Convexity and Two Piece Property
in n-Hyperbolic Spaces

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Abstract

A subset $A$ of $H^n$, $n \geq 2$, has horosphere two piece property if any horosphere cuts $A$ into at most two pieces. A subset $A$ of $H^n$ has totally geodesic two piece property if any totally geodesic hypersurface in $H^n$, $n \geq 2$, cuts $A$ into at most two pieces. In this article we study some geometric properties of convexity and these types of two piece property in $H^n$ and the relations between them.

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1 Introduction

The notions of convexity and two piece property are very interesting, useful and have many applications in both geometry and analysis [1, 3]. Convexity and two piece property have been generalized in many aspects and for different reasons [1, 3, 4]. In this article we introduce two generalizations of two piece property in the hyperbolic space $H^n$, namely, horosphere and totally geodesic two piece properties and we study some important related concepts in geometry.

2 Notations and Definitions

Let $p$ and $q$ be two points in $H^n$, $n \geq 2$, the distance between $p$ and $q$ is denoted by $d(p, q)$. The geodesic sphere $S(p, r)$ of center $p$ in $H^n$ and radius $r$ is given by $S(p, r) = \{ x \in H^n : d(p, x) = r \}$.

Let $p$, $q$ be two points in $H^n$. For each $v \in T_pH^n$ and each $s \geq 0$ define

$$b_{vs}(q) = s - d(\alpha(s), q)$$

where $\alpha(s)$ is a unit speed geodesic with properties $\alpha(0) = p$ and $\alpha'(0) = v$. The functions $b_{vs}$ are all smooth except at $\alpha(s)$, increasing with $s$ and absolutely bounded by $d(\alpha(0), q)$. Hence the function $b_v = \lim_{s \to \infty} b_{vs}$ is defined everywhere on $H^n$. Call $H_v = b_v^{-1}(0)$ the horosphere and $D_v = b_v^{-1}((0, \infty))$ the horodisc characterized by $v$. The function $b_v$ is called the Busemann function [5].

An alternative geometric definition of horosphere is introduced as follows. Let $\alpha$ be the geodesic ray starting at $p = \alpha(0)$ with $\alpha'(0) = v$, and consider the geodesic spheres through $p$ with center $\alpha(t)$, $t > 0$. As $t$ goes to infinity, these spheres converge to the horosphere $H_v$ through $p$. Roughly speaking, we can define horospheres in $H^n$ as the limit of a specific geodesic sphere sequence as the radius tends to infinity [5].

Let $A$ be a subset of $H^n$, $n \geq 2$. $A$ is said to be convex if for each pair of points $x$, $y$ in $A$, the geodesic segment joining $x$ and $y$ is contained in $A$. A convex subset $A \subset H^n$ is said to be strictly convex if its boundary $\partial A$ contains no geodesic segments, for example closed geodesic balls and closed horodesics in $H^n$ are strictly convex sets [2]. It is clear that the intersection of two convex sets in $H^n$ is again convex [4]. Any point in $H^n$ is a pole i.e all geodesics from a point in $H^n$ do not intersect.

$A$ is said to have totally geodesic two piece property (in brief TTPP) if any totally geodesic hypersurface in $H^n$ cuts $A$ into at most two pieces. $A$ is said
to have horosphere two piece property (in brief HTPP) if any horosphere cuts $A$ into at most two pieces.

A horosphere $H_v$ is said to be a supporting horosphere of a subset $A$ in $H^n$ if

1. the intersection of $H_v$ and $\bar{A}$ is non empty.

2. $A$ is a subset of $\bar{D}_v$, where $\bar{D}_v$ is the corresponding closed horodisc of $H_v$ [4], for example any geodesic ball in $H^n$ is supported at each boundary point by a horosphere.

Any horodisc $D_v$ is supported at each boundary point by a horosphere.
A totally geodesic hypersurface $T$ in $H^n$ is said to be a supporting totally geodesic hypersurface of a subset $A$ in $H^n$ if

1. the intersection of $T$ and $\bar{A}$ is non empty.

2. $A$ lies on one side of $T$, for example any geodesic ball in $H^n$ is supported at each boundary point by a totally geodesic hypersurface.

**Definition 1** A subset $A$ of $H^n$ with $\text{int}(A) \neq \emptyset$ is said to be $h$-convex if $A$ is supported at each boundary point of $A$ by a horosphere $H_v$ for some $v$.

