cos \theta_1 A_1' - \lambda_0 \cos \theta_0 B_1')^2},$$

$$u'_{2-2} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2 A_2'^2 - \lambda_0^2 B_1'^2) - \lambda_0 \lambda_1 (\cos \theta_1 A_2' - \sin \theta_0 B_1')^2}{x^2(\lambda_1 \cos \theta_1 A_2' - \lambda_0 \sin \theta_0 B_1')^2},$$

$$u'_{2-3} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2 A_1'^2 - \lambda_0^2 B_2'^2) - \lambda_0 \lambda_1 (\sin \theta_1 A_1' - \cos \theta_0 B_2')^2}{x^2(\lambda_1 \sin \theta_1 A_1' - \lambda_0 \cos \theta_0 B_2')^2}$$

together with

$$u'_{2-4} = 2 \frac{(\lambda_1 - \lambda_0)x^2(\lambda_1^2 A_2'^2 - \lambda_0^2 B_2'^2) - \lambda_0 \lambda_1 (\sin \theta_1 A_2' - \sin \theta_0 B_2')^2}{x^2(\lambda_1 \sin \theta_1 A_2' - \lambda_0 \sin \theta_0 B_2')^2},$$

where $A_1' = \cos \theta_0 - \sqrt{-\lambda_0}x \sin \theta_0$, $B_1' = \cos \theta_1 - \sqrt{-\lambda_1}x \sin \theta_1$, $A_2' = \sin \theta_0 + \sqrt{-\lambda_0}x \cos \theta_0$, $B_2' = \sin \theta_1 + \sqrt{-\lambda_1}x \cos \theta_1$.

Finally, from (4) and (12), we get nonlinear rational solution

$$u'' = \frac{8\mu_0 x(12\mu_0 t + \gamma_0)}{(\mu_0 x^3 + 12\mu_0 t + \gamma_0)^2}.$$

References

- [1] J. L. Varez, A. Durán, Error propagation when approximating multi-solitons: The KdV equation with as a case study, *Applied Mathematics and Computation*, 217 (2010), 1522-1539.
- [2] J. l. Yin, L.X. Tian, Classification of the traveling waves in the nonlinear dispersive KdV equation. *Nonlinear Analysis*, 73(2010), 465-470.
- [3] A. Biswas, M. D. Petkovic, D. Milovic, Topological and non- Topological exact soliton solution of the KdV equation. *Commun Nonlinear Sci Numer Simulat*, 15(2010), 3263-3269.
- [4] M. Nivala, B. Deconinck. Periodic finite-genus solutions of the KdV equation are orbitally stable. *Physic D*, 239(2011), 1147-1158.
- [5] Y. Yamamoto, T. Nagase, M. Ohmiya, Appell's lemma and conservation laws of KdV equation, *Journal of Computational and Applied Mathematics*, 233(2010), 1612-1618.
- [6] N. K. Ameine, M. A. Ramadau, A small time solutions for the KdV equation using Bubnov-Galerkin finite element method. *Journal of the Egyptian Mathematical Society*, 9(2011), 118-123.
- [7] M.J. Ablowitz, P. A. Clarkson, *Solitons, Nonlinear Evolutions and Inverse Scattering*. Cambridge University Press, 1991, 23-99.
- [8] X. M Li, A. H. Chen, Darboux transformation and multi-soliton solutions of Boussinesq-Burgers equation *Physics Letters A*, 342(2005), 413-420.
- [9] Y. Wang, L. J. Shen, Dianlou Du. Darboux transformation and explicit solutions for some (2+1)-dimensional equation. *Physics Letters A*, 366(2007), 230-240.
- [10] H. X. Wu, Y. B. Zeng, T. Y. Fan, Complexions of the modified KdV equation by Darboux transformation. *Applied Mathematics and Computation*, 196(2008), 501-510.
- [11] H. A. Zedan, Exact solutions for the generalized KdV equation by using Backlund transformation, *Journal of the Franklin institute*, 348(2011), 1751-1768.
- [12] Y. Huang, New no-traveling wave solutions for the Liouville equation by Bäcklund transformation method, *Nonlinear Dynamics* 72(2013), 87-90.

- [13] C.H. Gu, H.S. Hu, Z.X. Zhou, Darboux Transformation In Soliton Theory And Its Applications On Geometry. Shanghai Scientific and Technical Publishers, China, 2005.

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