A point $p$ in a subset $A$ of $H^n$ is said to be $H$-exposed point of $A$ if $A$ is supported by a horosphere $H_v$ at $p$, for some $v$, and $H_v$ intersects $A$ only at $p$ [4]. A point $p$ in a subset $A$ of $H^n$ is said to be $T$-exposed point of $A$ if $A$ is supported by a totally geodesic hypersurface $T$ at $p$ and $T$ intersects $A$ only at $p$.

For example all boundary points of a geodesic ball are $H$-exposed and $T$-exposed points, while the boundary points of a horodisc are $T$-exposed but not $H$-exposed.

One may say that Busemann functions in Hyperbolic space $H^n$ play the same role of height functions in Euclidean space $E^n$. Consequently we may state the following definition.

**Definition 2** A subset $A$ of $H^n$ is said to be tight if each Busemann function $b_v$ when restricted to $A$ has one strict local maximum at most.
3 Results

In this section we present the main results of this paper and we begin with the following lemmas.

**Lemma 3** Any convex subset of \( H^n \) has TTPP.

**Proof 4** Suppose that a convex subset \( A \) of \( H^n \) does not have TTPP. Then there exists a totally geodesic hypersurface \( T \) which cuts \( A \) into more than two pieces. Suppose that \( T \) cuts \( A \) into three pieces \( A_1, A_2 \) and \( A_3 \), then at least two of the pieces, say \( A_1 \) and \( A_2 \), lie on one side of \( T \). Let \( p \) and \( q \) be two points in \( A_2 \) and \( A_1 \) respectively as indicated in Figure 1. Since \( T \) is convex and each point in \( H^n \) is a pole, the geodesic segment \([pq]\) lies completely in one side of \( T \). Thus \([pq]\) \( \not\subset \) \( A \) i.e \( A \) is not convex which is a contradiction and the proof is complete.

![Figure 1:](image)

**Remark 5** The converse of the above lemma is not generally true. For example the area between two concentric geodesic circles (annulus) in \( H^2 \) has TTPP but it is not convex as in Figure 2.

**Lemma 6** Let \( A \) be a closed subset of \( H^n \) with smooth boundary and \( \text{int} \; A \neq \emptyset \). \( A \) is convex if and only if \( A \) is supported at each boundary point by a totally geodesic hypersurface.

**Proof 7** Suppose that \( A \) is supported at every boundary point by a totally geodesic hypersurface. Let \( y \) be any point in \( H^n \) and does not belong to \( A \). Since \( A \) is closed, there is a totally geodesic hypersurface \( T_y \) supports \( A \) and
separates \( y \) from \( A \). It is clear that \( A \) is the intersection of all closed half-spaces generated by \( T_y \) for every \( y \) not in \( A \). Thus \( A \) is convex.

Conversely suppose that \( A \) is not supported at a boundary point \( p \) by a totally geodesic hypersurface \( T \), then \( A \) does not lie in one side of \( T \); i.e., \( T \) must cut the interior of \( A \). Let \( q \) be a point in \( \text{int} A \) and lies on \( T \). Let \( B(q, \epsilon) \) be a geodesic ball about \( q \) with sufficiently small radius \( \epsilon \). The geodesic cone with base \( B(q, \epsilon) \) and vertex \( p \) shows that \( A \) is not convex. This completes the proof.

From Lemma 3 and Lemma 5 we have

**Corollary 8** Suppose that \( A \) is a closed subset of \( H^n \) with smooth boundary and \( \text{int} A \neq \emptyset \). If \( A \) is supported at each boundary point by a totally geodesic hypersurface \( T \), then \( A \) has TTPP.

**Lemma 9** Suppose that \( A \) is a closed subset of \( H^n \) with smooth boundary and \( \text{int} A \neq \emptyset \). If each boundary point is \( T \)-exposed, then \( A \) is strictly convex.

**Proof 10** Suppose that each boundary point is \( T \)-exposed. Then \( A \) is supported at each boundary point by \( T \) and \( T \cap A \) is a singleton. From the above lemma \( A \) is convex. Since \( T \cap A \) is a singleton and \( T \) is convex, there is no geodesic segment in the boundary of \( A \); i.e., \( A \) is strictly convex.

From Lemma 3 and Lemma 7 we have

**Corollary 11** Suppose that \( A \) is a closed subset of \( H^n \) with smooth boundary and \( \text{int} A \neq \emptyset \). If each boundary point is \( T \)-exposed, then \( A \) has TTPP.

**Lemma 12** If \( A \) is a subset of \( H^n \) with smooth boundary. Then any \( H \)-exposed point of \( A \) is \( T \)-exposed.
**Proof 13** Suppose that $p$ is an arbitrary $H$-exposed point of a subset $A$ of $H^n$. Then there exists a supporting horosphere $H_v$, for some $v$, with $H_v \cap A = \{p\}$ and $A$ is a subset of $D_v$. Since $D_v$ is supported by a totally geodesic hypersurface $T$ at $p$, $A$ is also supported by $T$ at $p$ and $T \cap A = \{p\}$ i.e $p$ is $T$-exposed point.

**Remark 14** The converse of the above lemma is not generally true, for example, as indicated in Figure 3, $p$ is $T$-exposed point but it is not $H$-exposed.

![Figure 3: p is T-exposed but it is not H-exposed.](image)

From Lemma 7 and Lemma 9 we have

**Corollary 15** Let $A$ be a closed subset of $H^n$ with smooth boundary and $\text{int} A \neq \emptyset$. If each boundary point is $H$-exposed, then $A$ is strictly convex.

**Theorem 16** Let $A$ be a subset of $H^n$ with smooth boundary and $\text{int} A \neq \emptyset$. If $A$ has HTTP, then $A$ has TTTP.

**Proof 17** Suppose that $A$ does not have TTTP. Then there exists a totally geodesic hypersurface $T$ which cuts $A$ into at least three pieces. Let $p$ be a point belonging to $\partial A$ and lies on $T$. Let $H_v(p)$ be the horosphere passing through $p$, where $v$ is a normal vector to $\partial A$ at $p$, then $H_v(p)$ cuts $A$ into at least three pieces since $H_v$ lies on one side of $T$ and it cuts at least one of the pieces in this side to at least two pieces. Thus $A$ does not have HTTP.

**Remark 18** The converse of the above theorem is not generally true. For example a triangle in $H^2$ has TTTP and does not have HTTP as shown in Figure 4.

**Proposition 19** Let $A$ be a closed subset of $H^2$ having TTTP, and $\text{int} A \neq \Phi$. Let $p$ and $q$ be any two points in $\text{int} A$. If $[pq] \subset \text{int} A$, then $A \setminus [pq]$ has TTTP.
Figure 4: TTPP but not HTPP

**Proof 20** Suppose that $A \setminus [pq]$ does not have TTPP, then there exists a geodesic $\gamma$ that cuts $A \setminus [pq]$ into more than two pieces. But $\gamma$ cuts $A$ into at most two pieces, thus $[pq]$ cuts one of the two pieces. Therefore $[pq]$ must cut the boundary of $A$ since $[pq]$ meets $T$ once. Thus $[pq]$ is not contained in the interior which is a contradiction.

Using a similar proof, we get the following proposition:

**Proposition 21** Let $A$ be a closed subset of $H^2$ having HTPP, and $\text{int}A \neq \emptyset$. Let $p$ and $q$ be any two points in $\text{int}A$. If $[pq] \subset \text{int}A$, then $A \setminus [pq]$ has HTPP.

**Proposition 22** Let $A$ be a closed subset of $H^2$ having TTPP, $B$ be a convex subset of $A$ and both $\text{int}A$ and $\text{int}B$ are non-empty. If $B \cap \partial A = \emptyset$, then $A \setminus B$ has TTPP.

**Proof 23** Suppose that $A \setminus B$ does not have TTPP, then there exists a geodesic $\gamma$ that cuts $A \setminus B$ into more than two pieces. But $\gamma$ cuts $A$ into at most two pieces, therefore $B$ intersects one of the two pieces into at least two pieces i.e. $B$ must intersects the boundary of $A$ since $B$ is convex. This contradiction completes the proof.

Using a similar proof, we get the following proposition:

**Proposition 24** Let $A$ be a closed subset of $H^2$ having HTPP, $B$ be a convex subset of $A$ and both $\text{int}A$ and $\text{int}B$ are non empty. If $B \cap \partial A = \emptyset$, then $A \setminus B$ has HTPP.

**Remark 25** In Proposition 15 if we replace convexity of $B$ by TTPP, then $A \setminus B$ does not necessarily have TTPP. For example if $A$ is a geodesic ball in $H^2$ and $B$ is the area between two concentric geodesic balls in $\text{int}A$, then $A \setminus B$ does not have TTPP as shown in Figure 5.
Theorem 26 Suppose that $A$ is a closed subset of $H^n$ with smooth boundary and $\text{int} A \neq \emptyset$. If $A$ is h-convex, then $A$ is strictly convex.

Proof 27 First suppose that $A$ is h-convex. Then $A$ is supported at each boundary point by a horosphere. Since $D_v$ is supported by a totally geodesic hypersurface $T$, $A$ is also supported by $T$ at each boundary point. From Lemma 5 $A$ is convex.

Second suppose that $A$ is not strictly convex. Then there is a geodesic segment, say $[pq]$ in the boundary of $A$ and $[pq]$ is contained in a totally geodesic hypersurface. Let $r$ be any point on $[pq]$ and $H_v$ be a horosphere at $r$ where $v$ is in the direction of the interior of $A$. Since $D_v$ is supported by a totally geodesic hypersurface $T$ at $r$, $A$ is not supported by a horosphere $H_v$ at $r$. Therefore $A$ is not h-convex.

Remark 28 The converse of the above lemma is not generally true, for example the closed half space generated by any totally geodesic hypersurface $T$ is not h-convex but it is convex.

From Lemma 3 we have

Corollary 29 Suppose that $A$ is a closed subset of $H^n$ with smooth boundary and $\text{int} A \neq \emptyset$. If $A$ is h-convex, then $A$ has TTPP.

From the definition of horosphere we conclude that any geodesic ball $B(x, r)$ in $H^n$ is h-convex and hence it has TTPP.

Theorem 30 Suppose that $A$ is a compact subset of $H^n$ with smooth boundary and $\text{int} A \neq \emptyset$. If $A$ has HTPP, then $A$ is tight.

Proof 31 Suppose that $A$ is not tight. Then there exist at least two strict local maxima (say $p$ and $q$) of a Busemann function $b_v$, for some $v$ restricted on $A$. We have two cases:
1. If \( p \) and \( q \) satisfy \( b_v(p) = b_v(q) \) as shown in Figure 6 case 1. Let \( H_v \) be the base of the Busemann function and \( H'_v \) be a parallel horosphere to \( H_v \) which is tangent to \( \partial A \) at \( p \) and \( q \). Moving \( H'_v \) parallel to itself in a sufficiently small neighborhood in the direction of interior of \( A \), we find that the resulting horosphere cuts \( A \) into more than two pieces i.e \( A \) does not have HTPP.

2. If \( p \) and \( q \) satisfy \( b_v(p) \neq b_v(q) \) as shown in Figure 6 case 2. As in the first case let \( H_v \) be the base of the Busemann function and \( H'_v \) be a parallel horosphere to \( H_v \) which is tangent to \( \partial A \) only at \( p \). Moving \( H'_v \) parallel to itself in a sufficiently small neighborhood in the direction of interior of \( A \), we find that the resulting horosphere cuts \( A \) into more than two pieces i.e \( A \) does not have HTPP.

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Theorem 32 Suppose that \( A \) is a closed subset of \( H^n \) with smooth boundary and \( \text{int} \ A \neq \Phi \). If \( A \) is \( h \)-convex, then \( A \) is has HTPP.

Proof 33 Suppose that \( A \) does not have HTPP. Then there exists a horosphere \( H_v \) which cuts \( A \) into more than two pieces and at least two pieces in one side of \( D_v \). Suppose that \( H_v \) is the base of the Busemann function when restricted on \( A \). Let \( p \) and \( q \) be two points on the boundary of \( A \) and each in one piece we have two cases:

1. If \( p \) and \( q \) satisfy \( b_v(p) \neq b_v(q) \) as shown in Figure 6 case 2. Let \( H_v \) be the base of the Busemann function when restricted on \( A \) and \( H'_v \) be a parallel horosphere to \( H_v \) which is tangent to \( \partial A \) only at \( p \) and cuts the interior of \( A \) i.e \( A \) is not supported by \( H'_v \) at \( p \).

2. If \( p \) and \( q \) satisfy \( b_v(p) = b_v(q) \) as shown in Figure 6 case 1. Let \( H_v \) be the base of the Busemann function when restricted on \( A \). Make a small variation of \( v \) to \( v_1 \) and draw a horosphere \( H_{v_1} \) at a point \( p' \in \partial A \)
sufficiently close to p such that $b_{v_1}(p') \neq b_{v_1}(q)$. As in the first case A is not supported by $H_{v_1}$ at p.

The two cases imply that A is not h-convex.

From Theorem 22 and Theorem 23 we have

**Corollary 34** If A is a closed h-convex subset of $H^n$ with smooth boundary and int$A \neq \Phi$ then A is tight.

**Remark 35** The following suggests topics may be considered in future as open problems:

1. Almost all results of this article can be proved in any complete simply connected Riemannian manifold without focal points.

2. Two piece property in $S^n$ deserves a deep discussion as $S^n$ is not free from focal points.

**References**


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